Some numerical topics in stellarator optimization

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Magnetic fields can be used to confine charged particles & plasma.

There is no confinement along the magnetic field, so bend it into a torus.
A *stellarator* is a configuration of magnets for confining plasma without continuous rotation symmetry.

Axisymmetry (tokamak) requires large current in plasma: hard to sustain, unstable, sometimes impossible.
A stellarator is a configuration of magnets for confining plasma without continuous rotation symmetry.

Applications:
- Fusion energy
- Particle trap for basic physics
Example: W7-X (Germany)

Clery, Science (2022)
Fusion startup Type One Energy gets $29M seed round to fast-track its reactor designs

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Stellarator plasma & coil shapes can be optimized for several objectives

- Field lines lie on nested 2D toroidal surfaces, not volume-filling
- Good confinement of particle trajectories
- Buildable coil shapes
- Any plasma current doesn’t modify $B$ too much
- Magnetohydrodynamic stability
- Reduced turbulence
• Optimizing plasma shape, directly targeting particle confinement.
• Optimizing plasma shape using a surrogate (“quasisymmetry”)
• Optimizing plasma shape to reduce turbulence
• Topology optimization for magnets
• Optimizing plasma shape, directly targeting particle confinement.

• Optimizing plasma shape using a surrogate ("quasisymmetry")

• Optimizing plasma shape to reduce turbulence

• Topology optimization for magnets
For plasma shape optimization, parameter space is usually Fourier amplitudes of the boundary surface.

\[ R(\theta,\phi) = \sum_{m,n} R_{m,n} \cos(m\theta - n\phi), \]
\[ Z(\theta,\phi) = \sum_{m,n} Z_{m,n} \sin(m\theta - n\phi) \]

Room for innovation here...

Shape of boundary surface determines the magnetic field \( \mathbf{B} \) inside (up to scale factor).

- Without plasma: \( \mathbf{B} = \nabla V, \nabla^2 V = 0 \) in \( \Omega \), \( \mathbf{n} \cdot \nabla V = 0 \) on \( \partial \Omega \).
- With plasma: \( \nabla \times (\nabla \times \mathbf{B}) = \nabla p \). Also need to specify pressure \( p \) & current on each interior surface.
Charged particle trajectories can be simulated accurately using “guiding center” ODEs.

\[
\frac{dR}{dt} = v_\parallel \mathbf{b} + \frac{mv_\parallel^2}{qB^3} \left( v_\parallel^2 + \frac{v_\perp^2}{2} \right) \mathbf{B} \times \nabla B,
\]

\[
\frac{dv_\parallel}{dt} = -\frac{v_\perp^2}{2B} \mathbf{b} \cdot \nabla B.
\]
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\]
Direct optimization of particle confinement is challenging due to computational cost & noise

![Graph showing energy loss and integrating over initial conditions (objective function)]

Energy loss, integrating over initial conditions (objective function)

Bindel, ML, Padidar, arXiv (2023)
Improve;ment in confinement can be achieved with this direct approach

\[ \text{minimize} \quad J_{\text{penalty}}(w) := J(w) + \sum_{i=1}^{n_{\text{grid}}} \max(B(x_i) - B^*_+, 0)^2 + \max(B^*_- - B(x_i), 0)^2 \]

Fraction of particles lost

Other stellarator shapes

Energy losses
• Optimizing plasma shape, directly targeting particle confinement.
• Optimizing plasma shape using a surrogate ("quasisymmetry")
• Optimizing plasma shape to reduce turbulence
• Topology optimization for magnets
Quasisymmetry is a sufficient (though not necessary) condition for confinement, & a useful surrogate

$B = B(s, \theta - N \varphi)$

“Boozer angles”

Objective: $f_{qs} = \int d^3x \left( \frac{1}{B^3} \left[ (N-t) \mathbf{B} \times \nabla \cdot \nabla \psi - (G+NI) \mathbf{B} \cdot \nabla \mathbf{B} \right]^2 \right)$

For quasi-axisymmetry, $N = 0$. For quasi-helical symmetry, $N$ is the number of field periods, here $N = 4$.

$f_{qh} = (A - A^*)^2 + f_{qs}$

Boundary aspect ratio
Example of quasi-axisymmetry optimization

Optimization details:
• Default algorithm in scipy for nonlinear least-squares ("trust region reflective")
• Initialize with circular cross-section torus.
• Gradients from finite differences or automatic differentiation.
• Expand the number of Fourier modes in the parameter space in steps. 8-120 design variables.
Example of quasi-helical symmetry optimization

ML & Paul, Phys Rev Lett (2022)

|B| on flux surfaces of the quasi-helically symmetric field
Quasisymmetry has been an effective surrogate for finding plasma shapes with good confinement, though it might be too limiting.

Trajectories of fusion-produced distribution of alpha particles followed in many magnetic configurations, all scaled to reactor size and $|B|$: Since 2021

Multi-fidelity method with quasisymmetry + direct ODE integration?
• Optimizing plasma shape, directly targeting particle confinement.

• Optimizing plasma shape using a surrogate ("quasisymmetry")

• Optimizing plasma shape to reduce turbulence

• Topology optimization for magnets
Shape optimization to reduce plasma turbulence would be valuable but is hard

- Gradients of plasma density & temperature cause instabilities & turbulence.
- Causes heat to leak out.
- Known to depend on plasma shape.
- Turbulence simulations: ~10 GPU-min to $10^6$ s of CPU-hours.
- 5D + time
- No agreed-upon surrogate like Reynolds-averaged Navier-Stokes + closure yet.

Nunami 2012
One approach to optimizing turbulence:
minimize linear growth rate of instabilities

Linear growth rate gives a smooth, deterministic, non-chaotic objective

Jorge et al, arXiv (2023)

Heat flux out of the plasma

Stellarator optimized for only quasisymmetry
Stellarator also optimized for linear growth rate

But linear growth rate is not a perfect predictor of the nonlinear heat flux...
First optimizations with nonlinear turbulence calculations in the objective

By Patrick Kim (UMD undergraduate!)

Using stochastic gradient descent with momentum

Bayesian optimization with 20-100 design variables would be nice...
• Optimizing plasma shape, directly targeting particle confinement.
• Optimizing plasma shape using a surrogate ("quasisymmetry")
• Optimizing plasma shape to reduce turbulence
• Topology optimization for magnets
An established method for electromagnetic coil design is available, but has some limitations.

Assume plasma shape has already been optimized, so target B field is known.

Coils represented as space curves. Design variables: Fourier modes of Cartesian components.

\[ x(t) = x_{c,0} + \sum_{n=1}^{N_F} [x_{c,n} \cos(nt) + x_{s,n} \sin(nt)] \]

Objective:

\[ f = \int_{\text{plasma}} (\mathbf{B} \cdot \mathbf{n})^2 + \lambda (\text{length} - \text{target})^2 + \ldots \]

Issues:

- Non-convex, so there are multiple local minima. Need good initial guess.
- Coil topology & number of coils are fixed by initial guess.
Pre-define grid of voxels where current might flow.

Current density $\mathbf{J}$ in each voxel represented by basis of 11 divergence-free functions. Amplitudes are the design variables.

Current conservation at each cell face gives linear equality constraints.

Given basis functions, $\mathbf{B}$ can be computed by Biot-Savart Law:

$$
\mathbf{B}(r) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J'(r')} \times (r - r')}{|r - r'|^3}
$$

Objective:

$$
f = \int_{\text{plasma surf}} (\mathbf{B} \cdot \mathbf{n})^2 + \lambda \int_{\text{voxels}} \|\mathbf{J}\|_2^2 + \nu \|\mathbf{J}\|_1 + \eta \|\mathbf{J}\|_0
$$

- Without sparsity-promoting terms, problem is linear.
- With l1 sparsity, problem remains convex.
- With l0 sparsity, good algorithms exist.
Preliminary results from electromagnet topology optimization: In axisymmetry, expected currents are recovered.
Preliminary results from electromagnet topology optimization: With sparsity objective, currents coalesce into discrete coils.
Preliminary results from electromagnet topology optimization: Starting to look at non-axisymmetric geometries
Conclusions: stellarator optimization is a subject with many challenging numerical problems

This talk:
• Shape optimization for confining particle trajectories, with PDE & possibly ODE solves.
• Shape optimization with linear stability or nonlinear turbulence simulations in the loop.
• Topology optimization for electromagnets.

Many other directions:
• Other surface & curve discretizations
• Multi-fidelity
• Surrogates for confinement or turbulence
• ≥ 4D advection-dominated PDEs
• Uncertainty quantification
• ...
Extra slides
A stellarator is a configuration of magnets for confining plasma without continuous rotation symmetry.

- Applications: fusion energy, or electron-positron plasmas (astrophysics)
- Shape of the plasma and of the coils is chosen by numerical optimization.
Direct optimization of particle confinement is challenging due to computational cost & noise

Objective function to measure energy loss:

$$\mathcal{J}(\mathbf{w}) := \int_x \int_{v_{||}} 3.5 e^{-2T(x, v_{||}, \mathbf{w})/t_{\text{max}}} f(x, v_{||}) \, dv_{||} \, dx.$$
Divergence-free basis functions for current density $J$

$$\phi_i^{(k)} = \begin{bmatrix} 1 & 0 & 0 & Y_k & Z_k & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & X_k \\ 0 & 0 & Z_k & X_k & \frac{\sqrt{3}}{2} & \frac{X_k}{-Y_k} \\ Y_k & 0 & X_k & 0 & \frac{\sqrt{3}}{2} & \frac{X_k}{0} \\ 0 & Z_k & 0 & 0 & -Z_k & 0 \end{bmatrix},$$

where

$$X_k \equiv \frac{x-x_k}{\Delta x_k}, \quad Y_k \equiv \frac{y-y_k}{\Delta y_k}, \quad Z_k \equiv \frac{z-z_k}{\Delta z_k}.$$