The magnetic field scale length:
An influential property of stellarators

$$\nabla B \sim B / L$$

Matt Landreman, John Kappel
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\[ \nabla B \sim \frac{B}{L} \]

• Definitions of \( \nabla B \) scale length

• Expanding volume of quasisymmetry in near-axis expansion
  “Mapping the space of quasisymmetric stellarators using optimized near-axis expansion”, arXiv:2209.11849 (2022)

• Limits on the coil-to-plasma distance
  Why is it hard to find coils for Wistell-B?

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At any point, a magnetic field has multiple gradient length scales
\[ \nabla, \nabla_B, \nabla_{\perp B}, \mathbf{b} \cdot \nabla \mathbf{b} \quad \text{eigenvalues of } \nabla \mathbf{B}, \quad \| \nabla \nabla \mathbf{B} \| \ldots \]
\[ (B = |\mathbf{B}|, \quad \mathbf{b} = \mathbf{B}/B) \]
\|\nabla \mathbf{B}\| \text{ captures largest gradient } \Rightarrow \text{ shortest length scale}

We can get some insights by considering vacuum fields:

\[ \mathbf{B} = \nabla \Phi \text{ so } \nabla \mathbf{B} = \nabla \nabla \Phi \text{ is a symmetric } 3 \times 3 \text{ matrix } \Rightarrow \text{ 6 degrees of freedom.} \]

\[ \nabla \mathbf{B} = \begin{pmatrix} \partial_{xx} \Phi & \partial_{xy} \Phi & \partial_{xz} \Phi \\ \partial_{yx} \Phi & \partial_{yy} \Phi & \partial_{yz} \Phi \\ \partial_{zx} \Phi & \partial_{zy} \Phi & \partial_{zz} \Phi \end{pmatrix} \]

-1 degree of freedom since \( \nabla \cdot \mathbf{B} = 0 \).

Some entries can be made to vanish by rotating the coordinate system.
The $\nabla B$ scale lengths can be normalized so that in the case of an infinite straight wire, they give the distance to the wire.

$$L_{\nabla B} = \frac{\sqrt{2}B}{||\nabla B||}$$
The different $B$ scale lengths are not identical, but have similarities, e.g. all are small on the inside of concave regions

\[ \frac{\sqrt{2B}}{||\nabla B||} \quad \frac{B}{|\nabla B|} \quad \frac{B}{\max(n \cdot \nabla B, 0)} \]

\[ \frac{1}{|b \cdot \nabla b|} \quad \frac{B}{\max|\lambda_j|} \quad B \frac{\sqrt{2}}{\sqrt{\sum \lambda_j^2}} \]

$\lambda_j$ = eigenvalues of $\nabla B$
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Expansion about the magnetic axis is a complementary method to traditional stellarator optimization

Traditional optimization: parameter space is the shape of toroidal boundary surface.

Near-axis expansion:
- Accurate in the core of any stellarator
- 3D PDEs → 1D ODEs in $\phi$.
- Opportunities for analytic insights.
- Can solve & characterize a configuration in < 1 ms
- Can generate new initial conditions that can be refined by optimization.

References:

ML, Sengupta, & Plunk (2019)  
Rodriguez & Bhattacharjee (2021)  
Jorge et al (2022)  
...
The expansion by Garren & Boozer (1991) has been converted into a practical algorithm for generating stellarator shapes.

- **Inputs:**
  - Shape of the magnetic axis.
  - 3-5 other numbers (e.g. current on the axis).

- **Outputs:**
  - Shape of the surfaces around the axis.
  - Rotational transform on axis.
  - ...

- Quasisymmetry can be guaranteed in a neighborhood of axis: \( B = B(r, \theta - N\varphi) \)

- Can pick any surface to pass to traditional 3D MHD fixed-boundary solve.
Problem: The radius of applicability of the expansion is typically small. ⇒ high aspect ratio

- Increasing scale length (decreasing $\nabla B$) may increase the minor radius over which the expansion is accurate. Expansion parameter is $\sim r/L_{\nabla B}$.
- Quasisymmetry fails at $O(r^3)$ [Garren & Boozer (1991)], so increasing the scale length may improve quasisymmetry.
∇B turns out to be a better cost function than aspect ratio: fewer local minima

Line through parameter space, linearly interpolating the Fourier modes in axis shape.

\[ R = 1 + 0.17 \cos 4\phi \]
\[ R = 1 + 0.17 \cos 4\phi + 0.01804 \cos 8\phi + 0.001409 \cos 12\phi + 0.00005877 \cos 16\phi \]
\( \nabla B \) turns out to be a better cost function than aspect ratio: fewer local minima

Lesson: To minimize some quantity \( Q \), even if \( Q \) is fast to compute, the best objective function is not necessarily \( Q \).
Complete objective function used for near-axis expansion

\[ f = \frac{1}{L} \int d\ell \| \nabla B \|^2 + \frac{w_{\nabla \nabla}}{L} \int d\ell \| \nabla^2 \nabla B \|^2 + w_{L} (L - L^*)^2 + w_{t} (t - t^*)^2 \]

\[ + \frac{w_{B_{20}}}{L} \int d\ell \left( B_{20} - \frac{1}{L} \int d\ell' B_{20} \right)^2 + w_{\text{well}} \left[ \max \left( 0, \frac{d^2 V}{d\psi^2} - W_* \right) \right]^2 \]

\[ w_{\nabla \nabla}, w_{L}, w_{t}, w_{B_{20}}, w_{\text{well}} : \text{Weights chosen by user} \]
The near-axis equations can be solved so quickly that tensor-product scans over many parameters are feasible.

- 5.2 × 10^5 optimized stellarators plotted
- 1.3 × 10^7 optimizations performed
- 1.4 × 10^{11} configurations computed
The near-axis equations can be solved so quickly that tensor-product scans over many parameters are feasible.
Some intriguing configurations from these near-axis optimizations

QH with \(\text{nfp}=2\):

QH with \(\text{nfp}=3\), \(\langle \beta \rangle = 4\%\):

Fewer field periods \(\Rightarrow\) fewer coils?

QH with reversed triangularity, no bean:

ML, arXiv:2209.11849 (2022)
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- Limits on the coil-to-plasma distance
  
  Why is it hard to find coils for Wistell-B?
In a reactor, must fit ~ 1.5m “blanket” between plasma and coils to absorb neutrons.

But at fixed plasma shape & size, coils shapes become impractical if they are too far away:

Coils offset a uniform distance from W7-X plasma:
- 25cm separation
- 50cm separation
- 65cm separation

So we must scale everything up:

In a reactor, must fit ~ 1.5m “blanket” between plasma and coils to absorb neutrons

But at fixed plasma shape & size, coils shapes become impractical if they are too far away:

**Hypothesis:**

The coil-to-plasma distance scale for which coils are feasible is ~ the $\nabla B$ scale length.
The small plasma-to-coil separation in stellarators is also a headache for engineering.

“Lesson 1: A lack of generous margins, clearances and reasonable tolerance levels implies an unnecessary increase of the complexity and leads to late design changes. This has a strong impact on schedule, budget, man-power and potentially sours the relationship to funding bodies.”

Klinger et al, Fusion Engineering & Design (2013)
To test hypothesis that $\nabla B$ is related to coil-plasma distance, scale length will be compared to “real” coil designs for a diverse set of 35 configurations.

NCSX (li383 & c09r00)  
ARIES-CS  
HSX  
W7-X (std, high-mirror, ...)  
LHD, R=3.5, 3.6, 3.75  
CFQS  
ML+Paul QA, QA+well  
ML+Paul QH, QH+well  
ML, Buller, Drevlak QH, QA  
Near-axis nfp=3 QH  
Near-axis nfp=4 QH  
Jorge et al nfp=1 QI  
Goodman et al nfp=1 QI  

ESTELL  
ITER  
Garabedian QA  
Henneberg et al QA  
Wistell-A  
Wistell-B  
Wechsung et al QA  
Wechsung et al QA+well  
Giuliani et al QA  
Ku & Boozer nfp=4 QH  
Nuhrenberg & Zille QH  
Drevlak QH  

All scaled to same minor radius (1.7 m) and $\langle B \rangle = 5.9$ T.
scale length will be compared to “real” coil designs from Regcoil

Regcoil: Consider sheet current on a “coil winding surface”

\[ K = n \times \nabla \phi \]

Surface current “current potential” Normal to winding surface

\[
\min_{\phi} \left( \int_{\text{Plasma surface}} d^2 x \left[ (B - B_{\text{target}}) \cdot n \right]^2 + \lambda \int_{\text{Coil surface}} \! d^2 x |K|^2 \right)
\]

\( K = n \times \nabla \phi \)

\( B \) field error

Regularization parameter

\( K = |K| \propto 1/\text{distance between coils} \)

\( \phi \) contours = coils

Regcoil is preferable to Focus/Simsopt for this study for comparison across many stellarators:

- \textit{Linear} least-squares: no local optima besides the global one.
- Only 2 parameters to vary: coil-to-plasma distance and \( \lambda \).
Methodology: Adjust regularization $\lambda$ and coil-to-plasma separation to match $B$ error and coil current density between configurations.

At fixed coil-to-plasma separation, $\lambda$ trades off between $B$ field error and coil complexity.

At the target $B$ field error, coil complexity increases with coil-to-plasma separation.
Main result: $\nabla B$ length is well correlated with real coil designs

$$\min \frac{\sqrt{2B}}{||\nabla B||} = L_{\nabla B} \text{ [m]}$$

All configurations scaled to same minor radius
Main result: $\nabla B$ length is well correlated with real coil designs

$$\min \frac{\sqrt{2B}}{||\nabla B||} = L_{\nabla B} [m]$$
Regcoil coil-to-plasma distance is actually well correlated with other definitions of B length scale too.

\[
\min \frac{1}{|b \cdot \nabla b|} [\text{m}]
\]

All configurations scaled to same minor radius.
Regcoil coil-to-plasma distance is actually well correlated with other definitions of B length scale too.

$$\text{min} \frac{B}{|\nabla B|} [\text{m}]$$
The location of limiting $\nabla B$ length and coil complexity are also correlated spatially.

$$L_{\nabla B} = \frac{\sqrt{2} B}{\| \nabla B \|} \text{[m]}$$

Limiting coil-to-coil distance occurs where scale length is smallest.

Current density $K$ [MA/m]
Conclusion: We should pay attention to $\nabla B$ length scales in stellarators

- Since quasisymmetry is allowed only to $O(r^2)$, reducing $\nabla B$ can expand the volume of good quasisymmetry.

- $\nabla B$ appears to explain the maximum coil-to-plasma distance
  - Driver of size and costs!
  - Significant variation between configurations. 1-field-period QIs look promising.

Future work:
- Compare & understand different scale lengths.
- For $\beta > 0$, check if $\|\nabla B_{\text{external}}\|$ is more meaningful than $\|\nabla B\|$.
- Optimize for small $\|\nabla B\|$ (already in StellaratorOptimizationMetrics.jl).
Extra slides
Main result: $\nabla B$ length is well correlated with real coil designs.

$$L_{\nabla B} = \frac{\sqrt{2B}}{||\nabla B||} \text{ [m]}$$