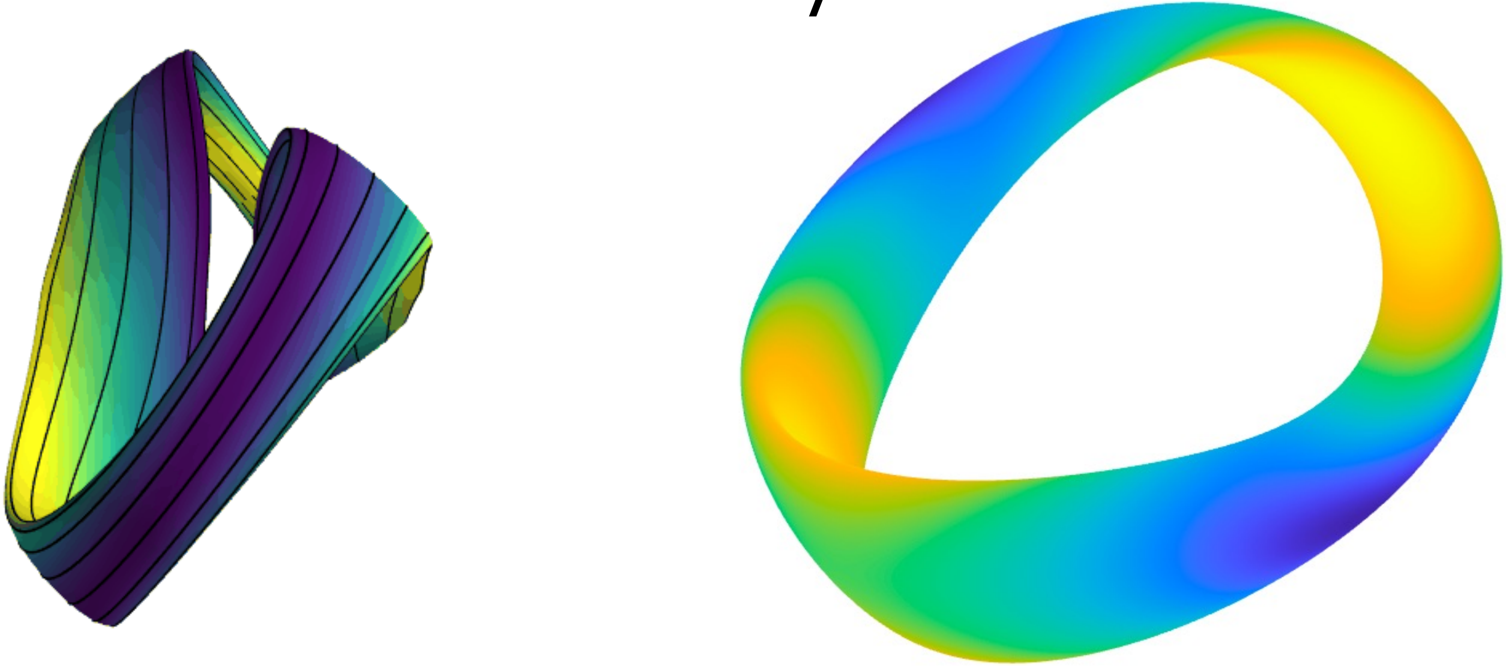
The magnetic field scale length: An influential property of stellarators

$$\nabla \mathbf{B} \sim B / L$$



Matt Landreman, John Kappel University of Maryland

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- Definitions of **∇B** scale length
- Expanding volume of quasisymmetry in near-axis expansion

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At any point, a magnetic field has multiple gradient length scales

∇B, $\nabla_{||} B$, $\nabla_{\perp} B$, $\boldsymbol{b} \cdot \nabla \boldsymbol{b}$, $||\nabla \boldsymbol{B}|| = \sqrt{\nabla \boldsymbol{B} : \nabla \boldsymbol{B}}$, eigenvalues of $\nabla \boldsymbol{B}$, $||\nabla\nabla \boldsymbol{B}||$…

"Frobenius norm"

$(B = |\boldsymbol{B}|, \quad \boldsymbol{b} = \boldsymbol{B}/B)$

$||\nabla \boldsymbol{B}||$ captures largest gradient $\Longrightarrow$ shortest length scale

We can get some insights by considering vacuum fields:

$\boldsymbol{B} = \nabla \Phi$ so $\nabla \boldsymbol{B} = \nabla\nabla\Phi$ is a symmetric 3×3 matrix $\Longrightarrow$ 6 degrees of freedom.

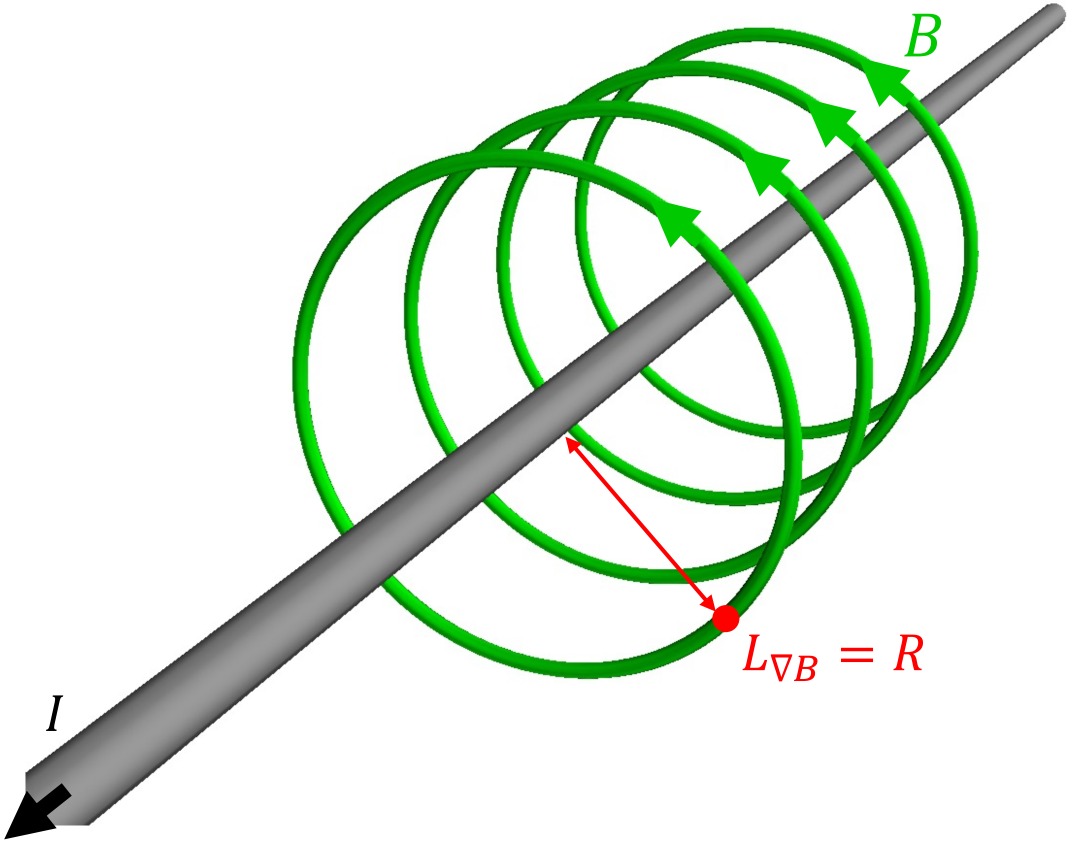
$$\nabla \boldsymbol{B} = \begin{pmatrix} \partial_{xx}\Phi & \partial_{xy}\Phi & \partial_{xz}\Phi \\ \partial_{yx}\Phi & \partial_{yy}\Phi & \partial_{yz}\Phi \\ \partial_{zx}\Phi & \partial_{zy}\Phi & \partial_{zz}\Phi \end{pmatrix}$$

-1 degree of freedom since $\nabla \cdot \boldsymbol{B} = 0$.

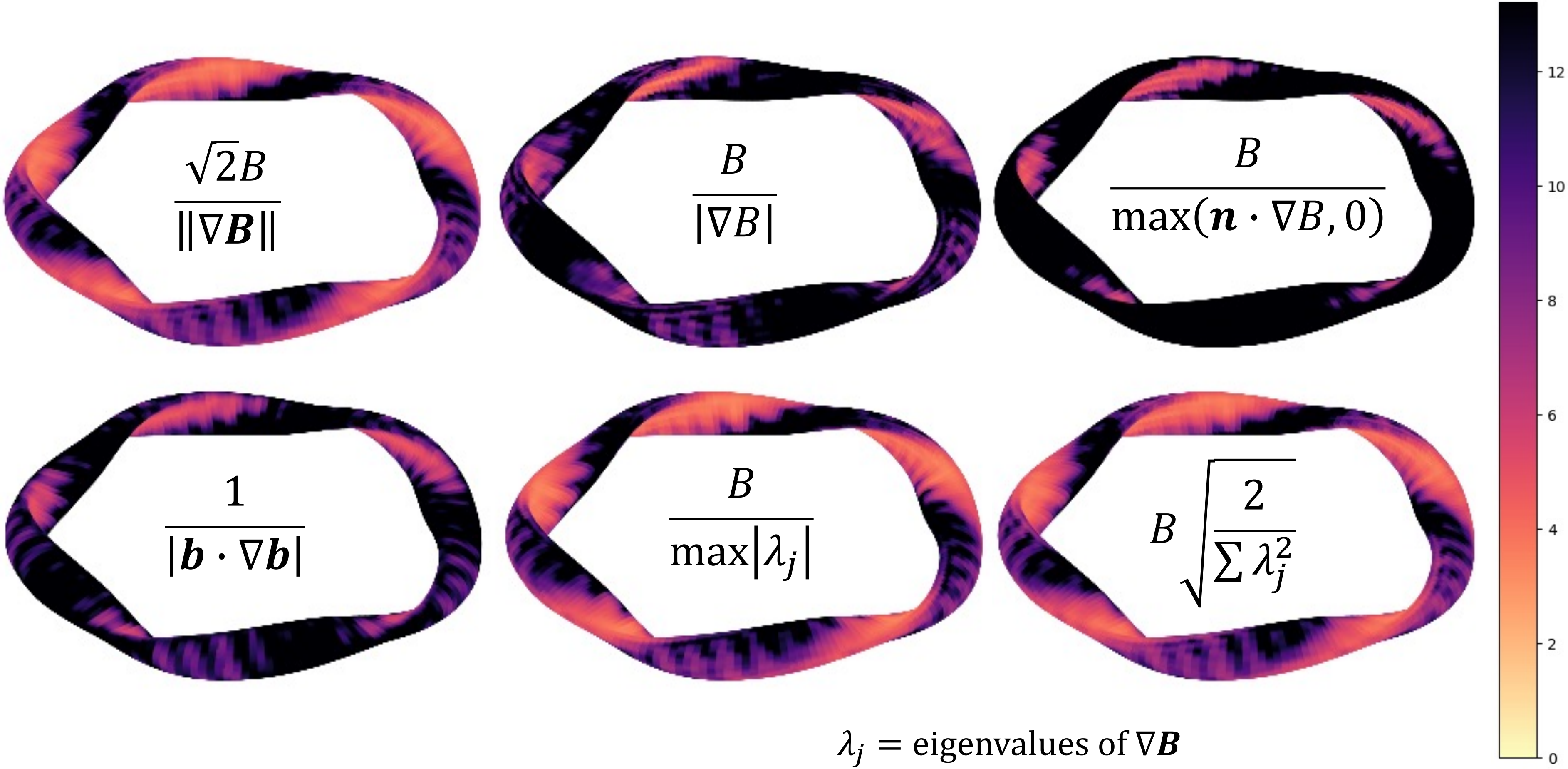
Some entries can be made to vanish by rotating the coordinate system.

The ∇B scale lengths can be normalized so that in the case of an infinite straight wire, they give the distance to the wire

$$L_{\nabla B} = \frac{\sqrt{2}B}{\|\nabla \boldsymbol{B}\|}$$



The different **B** scale lengths are not identical, but have similarities, e.g. all are small on the inside of concave regions



$$\frac{\sqrt{2}B}{\|\nabla \boldsymbol{B}\|}$$

$$\frac{B}{|\nabla B|}$$

$$\frac{B}{\max(\boldsymbol{n}\cdot\nabla B, 0)}$$

$$\frac{1}{|\boldsymbol{b}\cdot\nabla \boldsymbol{b}|}$$

$$\frac{B}{\max|\lambda_j|}$$

$$B\sqrt{\frac{2}{\sum \lambda_j^2}}$$

$\lambda_j =$ eigenvalues of $\nabla \boldsymbol{B}$

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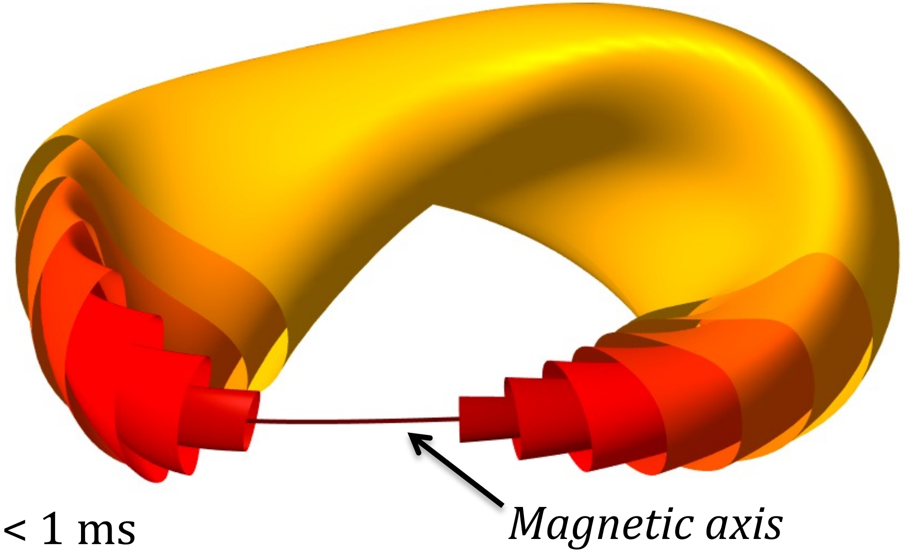
Expansion about the magnetic axis is a complementary method to traditional stellarator optimization



Traditional optimization: parameter space is the shape of toroidal boundary surface.

Near-axis expansion:

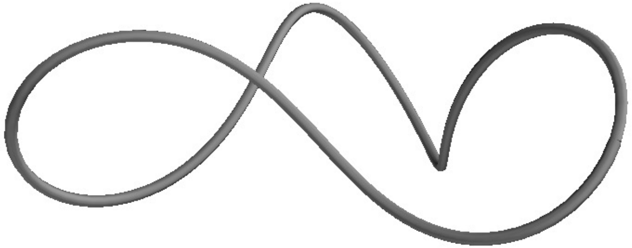
- Accurate in the core of any stellarator
- 3D PDEs → 1D ODEs in ϕ.
- Opportunities for analytic insights.
- Can solve & characterize a configuration in < 1 ms
- Can generate new initial conditions that can be refined by optimization.

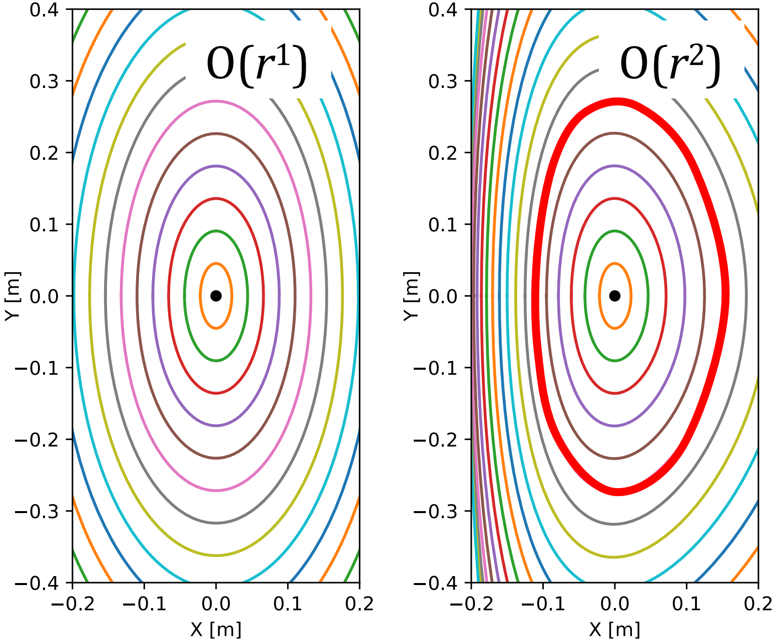
Mercier (1964),
Solov'ev & Shafranov (1970),
Lortz & Nührenberg (1976),
Garren & Boozer (1991)

ML, Sengupta, & Plunk (2019)
Rodriguez & Bhattacharjee (2021)
Jorge et al (2022)
…

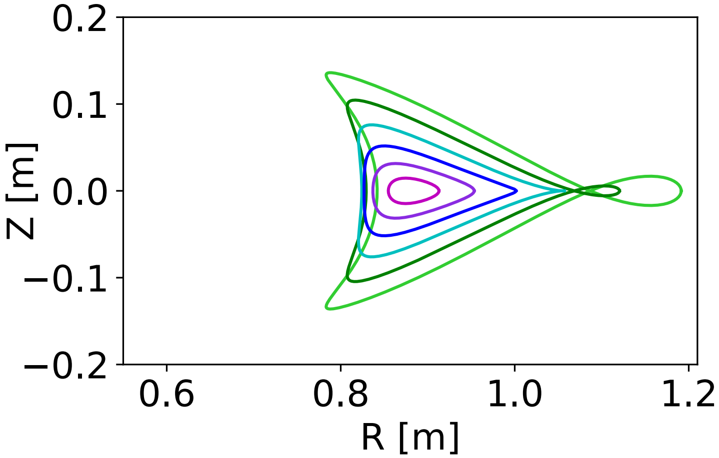
The expansion by Garren & Boozer (1991) has been converted into a practical algorithm for generating stellarator shapes

- Inputs:
  - Shape of the magnetic axis.
  - 3-5 other numbers (e.g. current on the axis).
- Outputs:
  - Shape of the surfaces around the axis.
  - Rotational transform on axis.
  - …
- Quasisymmetry can be guaranteed in a neighborhood of axis: $B = B(r, \theta\text{-}N\varphi)$
- Can pick any surface to pass to traditional 3D MHD fixed-boundary solve.

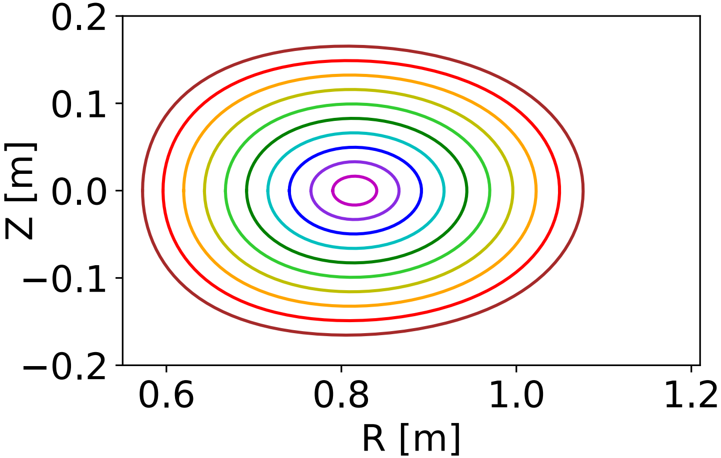




Problem: The radius of applicability of the expansion is typically small.
⇒ high aspect ratio



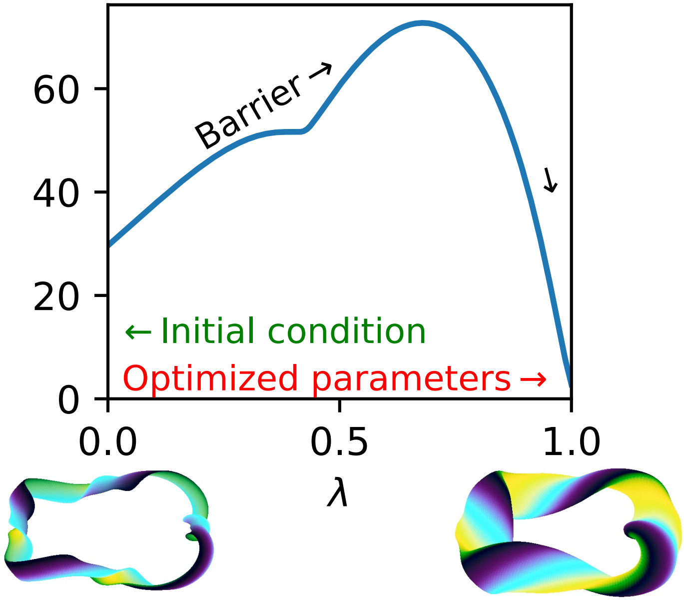
Optimize input parameters (axis shape, etc)?



- Increasing scale length (decreasing $\nabla\mathbf{B}$) may increase the minor radius over which the expansion is accurate. Expansion parameter is $\sim r/L_{\nabla \boldsymbol{B}}$.
- Quasisymmetry fails at $O(r^3)$ [Garren & Boozer (1991)], so increasing the scale length may improve quasisymmetry.

∇B turns out to be a better cost function than aspect ratio: fewer local minima

Min aspect ratio before surfaces become non-nested

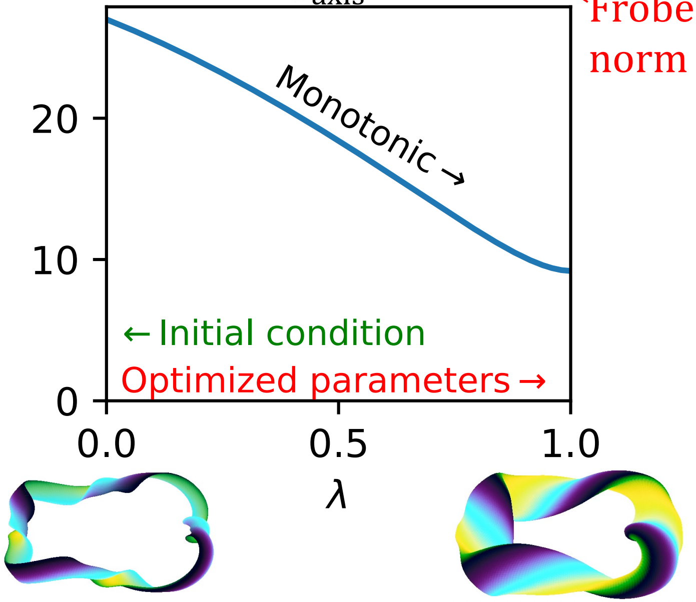


$R = 1 + 0.17\cos 4\phi$

$R = 1 + 0.17\cos 4\phi + 0.01804\cos 8\phi$
$+0.001409\cos 12\phi + 0.00005877\cos 16\phi$

$$f_\nabla = \frac{1}{L}\int_{\text{axis}} d\ell \, ||\nabla \boldsymbol{B}||^2$$

Frobenius norm



Line through parameter space, linearly interpolating the Fourier modes in axis shape.

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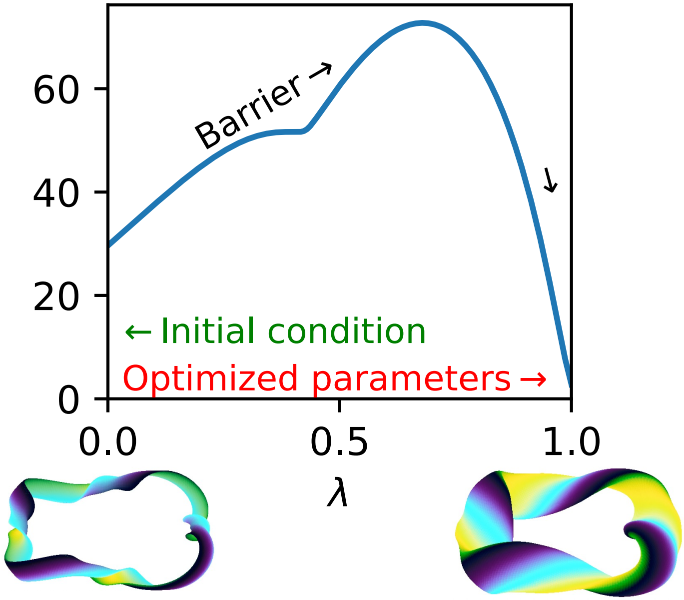
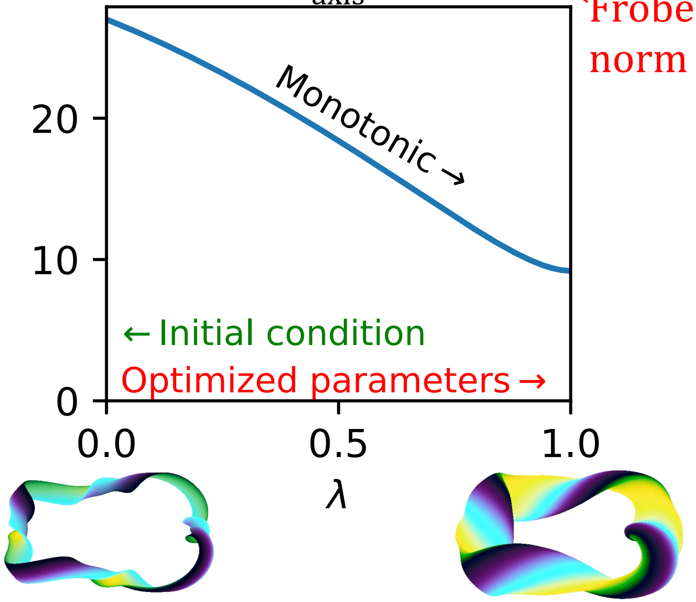


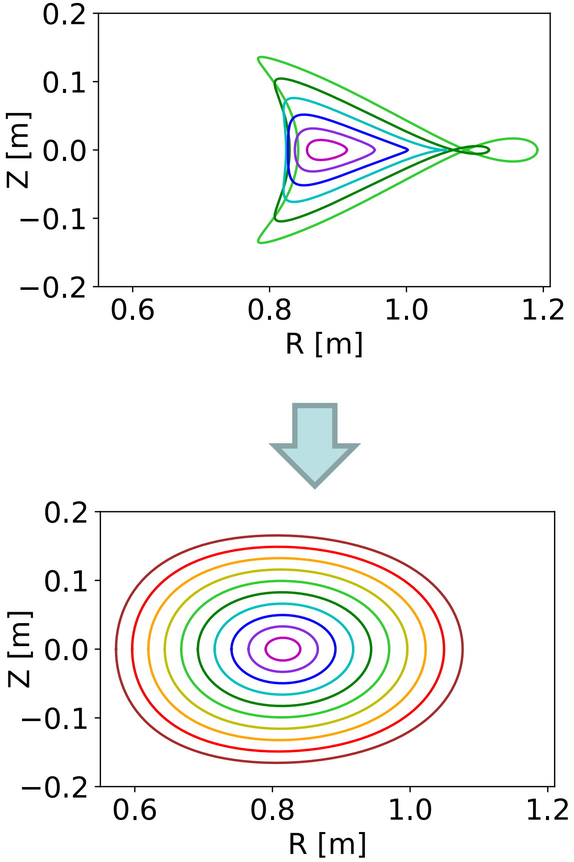
$$f_{\nabla} = \frac{1}{L}\int_{\text{axis}} d\ell\, \left|\left|\nabla \boldsymbol{B}\right|\right|^2$$

Frobenius norm



Lesson: To minimize some quantity Q, even if Q is fast to compute, the best objective function is not necessarily Q.

Complete objective function used for near-axis expansion

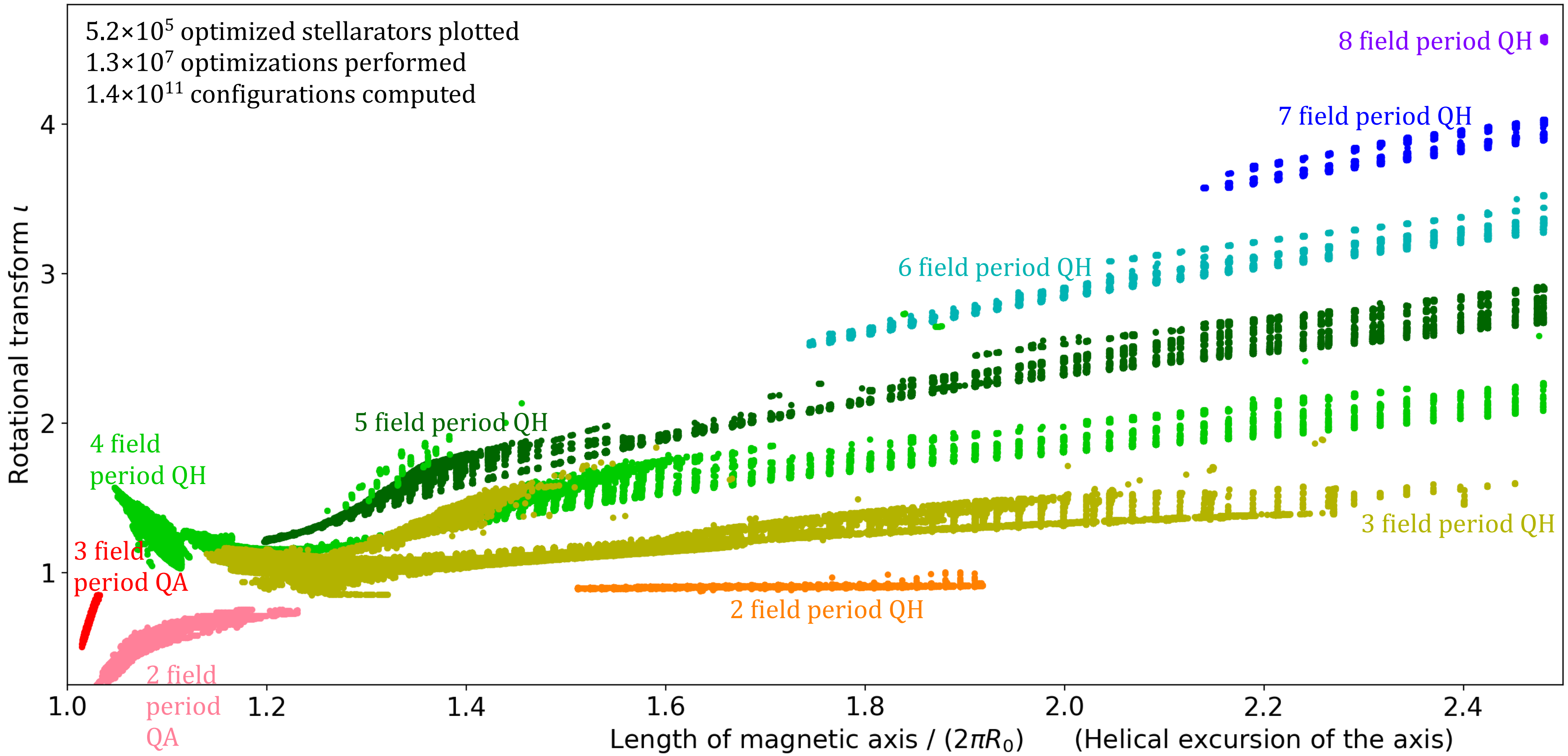


$$f = \overbrace{\frac{1}{L}\int d\ell}^{\text{Average along magnetic axis}} \left|\left|\nabla \mathbf{B}\right|\right|^2 + \frac{w_{\nabla\nabla}}{L}\int d\ell \left|\left|\nabla\nabla\mathbf{B}\right|\right|^2 + w_L\Big(\overset{\text{Axis length}}{L} - \overset{\text{Desired axis length}}{L_*}\Big)^2 + w_\iota\Big(\iota - \overset{\text{Desired rotational transform}}{\iota_*}\Big)^2$$

$$+ \frac{w_{B20}}{L}\int d\ell \underbrace{\left(B_{20} - \frac{1}{L}\int d\ell' B_{20}\right)^2}_{\text{Deviation from quasisymmetry at } O(r^2)} + w_{well}\left[\max\left(0,\ \underset{\text{Magnetic well}}{\frac{d^2V}{d\psi^2}} - \underset{\text{Desired well}}{W_*}\right)\right]^2$$

$w_{\nabla\nabla},\ w_L,\ w_\iota,\ w_{B20},\ w_{well}$: Weights chosen by user

The near-axis equations can be solved so quickly that tensor-product scans over many parameters are feasible



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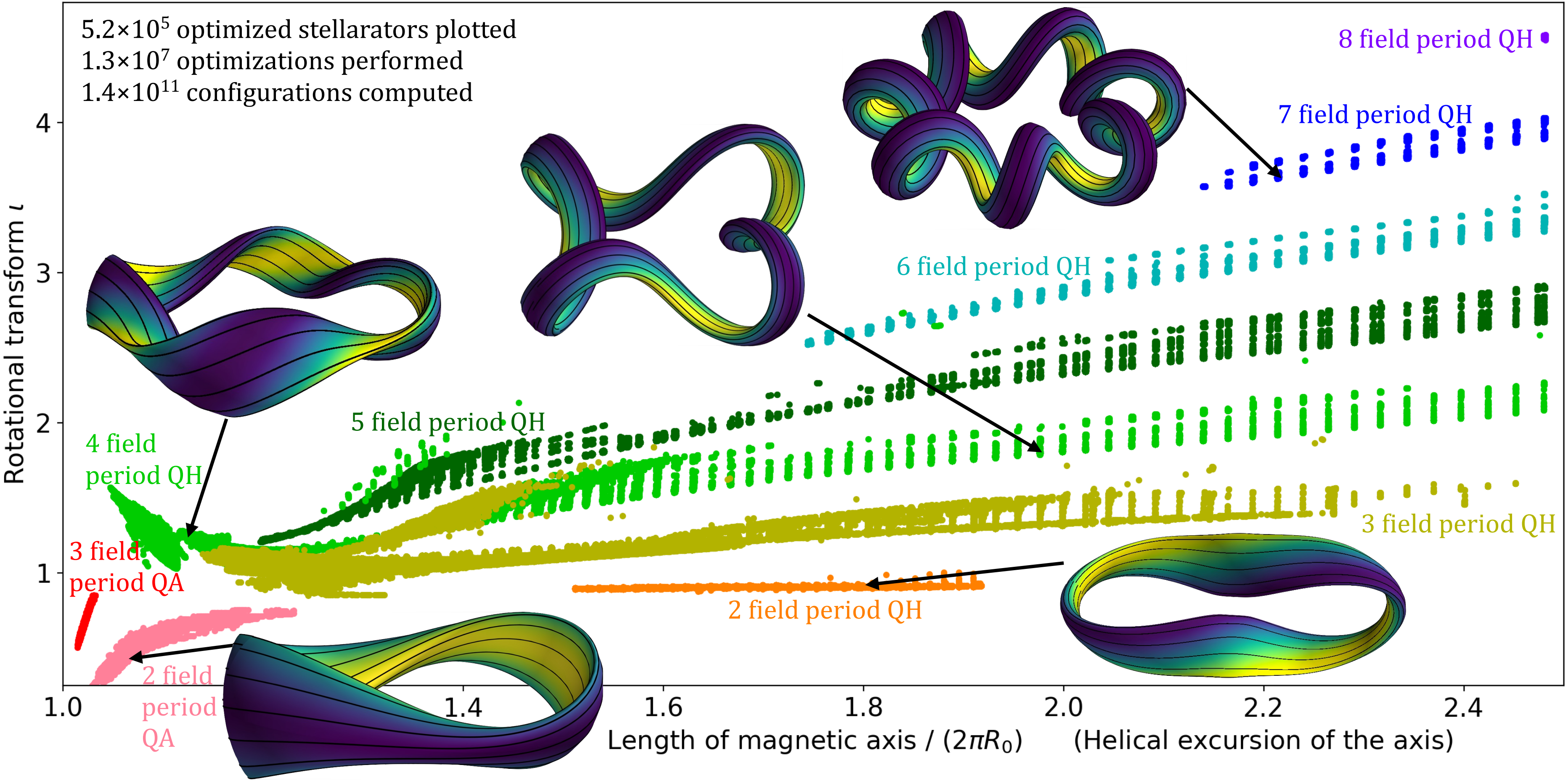
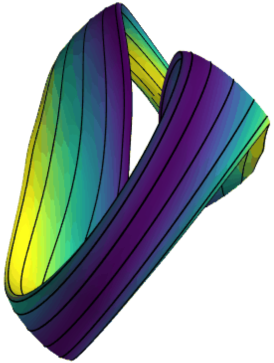


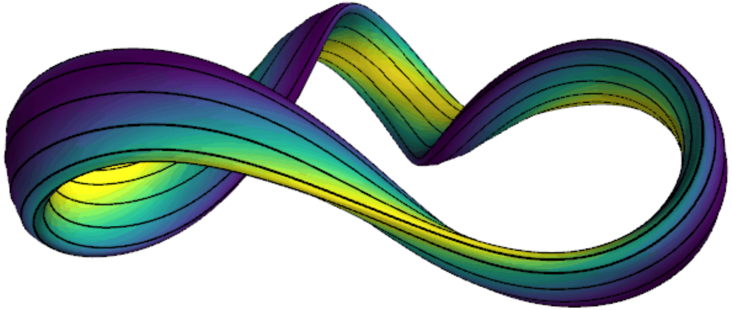
Some intriguing configurations from these near-axis optimizations

QH with nfp=2:

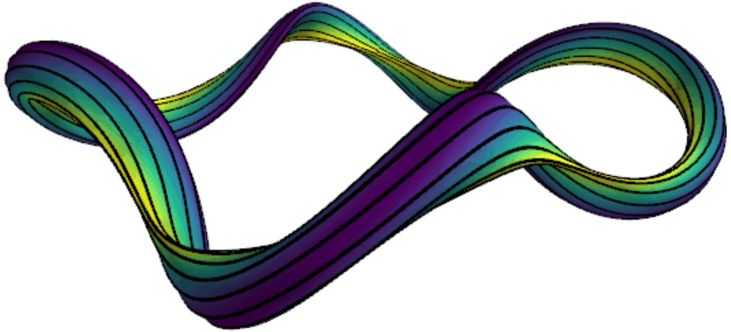


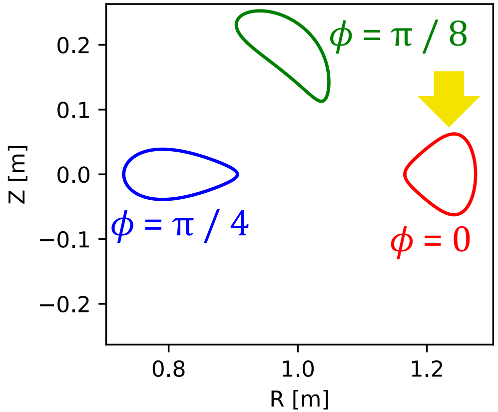
Fewer field periods
⟹ fewer coils?

QH with nfp=3, $\langle\beta\rangle$ = 4%:



QH with reversed triangularity, no bean:





ML, arXiv:2209.11849 (2022)

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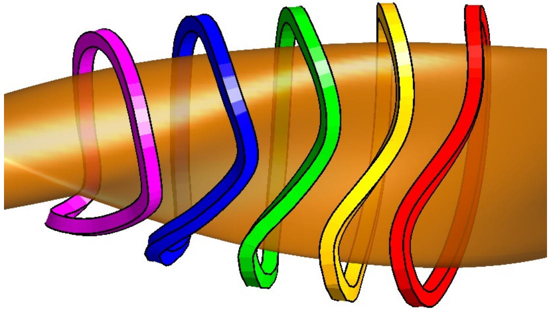
In a reactor, must fit ~ 1.5m "blanket" between plasma and coils to absorb neutrons

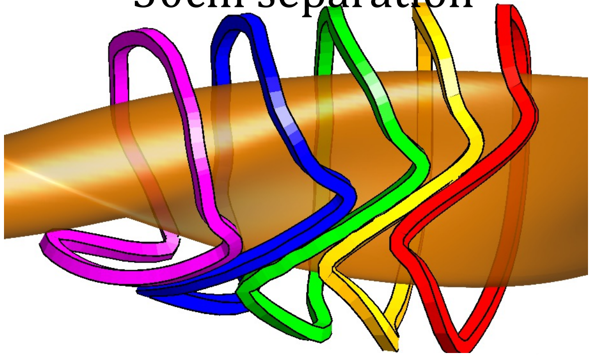
But at fixed plasma shape & size, coils shapes become impractical if they are too far away:

Coils offset a uniform distance from W7-X plasma:

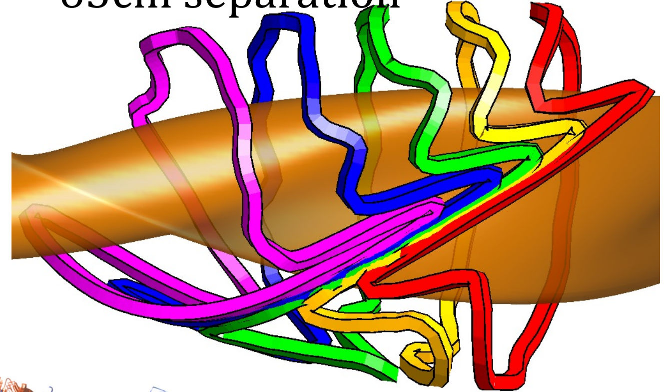
25cm separation



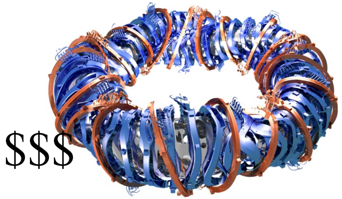
50cm separation

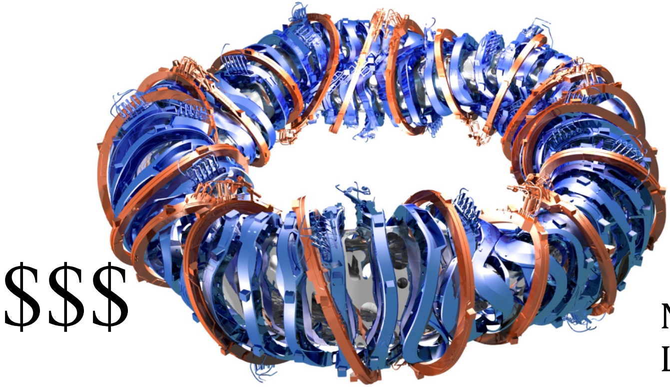


65cm separation



So we must scale everything up:





Najmabadi et al (2008),
Lion et al (2021)

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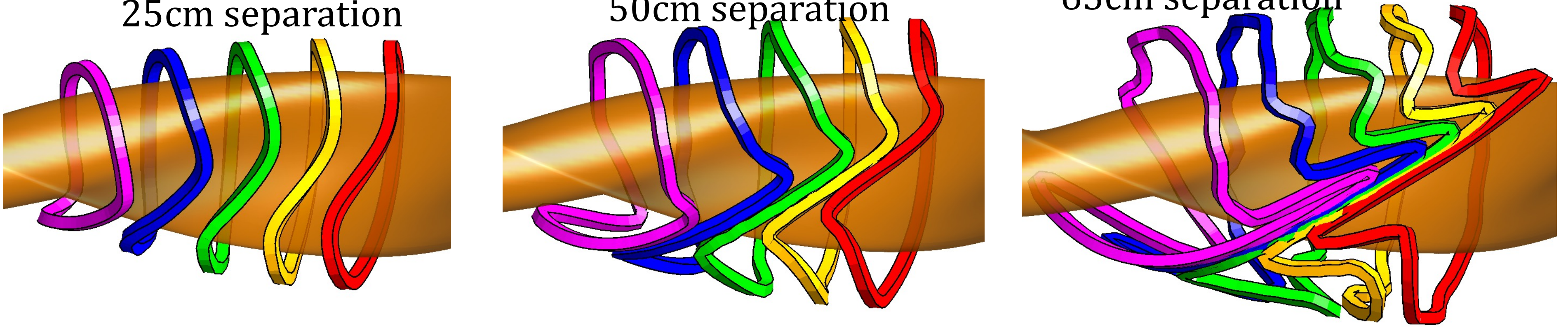


25cm separation

50cm separation

65cm separation

Hypothesis:

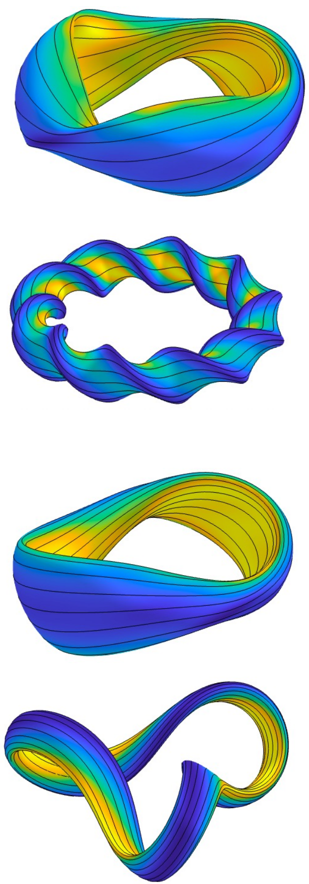
The coil-to-plasma distance scale for which coils are feasible is $\sim$ the $\nabla \mathbf{B}$ scale length

The small plasma-to-coil separation in stellarators is also a headache for engineering

“Lesson 1: A lack of generous margins, clearances and reasonable tolerance levels implies an unnecessary increase of the complexity and leads to late design changes. This has a strong impact on schedule, budget, man-power and potentially sours the relationship to funding bodies.”

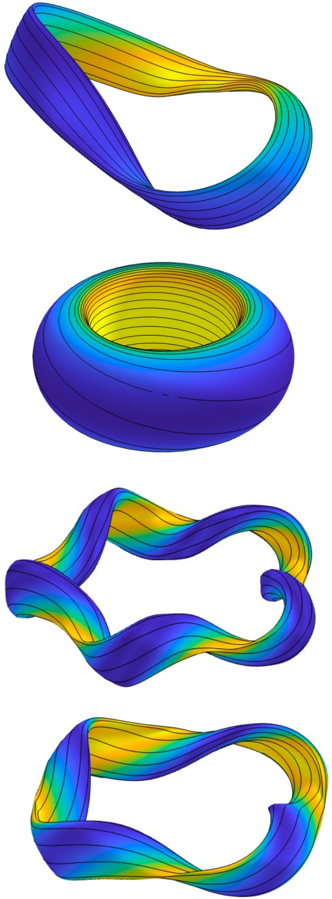
Klinger et al, Fusion Engineering & Design (2013)

To test hypothesis that ∇B is related to coil-plasma distance, scale length will be compared to “real” coil designs for a diverse set of 35 configurations



NCSX (li383 & c09r00)
ARIES-CS
HSX
W7-X (std, high-mirror, …)
LHD, R=3.5, 3.6, 3.75
CFQS
ML+Paul QA, QA+well
ML+Paul QH, QH+well
ML, Buller, Drevlak QH, QA
Near-axis nfp=3 QH
Near-axis nfp=4 QH
Jorge et al nfp=1 QI
Goodman et al nfp=1 QI

ESTELL
ITER
Garabedian QA
Henneberg et al QA
Wistell-A
Wistell-B
Wechsung et al QA
Wechsung et al QA+well
Giuliani et al QA
Ku & Boozer nfp=4 QH
Nuhrenberg & Zille QH
Drevlak QH
…



All scaled to same minor radius (1.7 m) and $\langle B \rangle$ = 5.9 T.

∇B scale length will be compared to "real" coil designs from Regcoil

Regcoil: Consider sheet current on a "coil winding surface"

ML, Nuclear Fusion (2017).

$$\boldsymbol{K} = \boldsymbol{n} \times \nabla \phi$$

Surface current

"current potential"

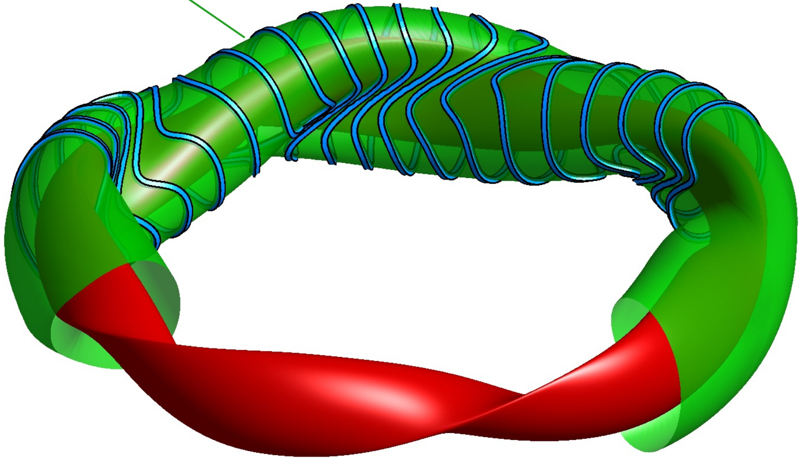
Normal to winding surface

$$\min_{\phi} \left(\int_{\substack{\text{Plasma} \\ \text{surface}}} d^2x \left[\left(\boldsymbol{B} - \boldsymbol{B}_{\text{target}} \right) \cdot \boldsymbol{n} \right]^2 + \lambda \int_{\substack{\text{Coil} \\ \text{surface}}} d^2x \left| \boldsymbol{K} \right|^2 \right)$$

B field error

Coil complexity

Regularization parameter



ϕ contours = coils

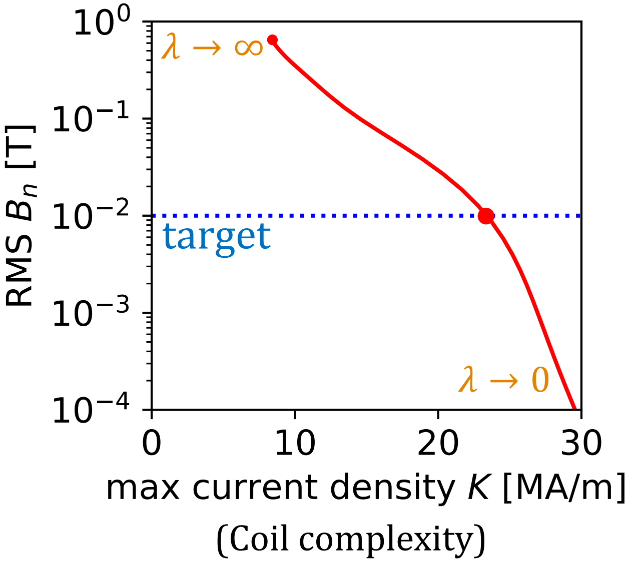
$K = |\boldsymbol{K}| \propto 1/\text{distance between coils}$

Regcoil is preferable to Focus/Simsopt for this study for comparison across many stellarators:

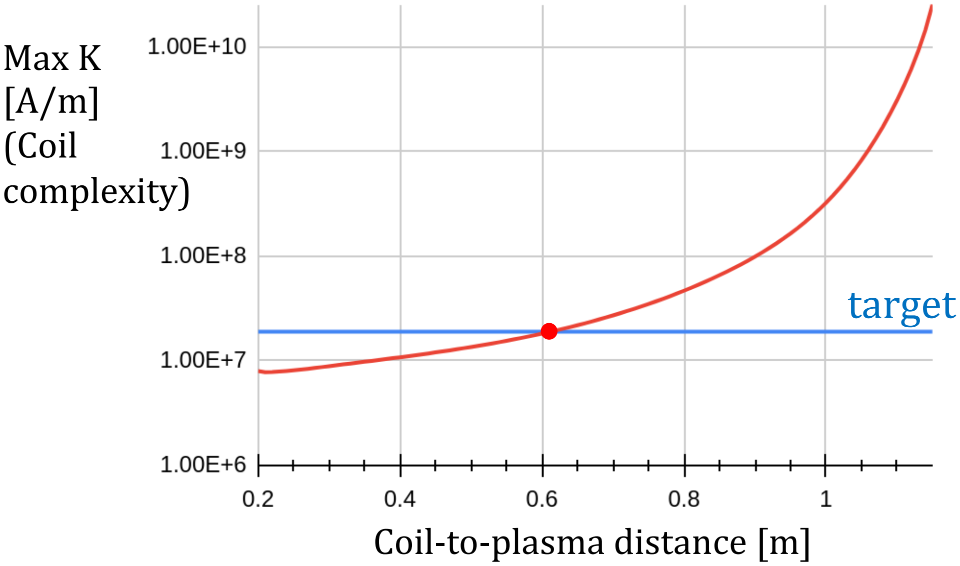
- *Linear* least-squares: no local optima besides the global one.
- Only 2 parameters to vary: coil-to-plasma distance and λ.

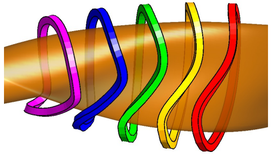
Methodology: Adjust regularization λ and coil-to-plasma separation to match ***B*** error and coil current density between configurations

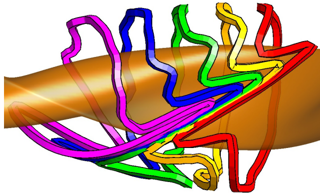
At fixed coil-to-plasma separation, λ trades off between ***B*** field error and coil complexity.



At the target ***B*** field error, coil complexity increases with coil-to-plasma separation

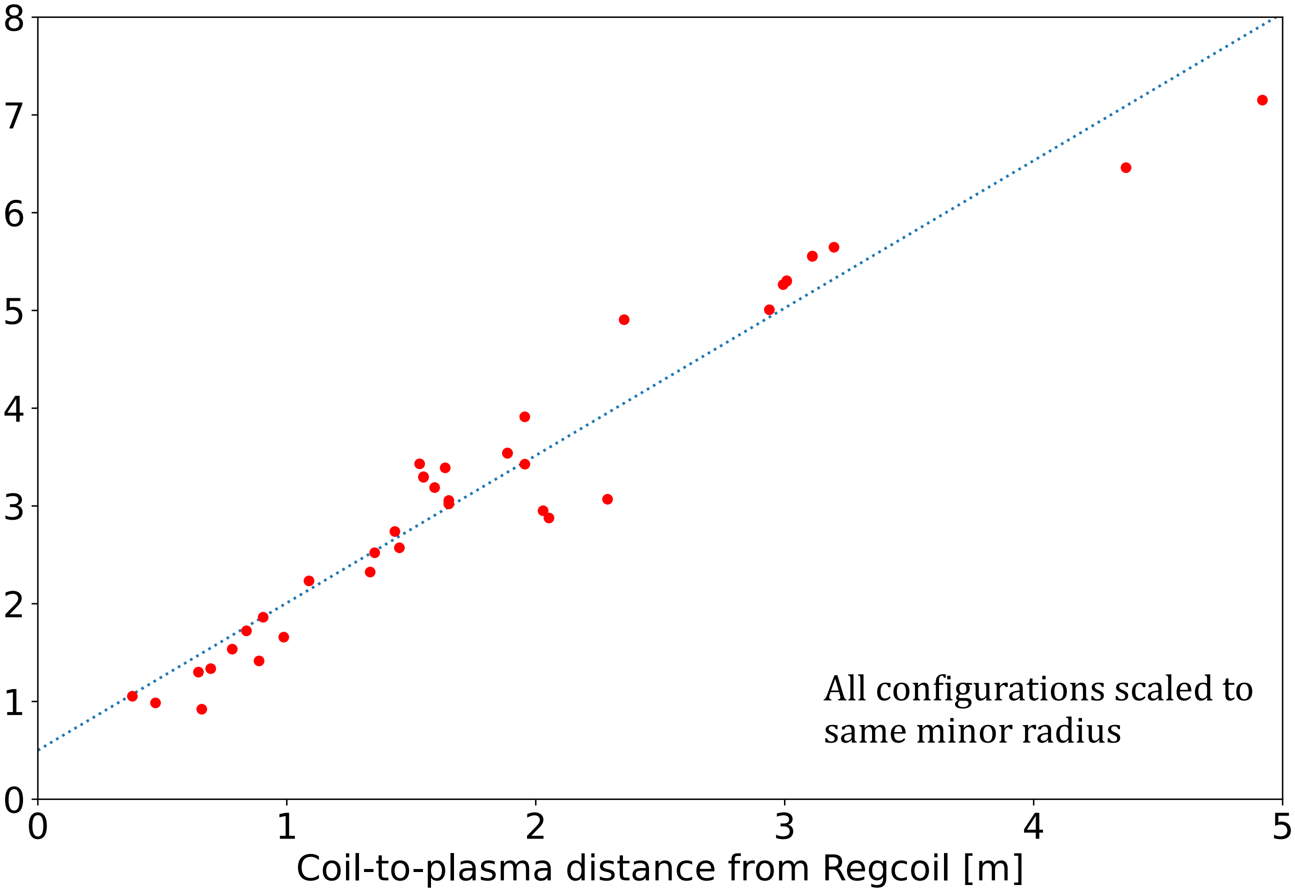






Main result: ∇B length is well correlated with real coil designs

$$\min \frac{\sqrt{2}B}{\|\nabla \boldsymbol{B}\|} = L_{\nabla \boldsymbol{B}}\ [\mathrm{m}]$$



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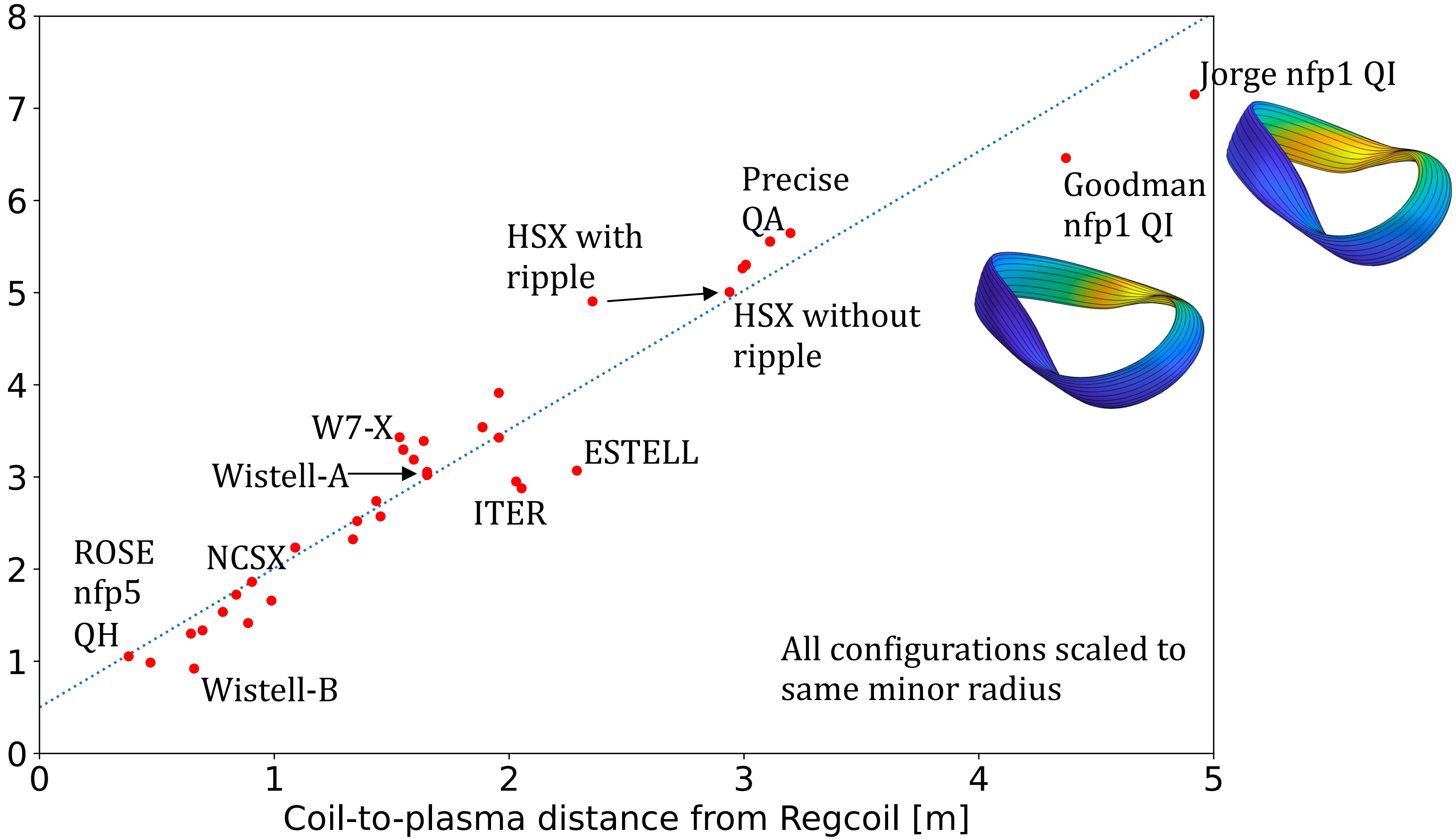
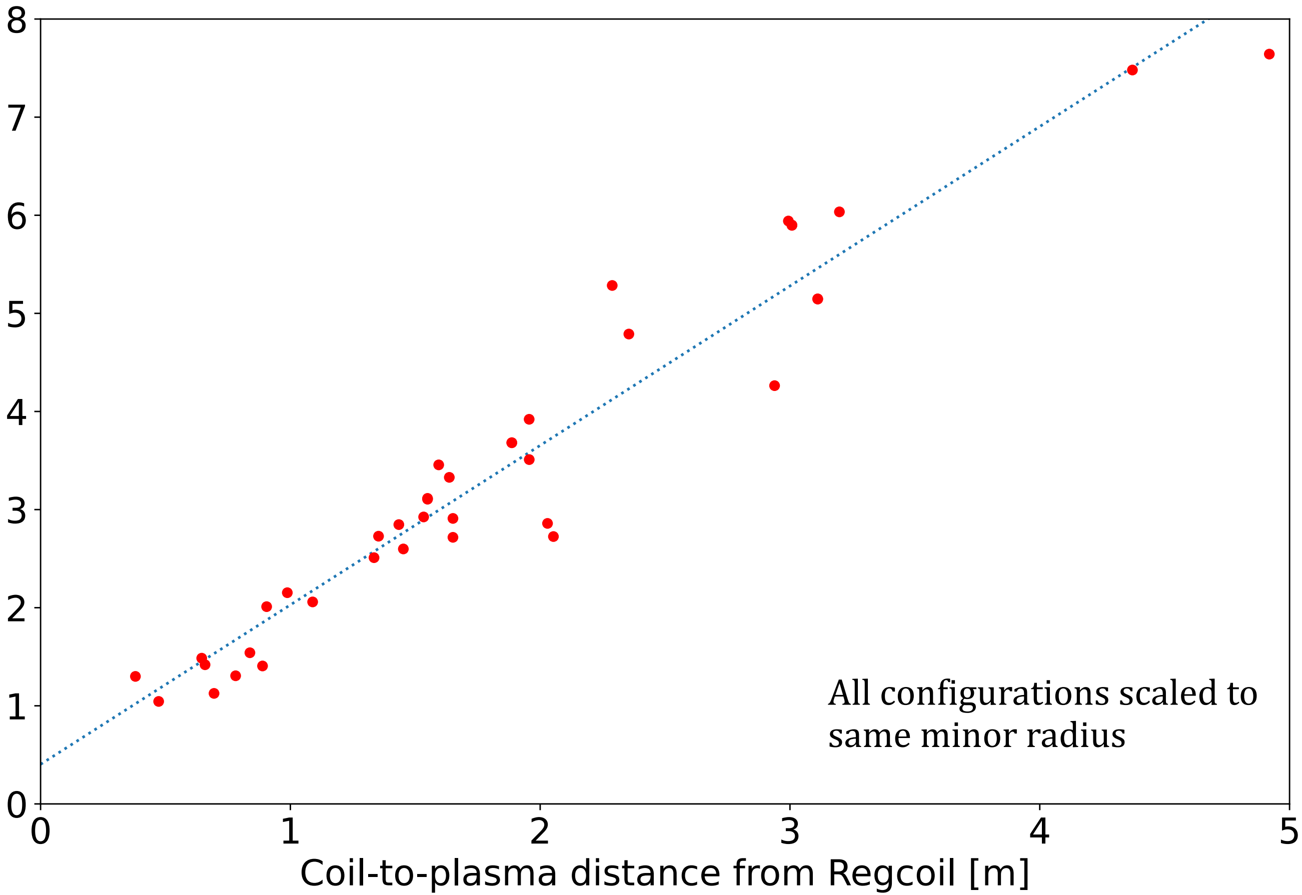


All configurations scaled to same minor radius

Regcoil coil-to-plasma distance is actually well correlated with other definitions of B length scale too

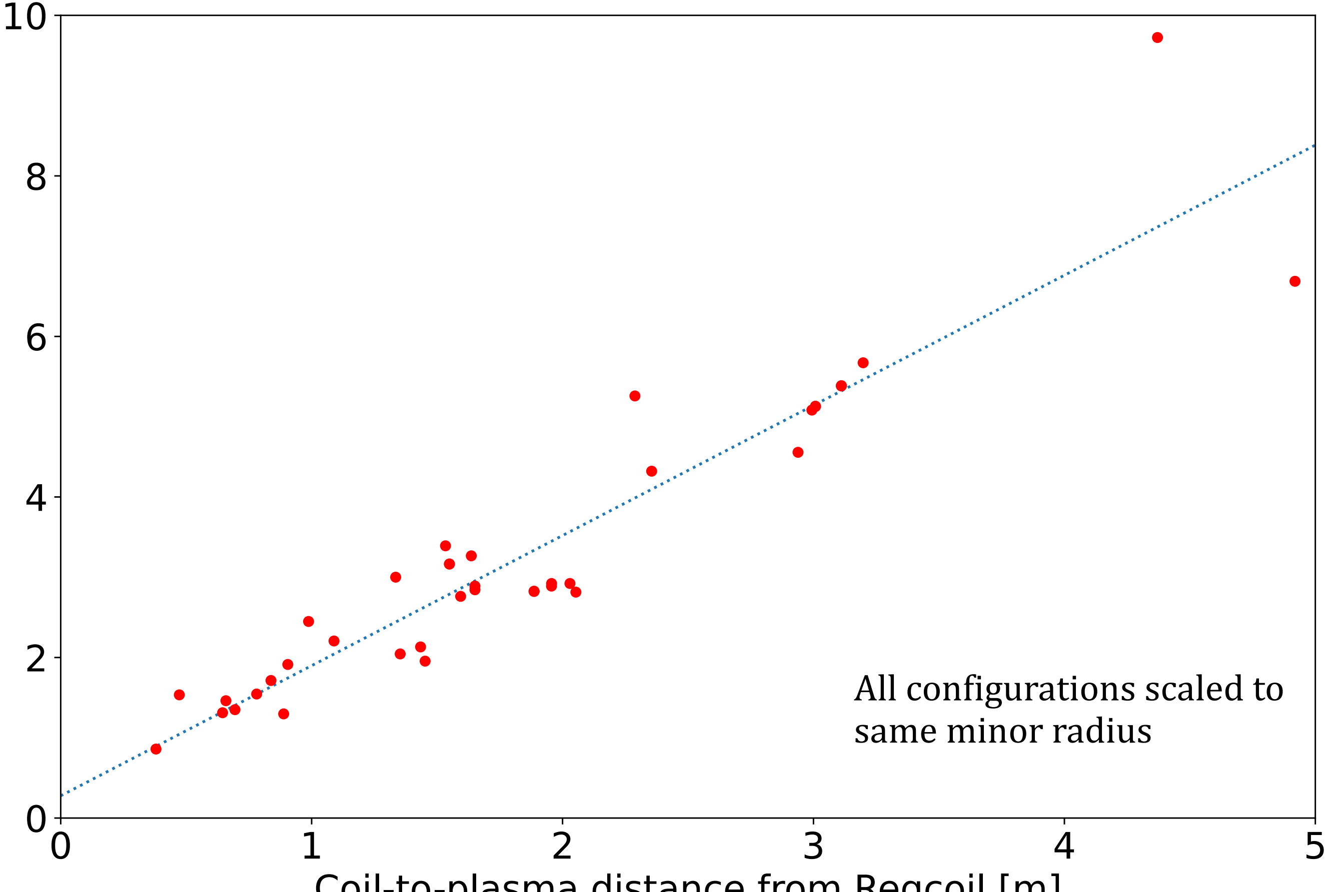


All configurations scaled to same minor radius

Regcoil coil-to-plasma distance is actually well correlated with other definitions of B length scale too

$$\min \frac{B}{|\nabla B|}$$

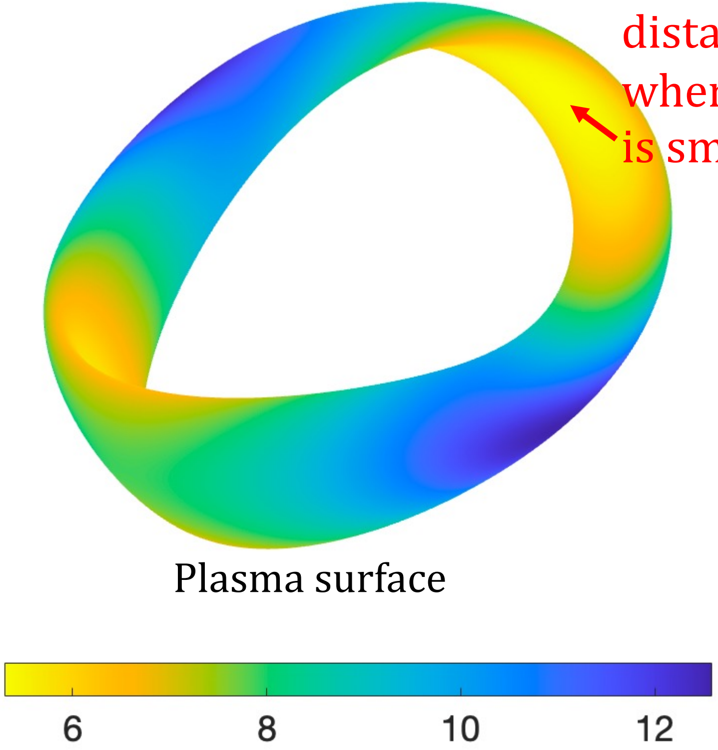
[m]



All configurations scaled to same minor radius

The location of limiting ∇B length and coil complexity are also correlated *spatially*

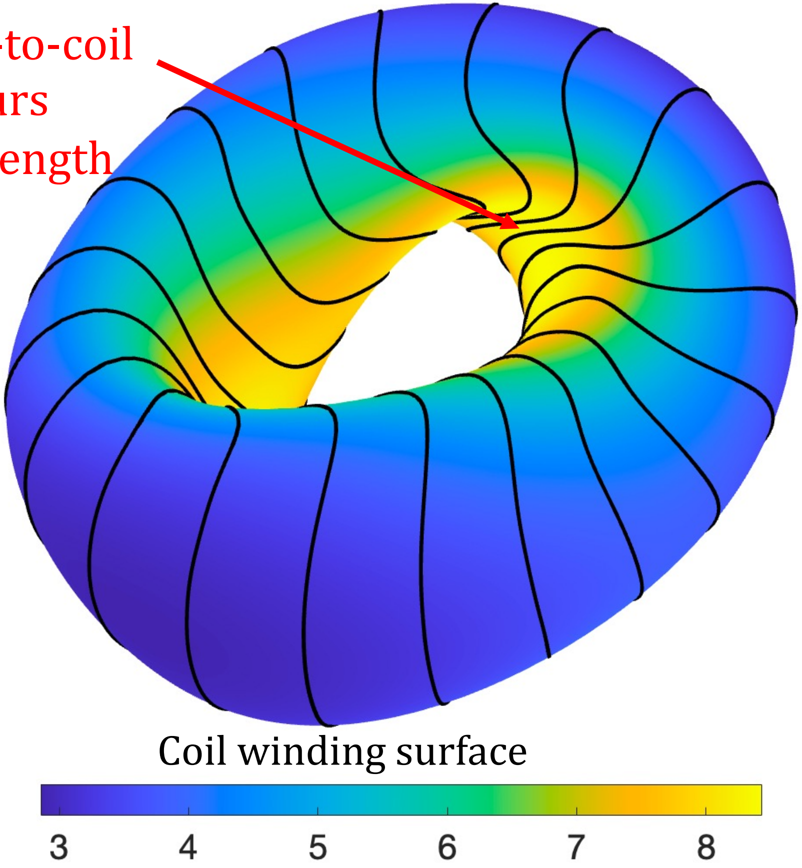
$$L_{\nabla \boldsymbol{B}} = \frac{\sqrt{2}B}{\|\nabla \boldsymbol{B}\|}\ [\mathrm{m}]$$



Plasma surface

Limiting coil-to-coil distance occurs where scale length is smallest

Current density K [MA/m]



Coil winding surface

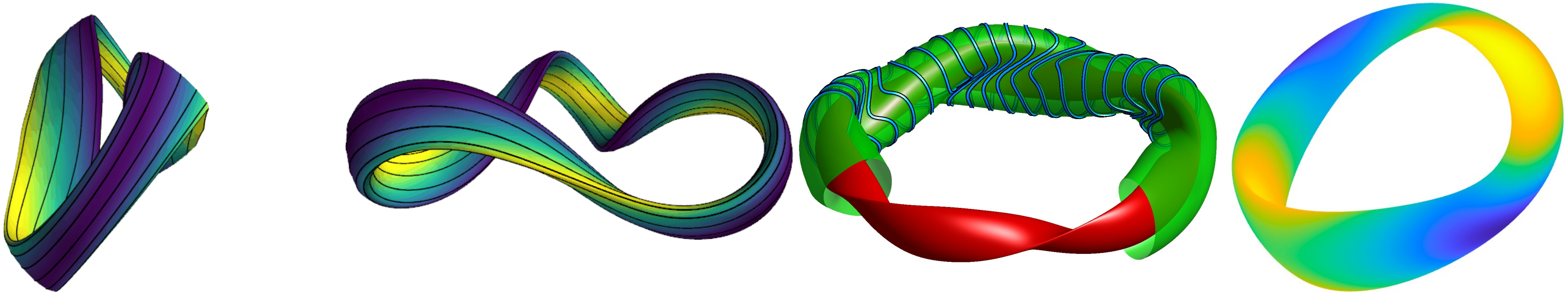
Conclusion: We should pay attention to ∇B length scales in stellarators

- Since quasisymmetry is allowed only to $O(r^2)$, reducing $\boldsymbol{\nabla B}$ can expand the volume of good quasisymmetry.

- $\boldsymbol{\nabla B}$ appears to explain the maximum coil-to-plasma distance
  - Driver of size and costs!
  - Significant variation between configurations. 1-field-period QIs look promising.

Future work:

- Compare & understand different scale lengths.
- For $\beta > 0$, check if $\left\|\nabla \boldsymbol{B}_{\text{external}}\right\|$ is more meaningful than $\|\nabla \boldsymbol{B}\|$.
- Optimize for small $\|\nabla \boldsymbol{B}\|$ (already in StellaratorOptimizationMetrics.jl).



Extra slides

Main result: ∇B length is well correlated with real coil designs

$$L_{\nabla B} = \frac{\sqrt{2}B}{\|\nabla \boldsymbol{B}\|}$$

[m]