New approaches to stellarator optimization using expansion in aspect ratio

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• Advantages of stellarators: steady-state, no disruptions, no Greenwald density limit, no power recirculated for current drive.
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• But, alpha losses & neoclassical transport would be too large unless you carefully choose the geometry.

\[ \oint \left( v_d \cdot \nabla r \right) dt = 0 \quad \text{in axisymmetry,} \quad \neq 0 \quad \text{in a general stellarator.} \]

• A solution: quasisymmetry

\[ B = B(r, \theta - N\zeta) \quad \Rightarrow \quad \oint \left( v_d \cdot \nabla r \right) dt = 0. \]

• How do you find configurations with quasisymmetry?

Guiding-center Lagrangian in Boozer coordinates depends on \((\theta, \zeta)\) only through \(B=|\mathbf{B}|\).
Until now, understanding of quasisymmetric plasmas has been limited by the method of finding them numerically.

Want $\mathbf{J} \times \mathbf{B} = \nabla p$ and $B = B(r, \theta - N\zeta)$. 
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minimize \( f(x) \)

Parameter space: \( x \in \{ \text{toroidal boundary shapes} \} \)

Objective: Solve \( \mathbf{J} \times \mathbf{B} = \nabla p \) numerically inside boundary,
\[ f = \text{departure from quasisymmetry in the result.} \]
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$f =$ departure from quasisymmetry in the result.

- Computationally expensive.
- What is the size & character of the solution space?
- Result depends on initial condition. ⇒ Cannot be sure you’ve found all solutions.
Expansion about the magnetic axis can be a powerful practical tool for generating quasisymmetric stellarator configurations.

- Accurate at least in the core of any configuration.
- Hasn’t been considered much since numerical optimization began.

Mercier (1964),
Solov’ev & Shafranov (1970),
Lortz & Nührenberg (1976),
Garren & Boozer (1991)

Revisit expansions with modern concepts (e.g. quasisymmetry, gyrokinetics), computing, & optimization.
• Accurate at least in the core of any configuration.

• Hasn’t been considered much since numerical optimization began.

• Complements the traditional optimization approach:
  – Many orders of magnitude faster.
  – Opportunities for analytic insights.
  – Can generate new initial conditions that can be refined by optimization.
• Constructing quasisymmetric stellarator shapes
• Evaluating other physics properties
• Optimizing configurations
• 1-stage coil optimization
Outline

- Constructing quasisymmetric stellarator shapes
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- 1-stage coil optimization
The expansion by Garren & Boozer (1991) has been converted into a practical algorithm for generating stellarator shapes.

- **Inputs:**
  - Shape of the magnetic axis.
  - 3-5 other numbers (e.g. current on the axis).

- **Outputs:**
  - Shape of the surfaces around the axis.
  - Rotational transform on axis.
  - ...

- Quasisymmetry guaranteed in a neighborhood of axis.

- Can pick any surface to pass to traditional 3D MHD fixed-boundary solve.
The construction can be verified by running an MHD equilibrium code (e.g. VMEC) which does not make the expansion.

Quasi-axisymmetry (QA)

\[ B = B(r, \theta) \]

Aspect ratio 6.0

Quasi-helical symmetry (QH)

\[ B = B(r, \theta - 3\zeta) \]

Aspect ratio 5.0
The construction can be verified by running an MHD equilibrium code (e.g. VMEC) which does not make the expansion.

Quasi-axisymmetry (QA)

\[ B = B(r, \theta) \]

Aspect ratio 12.0

Quasi-helical symmetry (QH)

\[ B = B(r, \theta - 3\zeta) \]

Aspect ratio 5.0
We can now numerically demonstrate Garren & Boozer’s scaling: $B_{\text{nonsymm}} \sim 1/A^3$

\[ S = \frac{1}{B_0} \sqrt{\sum_{m,n \neq Nm} B_{m,n}^2} = \text{Symmetry-breaking} \]
Accurate quasisymmetry is effective at curing alpha particle losses.

All configurations scaled to ARIES-CS minor radius (1.7 m) and |B| (5.7 T)

3.5 MeV alpha particles initialized at $\psi/\psi_{edge} = 0.3$. 

**Fraction of alpha particles lost**

**Time [sec]**

- Henneberg
- HSX
- ARIES-CS
- NCSX
- W7-X
- Garabedian
- CFQS
- Nuhrenberg-Zille
- Wistell-A

**Constructed nfp=3 (aspect = 5)**

**Constructed nfp=4 (aspect = 5)**
The near-axis analysis can be generalized to construct configurations with omnigenity.

\[ \oint (v_d \cdot \nabla r) \, dt = 0 \quad \forall \text{ magnetic moments & energies.} \]

- Weaker condition than quasisymmetry.
- \( B \) contours can close poloidally.
• Constructing quasisymmetric stellarator shapes
• Evaluating other physics properties
• Optimizing configurations
• 1-stage coil optimization
From a near-axis solution, analytic expressions exist for all the geometric quantities in gyrokinetics.

Rogerio Jorge & ML, PPCF 63, 014001 (2021), Rogerio Jorge & ML, PPCF 63, 074002 (2021)

\[
\nabla \alpha \cdot \nabla \alpha = \frac{1}{r^2 \eta^2 \kappa^2} \left[ \bar{\eta}^4 \cos^2 \theta + \kappa^4 \left( \sigma \cos \theta + \sin \theta \right)^2 \right]
\]

where \( \mathbf{B} = \nabla \psi \times \nabla \alpha \)

\[
\begin{align*}
|\mathbf{B}| \\
\mathbf{\bar{b}} \times \nabla \mathbf{B} \cdot \nabla \alpha \\
\mathbf{\bar{b}} \times \nabla \mathbf{B} \cdot \nabla \psi \\
\mathbf{\bar{b}} \times \kappa \cdot \nabla \alpha
\end{align*}
\]

\[
\begin{align*}
\nabla \alpha \cdot \nabla \alpha \\
\nabla \psi \cdot \nabla \alpha \\
\nabla \psi \cdot \nabla \psi
\end{align*}
\]

\( r/a = 0.5 \)
Mercier stability can be evaluated directly from a near-axis solution.
Many properties of a stellarator can be computed in ~ 1 ms directly from a solution of the near-axis equations

Available so far:
- Surface shapes
- Rotational transform
- Magnetic well
- Mercier stability
- All the geometric factors in the gyrokinetic equation & MHD ballooning equation
- $\mathbf{B}$ vector & 2 gradients
- Scale length in $\mathbf{B}$ (proxy for coil complexity?)

Not available now:
- Magnetic shear
- Ballooning growth rates
- Low-n MHD growth rates
- Gyrokinetic growth rates
• Constructing quasisymmetric stellarator shapes
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Though quasisymmetry is guaranteed in a neighborhood of the axis, optimization can greatly increase the volume of good symmetry.

Objective function to minimize:

\[
f = \frac{1}{L} \int d\ell \left\| \nabla B \right\|^2 + \frac{\omega_{\psi\psi}}{L} \int d\ell \left\| \nabla^2 \nabla B \right\|^2 + \omega_L \left( L - L^* \right)^2 + \omega_t \left( t - t^* \right)^2
\]

\[
+ \frac{\omega_{B20}}{L} \int d\ell \left( B_{20} - \frac{1}{L} \int d\ell' B_{20} \right)^2 + \omega_{\text{well}} \left[ \max \left( 0, \frac{d^2V}{d\psi^2} - W^* \right) \right]^2
\]

Deviation from quasisymmetry at \( O(r^2) \)

\( w_{\psi\psi}, w_L, w_t, w_{B20}, w_{\text{well}} \): Weights chosen by user

Desired rotational transform

Desired axis length

Axis length

Magnetic well

Desired well

Average along magnetic axis
The near-axis equations can be solved so quickly that tensor-product scans over many parameters are feasible.

3.1 \times 10^5 \text{ optimized stellarators shown } (O(r^2))
5.1 \times 10^6 \text{ optimizations computed}
6.0 \times 10^{10} \text{ equilibria computed}

8 field period QH
7 field period QH
6 field period QH
5 field period QH
4 field period QH
3 field period QH
2 field period QA
2 field period QH

Length of magnetic axis / (2\pi R) ("wiggliness")
The near-axis equations can be solved so quickly that tensor-product scans over many parameters are feasible.

3.1 × 10^5 optimized stellarators shown (O(r^2))
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6.0 × 10^{10} equilibria computed

Rotational transform \( \lambda \)

Length of magnetic axis / (2\pi R) ("wiggliness")
Discovery: quasi-helical symmetry with 2 field periods

Spitzer, 1958

Fig. 2. Top and end views of a figure-eight stellarator.

But now with quasisymmetry:
The near-axis expansion can yield configurations very similar to conventional optimization, but much faster.

<table>
<thead>
<tr>
<th></th>
<th>Conventional optimization</th>
<th>Optimized near-axis expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time for 1 equilibrium solve</td>
<td>50 CPU-sec</td>
<td>5e-4 CPU-sec</td>
</tr>
<tr>
<td>Total time for optimization (cold start)</td>
<td>8e+5 CPU-sec</td>
<td>1 CPU-sec</td>
</tr>
</tbody>
</table>

*ML & Paul, arXiv (2021)*
Outline

• Constructing quasisymmetric stellarator shapes
• Evaluating other physics properties
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Previous designs (HSX, W7-X) used a 2-stage approach:
1. Optimize plasma shape, ignoring coils.
2. Find coils to make the plasma shape from stage 1.

Downside: The result of stage 1 may be hard to produce with practical coils.

- Derivatives are available.
- Stochastic optimization yields wider coil tolerances
  \[ \Rightarrow \text{lower cost. } F \text{Wechsung et al, arXiv (2021)} \]
The near-axis expansion can form the basis of a new design formulation for coils

\[ \min f(X) \]

\[ X = \{ \text{Coil shapes, magnetic axis shape} \} \]

\[
 f = \oint d\ell \begin{vmatrix}
 B_{\text{Biot-Savart}} & -B_{\text{Near-axis quasisymmetry}} \\
 \end{vmatrix}^2 + \oint d\ell \begin{vmatrix}
 \nabla B_{\text{Biot-Savart}} & -\nabla B_{\text{Near-axis quasisymmetry}} \\
 \end{vmatrix}^2 + \left( \frac{I_{\text{axis}} - I_{\text{axis}}^{\text{target}}}{I_{\text{axis}}^{\text{target}}} \right)^2 + \left( \frac{L_{\text{coils}} - L_{\text{coils}}^{\text{target}}}{L_{\text{coils}}^{\text{target}}} \right)^2 + \left( \text{other terms} \right) 
\]

Analytic \( \partial f / \partial X \) is available!
Combined coil + quasisymmetry optimization using analytic derivatives successfully achieves flux surfaces & QA

\[ 0.4 < \ell < 0.5, \]
Decent surfaces out to aspect ratio 3.2

\[ |B| \text{ is quasisymmetric} \]

\[ |B| \text{ [T] along field lines} \]

< 2 minutes on a laptop
Ongoing & future work using this expansion

- Better understand the landscape of $O(r^2)$ quasisymmetric configurations.
- Evaluate more aspects of MHD & gyrokinetic stability.
- Higher order: compute magnetic shear & symmetry-breaking $B_3$.
- Construction to give quasisymmetry at an off-axis surface.
- Include off-axis quasisymmetry in the 1-stage coil optimization.
Closing thoughts

- The high-aspect-ratio expansion enables multiple new stellarator design paradigms:
  - Directly construct plasma geometries with good confinement.
  - Brute force parameter scans, enabled by orders-of-magnitude speed-up.
  - Derivative-based 1-stage optimization of coils for quasisymmetry.

- There is hope of definitively identifying all regions of parameter space with practical quasisymmetric fields.

- We may still discover qualitatively new magnetic confinement configurations.