# Stellarator figures of merit near the magnetic axis

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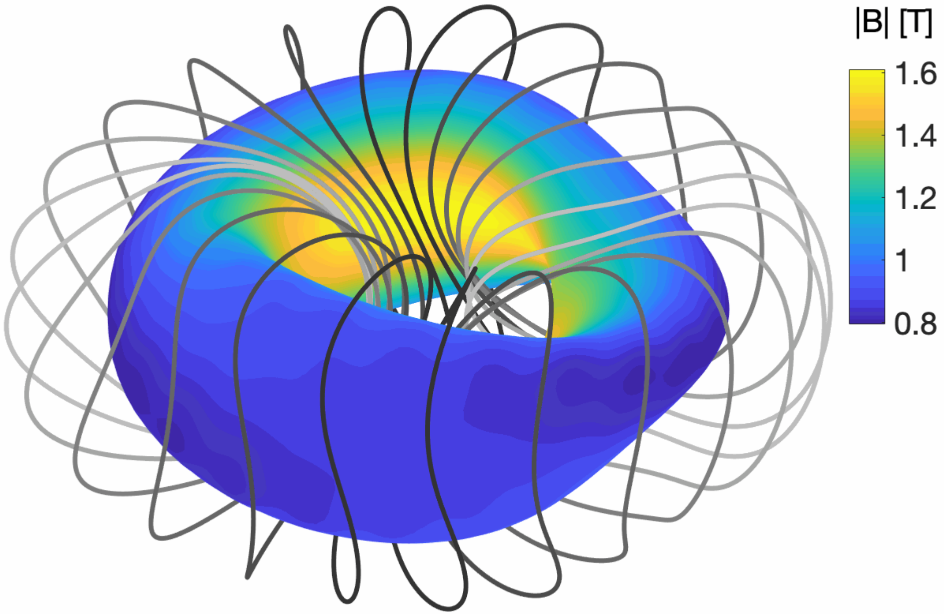


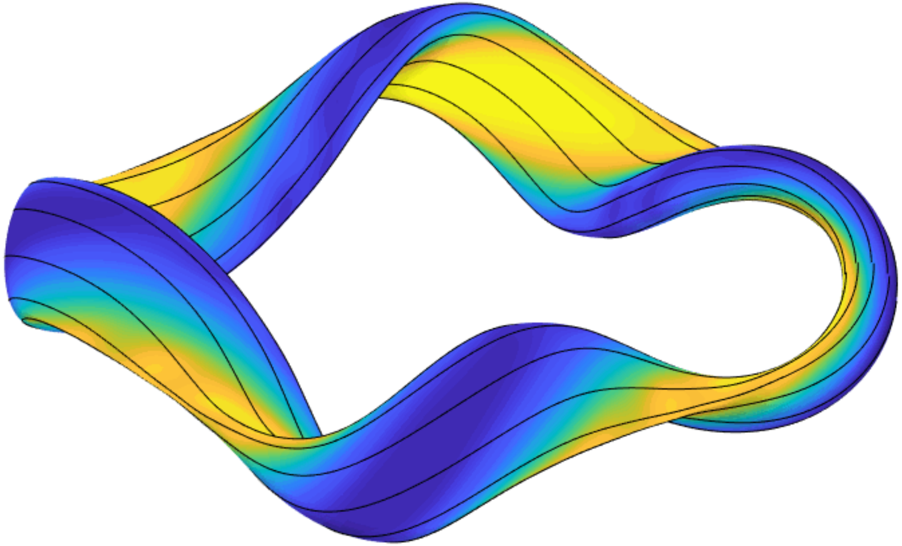

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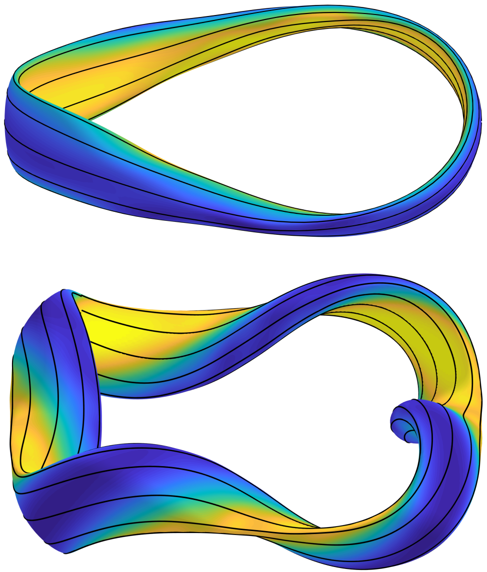
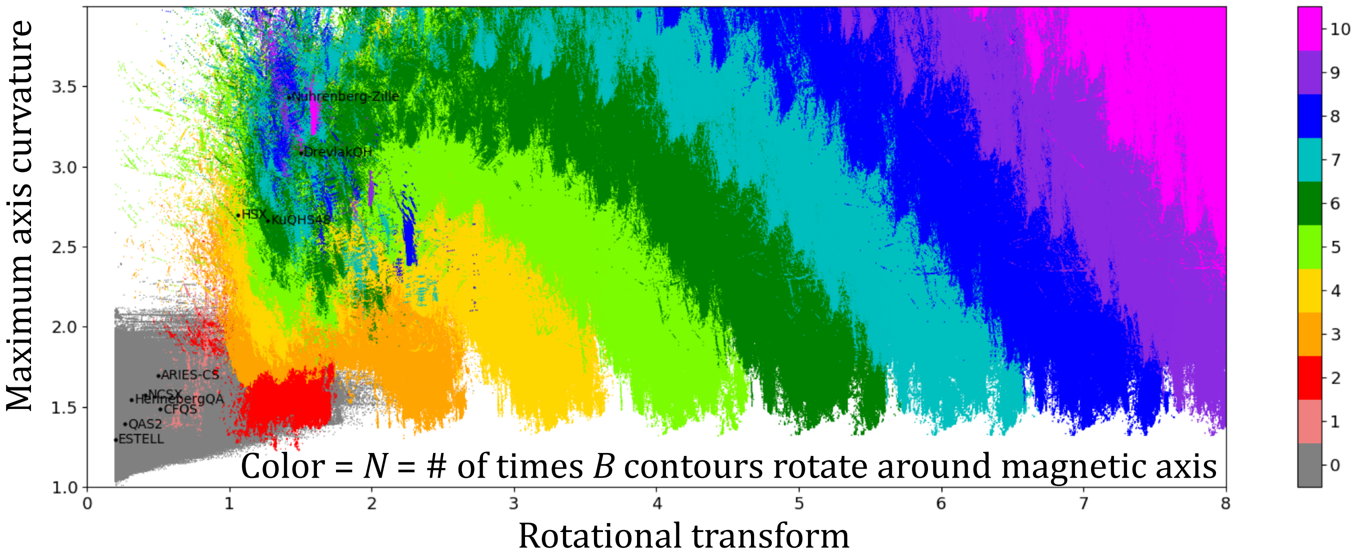
# Goal: Using rapid calculations high-aspect-ratio approximate equilibria instead of full 3D equilibria, survey all of stellarator shape-space

Previously we demonstrated that you can generate new quasisymmetric & omnigenous configurations & survey parameter space. [Landreman & Sengupta, JPP (2019)]






Takes only ~ 1-2 ms to compute & diagnose an equilibrium.

Here: what other properties can we diagnose at this speed?

# Figures of merit we can now compute near the magnetic axis

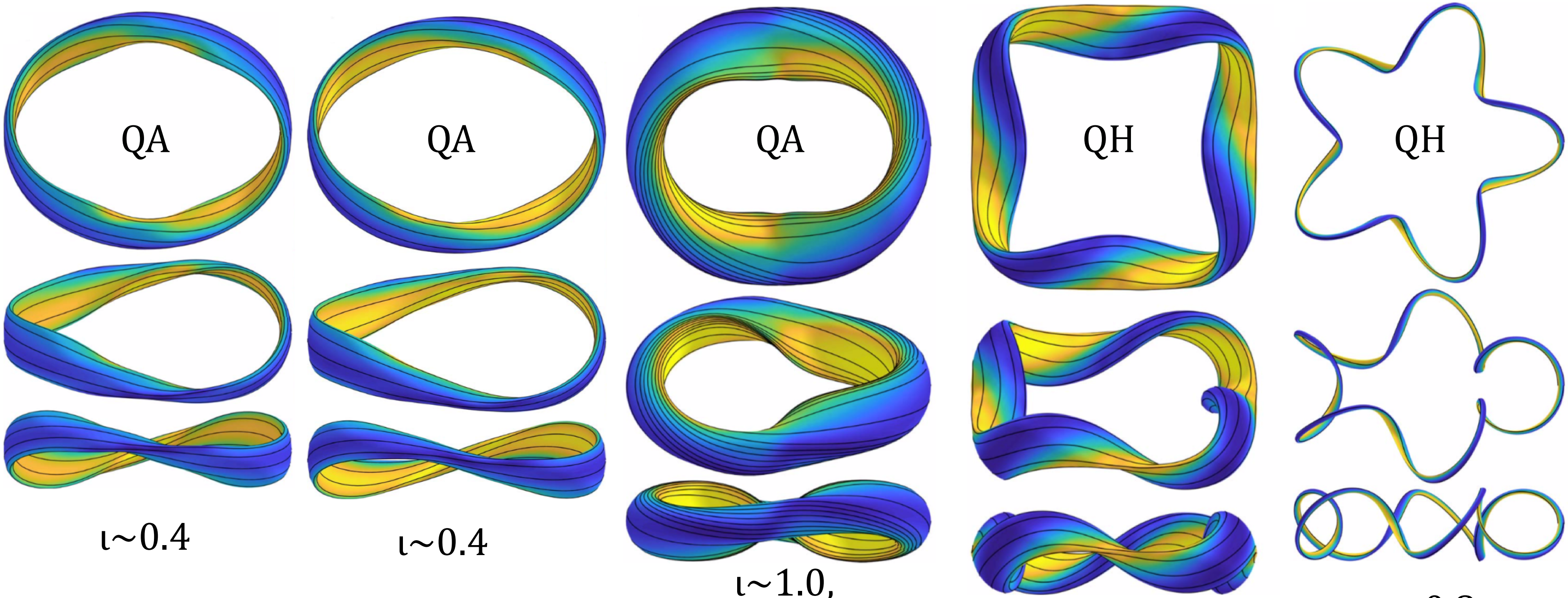
- Magnetic well
- Mercier & Glasser-Greene-Johnson stability criteria
- $\nabla \mathbf{B}$ and $\nabla\nabla \mathbf{B}$ tensors
- Departure from quasisymmetry
- Aspect ratio at which surfaces become singular.
- Geometry quantities for gyrokinetic stability/turbulence.

↖ Talk CO08.00003 by Rogerio Jorge et al, Monday 2:24pm & arXiv:2008.09057

# Here we demonstrate these figures of merit using 5 configurations

[Landreman & Sengupta, JPP (2019)]

| Section 5.1 | Section 5.2 | Section 5.3 | Section 5.4 | Section 5.5 |
|---|---|---|---|---|




| $\iota \sim 0.4$ | $\iota \sim 0.4$ | $\iota \sim 1.0$, $\iota_{vac} \sim 0.2$ | $\iota \sim 1.1$ | $\iota \sim 0.8$ |
|---|---|---|---|---|

# Magnetic well

- Related to MHD interchange stability.
- Dominant term in Mercier's criterion near the axis at low β.
- Usually included in stellarator design (W7-X, HSX, LHD, etc)
- Various definitions out there:

$$V'' = \frac{d^2V}{d\psi^2}, \text{ want} < 0.$$

$$\hat{W} = \frac{V}{\langle B^2 \rangle} \frac{d\langle B^2 \rangle}{dV}, \text{ want} > 0.$$

$$W = \frac{V}{\langle B^2 \rangle} \frac{d}{dV} \langle 2\mu_0 p + B^2 \rangle, \text{ want} > 0.$$

$V =$ Volume inside flux surface

$2\pi\psi =$ Toroidal flux

# Magnetic well can be computed directly from the near-axis expansion

For quasisymmetry:

$$V'' = \frac{16\pi^2 |G_0|}{B_0^3}\left[\frac{3}{4}\bar{\eta}^2 - \frac{B_{20}}{B_0} - \frac{\mu_0 p_2}{2B_0^2}\right] + O(\varepsilon^2)$$

where

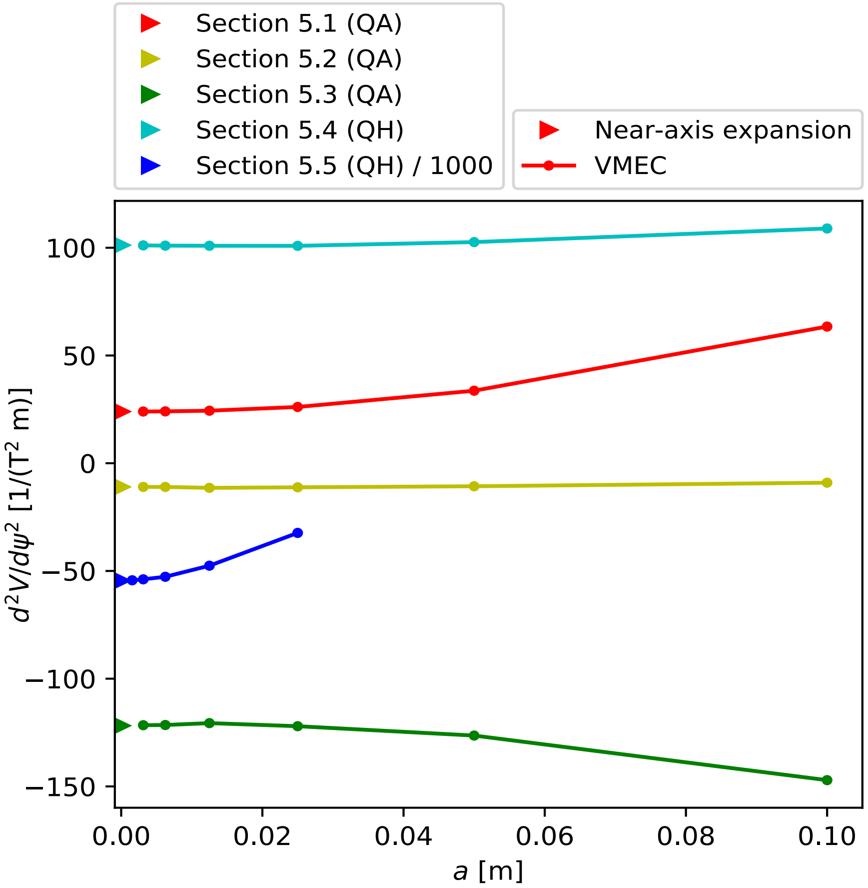
$$B(r,\theta,\varphi) = B_0 + r\bar{\eta}B_0\cos\theta + r^2\left[B_{20} + B_{2s}\sin 2\theta + B_{2c}\cos 2\theta\right] + O(\varepsilon^3)$$

$$\mathbf{B} = \beta\nabla\psi + I(\psi)\nabla\theta + G(\psi)\nabla\varphi$$

$$p(r) = p_0 + r^2 p_2 + O(\varepsilon^4)$$

[Landreman & Jorge, JPP (2020)]

Expansion agrees with VMEC



# Outline

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# Mercier criterion

Ideal MHD stability to radially localized perturbations (basically interchanges).

Mercier (1964):

$$M_G = \left[ \frac{s_G}{2} \frac{d\left(1/|\iota|\right)}{d\Phi} + \int \frac{\mathbf{B}\cdot\Xi\, dS}{|\nabla\Phi|^3} \right]^2 + \left[ \frac{s_\iota s_\psi}{\iota^2} \frac{dp}{d\Phi} \frac{d^2V}{d\Psi^2} - \int \frac{|\Xi|^2 dS}{|\nabla\Phi|^3} \right] \int \frac{B^2 dS}{|\nabla\Phi|^3} > 0$$

$$\Phi = \text{poloidal flux}, \quad \Psi = \text{toroidal flux}, \quad \Xi = \mathbf{J} - \mathbf{B}\frac{dI_{tor}}{d\Psi}, \quad s_G = \mathrm{sgn}(G), \quad s_\psi = \mathrm{sgn}(\Psi), \quad s_\iota = \mathrm{sgn}(\iota)$$

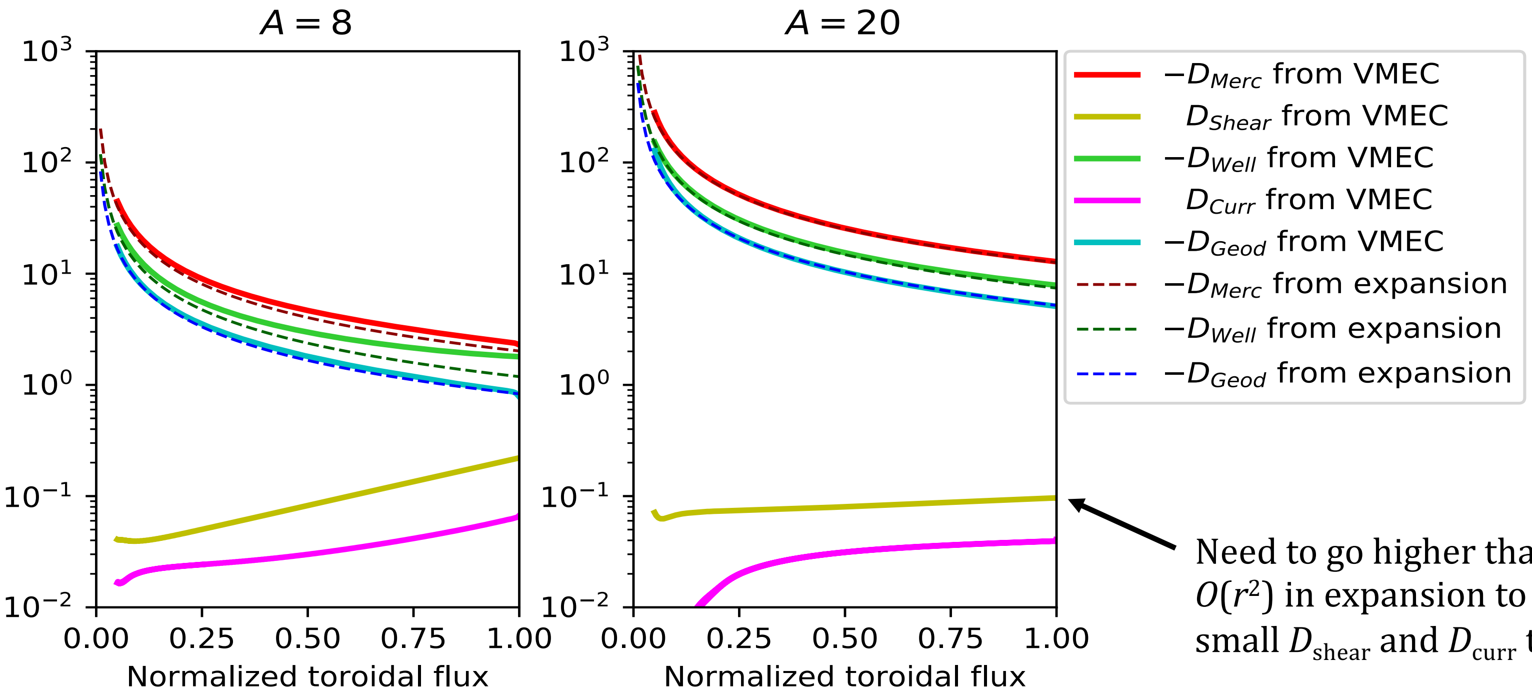
Equivalent expression in Bauer, Betancourt, & Garabedian (1984):

$$M_B = \frac{1}{4}\left(\frac{d\iota}{d\Psi}\right)^2 - s_G \frac{d\iota}{d\Psi} \iint \frac{d\theta d\varphi \left|\sqrt{g}\right| \mathbf{B}\cdot\Xi}{|\nabla\Psi|^2} + \frac{dp}{d\Psi}\left[ s_\psi \frac{d^2V}{d\Psi^2} - \frac{dp}{d\Psi} \iint \frac{d\theta d\varphi \left|\sqrt{g}\right|}{B^2} \right] \iint \frac{d\theta d\varphi \left|\sqrt{g}\right| B^2}{|\nabla\Psi|^2}$$
$$+ \left[ \iint \frac{d\theta d\varphi \left|\sqrt{g}\right| \mathbf{B}\cdot\mathbf{J}}{|\nabla\Psi|^2} \right]^2 - \left[ \iint \frac{d\theta d\varphi \left|\sqrt{g}\right| B^2}{|\nabla\Psi|^2} \right]\left[ \iint \frac{d\theta d\varphi \left|\sqrt{g}\right| (\mathbf{B}\cdot\mathbf{J})^2}{|\nabla\Psi|^2 B^2} \right] > 0$$

# Mercier stability can now be computed directly from the near-axis expansion

$$D_{\mathrm{Merc}} = \frac{|G_0|\mu_0 p_2}{8\pi^4 r^2 B_0^3}\left[\frac{d^2V}{d\psi^2} - \frac{8\pi^2|G_0|\mu_0 p_2}{B_0^5} - \frac{16\pi|G_0|^3\mu_0 p_2 \bar{\eta}^2}{B_0^7 \iota_{N0}^2}\int_0^{2\pi} d\varphi \frac{\bar{\eta}^4 + \kappa^4\sigma^2 + \bar{\eta}^2\kappa^2}{\bar{\eta}^4 + \kappa^4(1+\sigma^2) + 2\bar{\eta}^2\kappa^2}\right]$$

Section 5.4 configuration

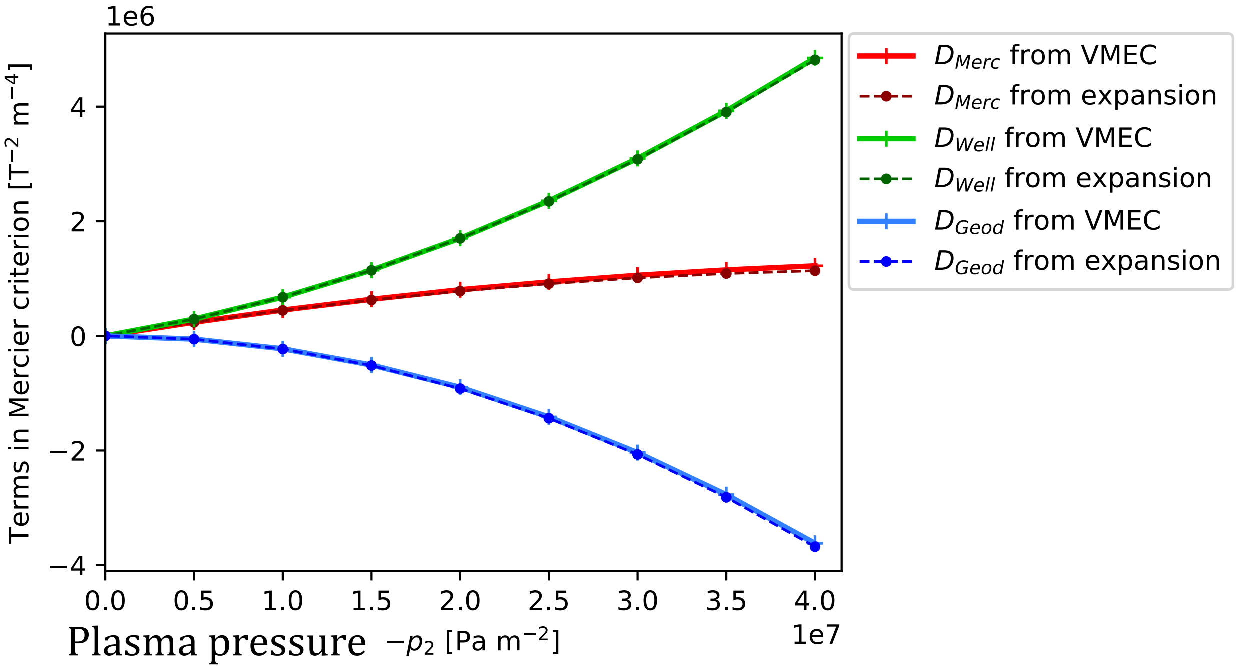



Need to go higher than $O(r^2)$ in expansion to get small $D_{\mathrm{shear}}$ and $D_{\mathrm{curr}}$ terms.

[Landreman & Jorge, JPP (2020)]

# Our expression for Mercier stability near the axis has been extensively benchmarked with VMEC

Section 5.5 configuration:




The criterion for *resistive* stability by Glasser, Greene, & Johnson [1975] turns out to be identical to Mercier's to the accuracy of our expansion.

[Landreman & Jorge, JPP (2020)]

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# ∇B and ∇∇B tensors

- Targeting **∇B** enables direct coil optimization for quasisymmetry. [Giuliani et al, arXiv:2010.02033 (2020)]

- These tensors contain all possible scale lengths in the 1st and 2nd derivatives of the field. These lengths should probably be large in order to make this **B** with distant coils.

$$L_{\nabla B} = B\sqrt{\frac{2}{\nabla\mathbf{B}:\nabla\mathbf{B}}} \qquad L_{\nabla\nabla B} = \sqrt{\frac{4B}{\sqrt{\sum_{i,j,k=1}^{3}\left(\nabla\nabla\mathbf{B}\right)_{i,j,k}^{2}}}}$$

At a distance $R$ from an infinite straight wire, $L_{\nabla B} = L_{\nabla\nabla B} = R$.

# Result for ∇B near the magnetic axis

$$\nabla \mathbf{B} = \frac{B_0}{\ell'}\Big[\left(X_{1c}' Y_{1s} + \iota X_{1c} Y_{1c}\right)\mathbf{nn} + \left(-\ell'\tau - \iota X_{1c}^2\right)\mathbf{bn}$$

$$+\left(Y_{1c}' Y_{1s} - Y_{1s}' Y_{1c} + \ell'\tau + \iota Y_{1s}^2 + \iota Y_{1c}^2\right)\mathbf{nb} + \left(X_{1c} Y_{1s}' - \iota X_{1c} Y_{1c}\right)\mathbf{bb}\Big] + \kappa B_0\left(\mathbf{tn} + \mathbf{nt}\right)$$

Frenet frame: $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ $\qquad$ $\ell' = (\text{axis length})/(2\pi)$ $\qquad$ $Y_{1s}' = dY_{1s}/d\varphi$

where the position vector is

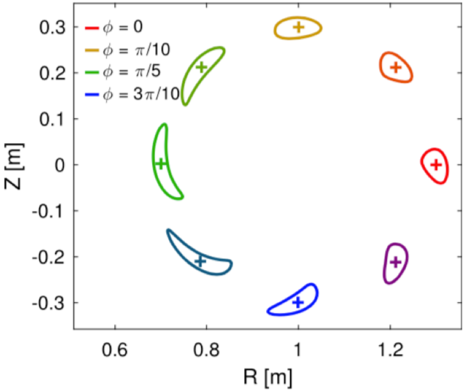
$$\mathbf{x}(r,\theta,\varphi) = \mathbf{x}_0(\varphi) + r X_{1c}(\varphi)\cos\theta\,\mathbf{n} + r\left[Y_{1c}(\varphi)\cos\theta + Y_{1s}(\varphi)\sin\theta\right]\mathbf{b} + O(r^2)$$

# These tensor norms seem correlated with intuition for how hard these configurations are to shape

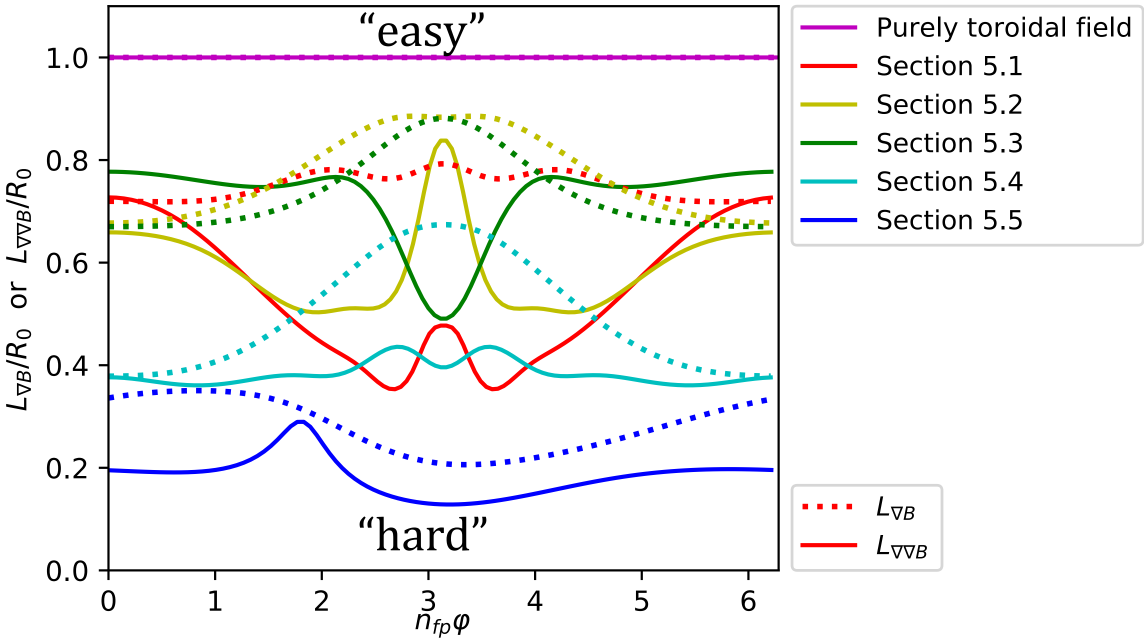
$$L_{\nabla B} = B\sqrt{\frac{2}{\nabla\mathbf{B}:\nabla\mathbf{B}}}$$

$$L_{\nabla\nabla B} = \sqrt{\frac{4B}{\sqrt{\sum_{i,j,k=1}^{3}\left(\nabla\nabla\mathbf{B}\right)_{i,j,k}^{2}}}}$$

Section 5.5 has very strong shaping




Scale lengths in the magnetic field, normalized to $R_0$




Is there anything else useful we can do with these tensors?

Are there other good measures of **B**-field complexity?

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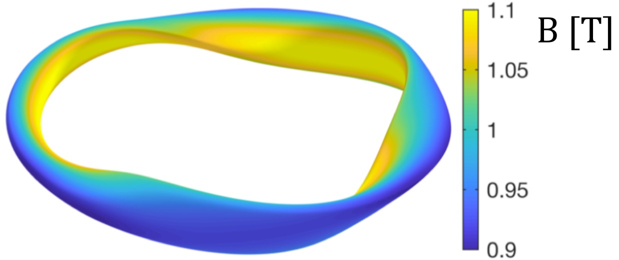
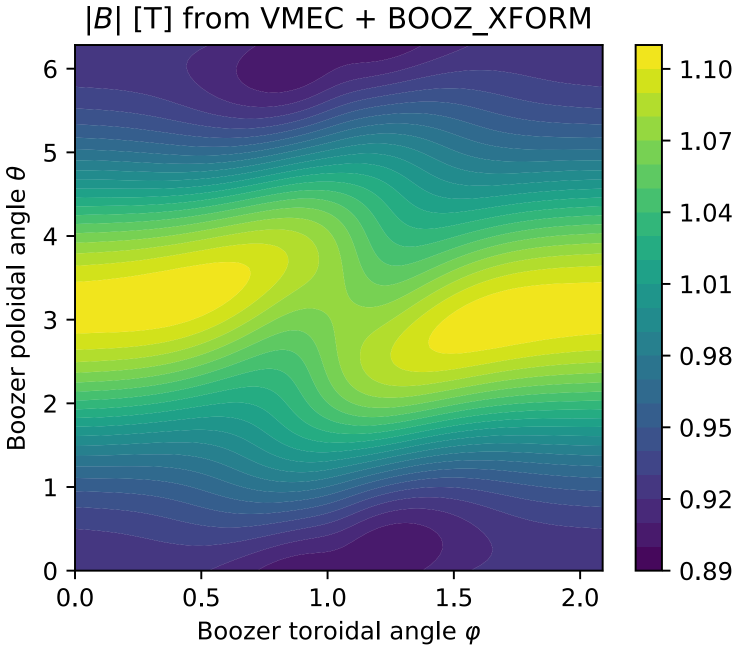
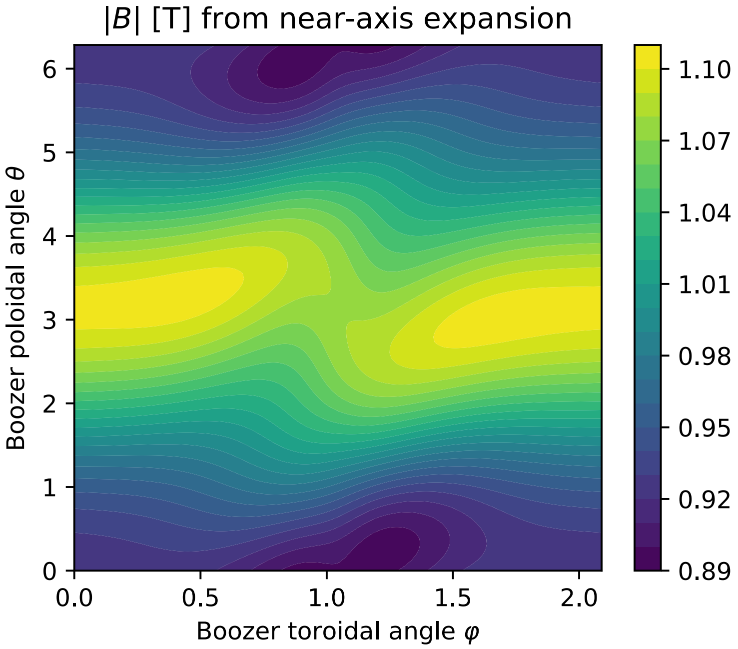
# If we strive for QS to O(r¹), we can compute the symmetry-breaking error at O(r²).



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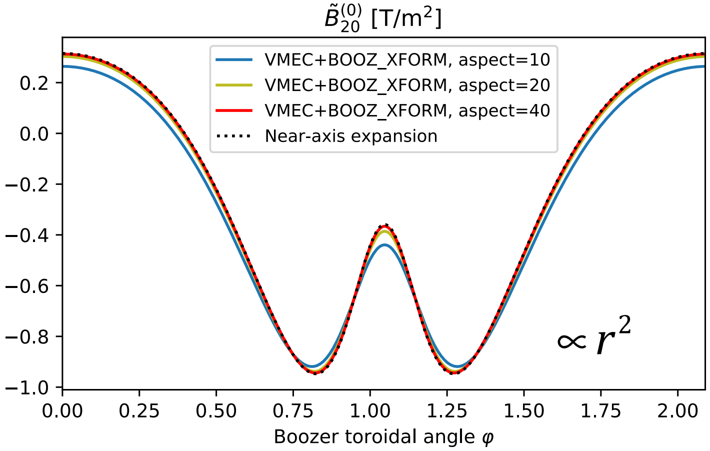
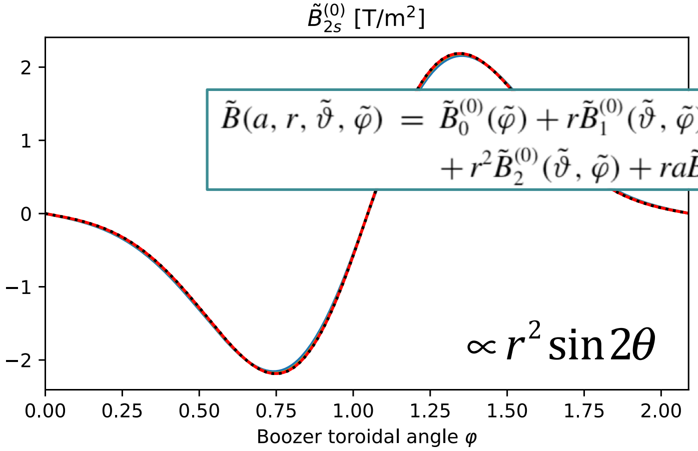
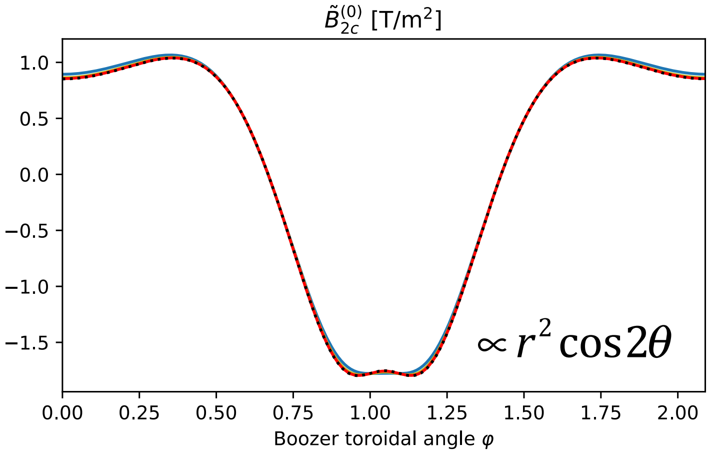
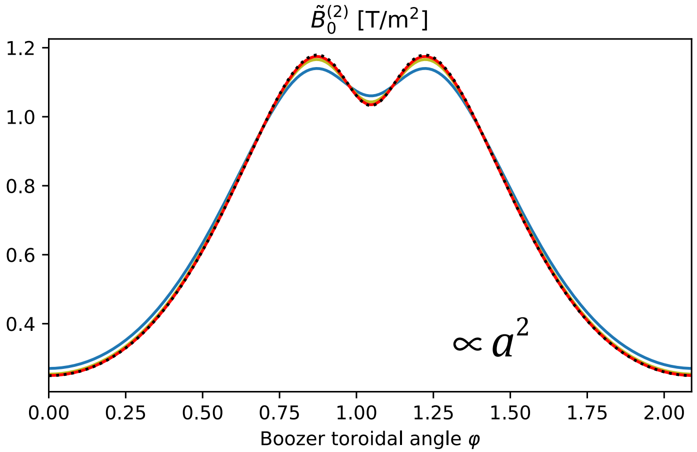



$$\tilde{B}(a, r, \tilde{\vartheta}, \tilde{\varphi}) = \tilde{B}_0^{(0)}(\tilde{\varphi}) + r\tilde{B}_1^{(0)}(\tilde{\vartheta}, \tilde{\varphi}) + a\tilde{B}_0^{(1)}(\tilde{\varphi}) + r^2\tilde{B}_2^{(0)}(\tilde{\vartheta}, \tilde{\varphi}) + ra\tilde{B}_1^{(1)}(\tilde{\vartheta}, \tilde{\varphi}) + a^2\tilde{B}_0^{(2)}(\tilde{\varphi}) + O((r/\mathcal{R})^3).$$




Next step: If we strive for QS to $O(r^2)$, can we compute the symmetry-breaking error at $O(r^3)$?

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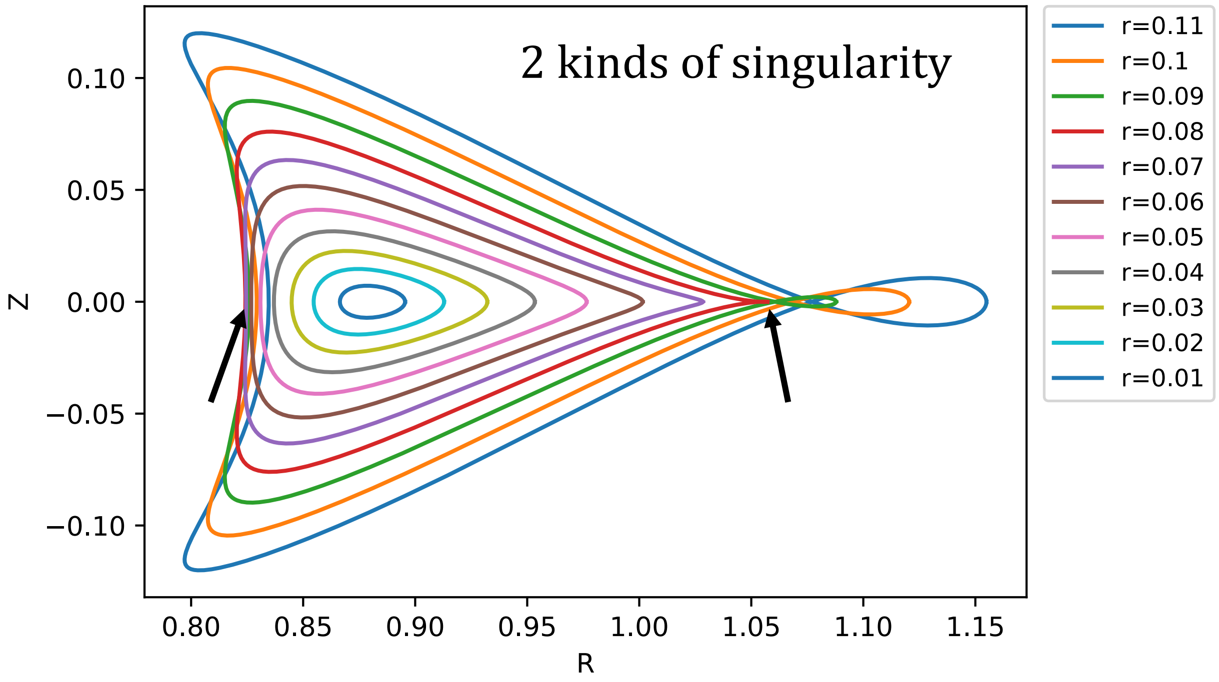
# Limiting factor for the aspect ratio: above some *r*, surfaces are no longer smooth & nested




$$\sqrt{g} = \frac{\partial \mathbf{x}}{\partial r} \cdot \frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \varphi} = 0$$

Minimize $r$ subject to $\sqrt{g} = 0$.

$$L = r + \lambda \sqrt{g}$$

$$\frac{\partial L}{\partial \lambda} = 0 \;\Rightarrow\; \sqrt{g} = 0$$

$$\frac{\partial L}{\partial \theta} = 0 \;\Rightarrow\; \frac{\partial \sqrt{g}}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \varphi} = 0 \;\Rightarrow\; \frac{\partial \sqrt{g}}{\partial \varphi} = 0$$

Uninteresting: $\frac{\partial L}{\partial r} = 0 \;\Rightarrow\; 1 + \lambda \frac{\partial \sqrt{g}}{\partial r} = 0$

# How can we compute the aspect ratio at which surfaces are no longer smooth & nested?

System of equations to solve: $\sqrt{g}=0 \qquad \frac{\partial\sqrt{g}}{\partial\theta}=0 \qquad \frac{\partial\sqrt{g}}{\partial\varphi}=0$

Form of Jacobian for $O(r^2)$ construction:

$$\sqrt{g}=r\left[g_0(\varphi)+rg_1(\theta,\varphi)+r^2g_2(\theta,\varphi)+r^3g_3(\theta,\varphi)+r^4g_4(\theta,\varphi)\right]$$

where $g_1(\theta,\varphi)=g_{1s}(\varphi)\sin\theta+g_{1c}(\varphi)\cos\theta$

$$g_2(\theta,\varphi)=g_{20}(\varphi)+g_{2s}(\varphi)\sin 2\theta+g_{2c}(\varphi)\cos 2\theta$$

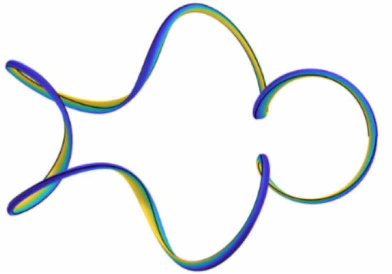
$$g_3(\theta,\varphi)=g_{3s1}(\varphi)\sin\theta+g_{3s3}(\varphi)\sin 3\theta+g_{3c1}(\varphi)\cos\theta+g_{3c3}(\varphi)\cos 3\theta$$

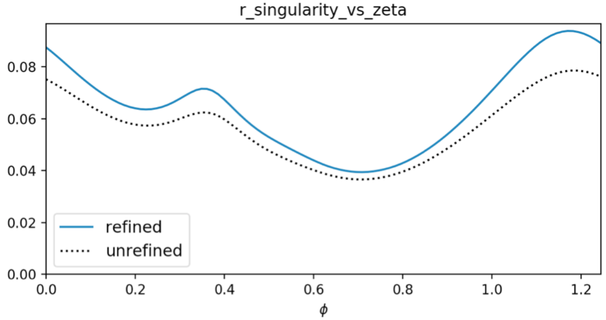
$$\ldots$$

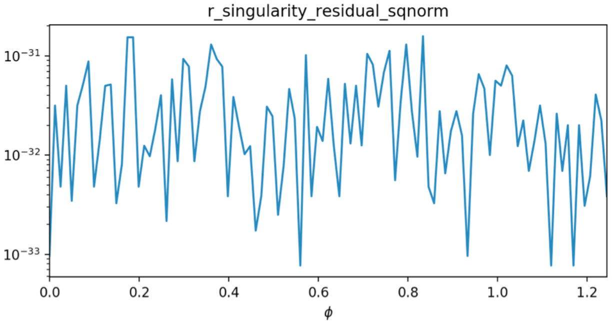
- Could solve with Newton method, but need good initial guess or else not robust.
- Worried most about small-$r$ solutions, so may be reasonable to set $g_3=g_4=0$?
- Then system has analytic solution. Can use as initial guess for Newton with $g_3$ & $g_4$.

# This approach of generating initial guesses for Newton iteration works sometimes but not always

Section 5.5:
Works well

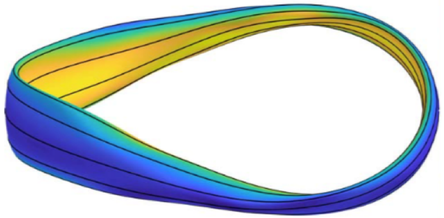


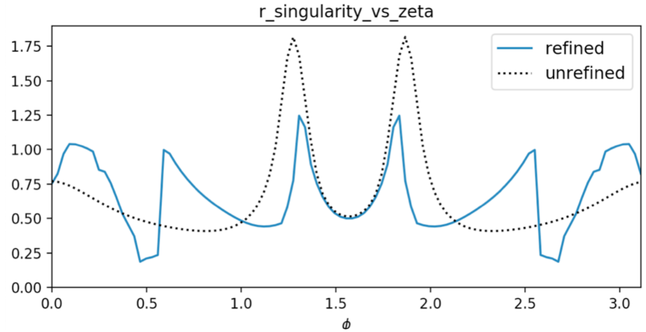


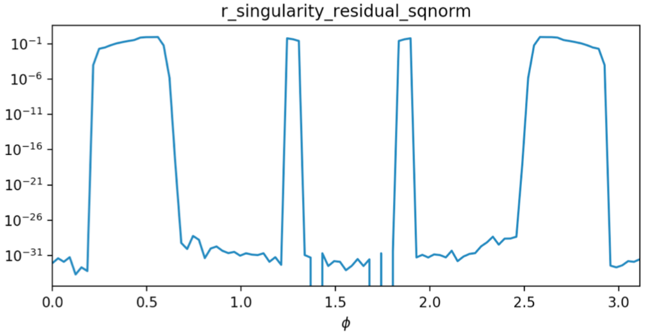




Newton converges to machine precision

Section 5.2:
Not so well






Newton fails to converge

# Questions for future work

- Is there anything else useful we can do with these $\nabla \mathbf{B}$ and $\nabla\nabla \mathbf{B}$ tensors?
- Are there other measures of **B** field complexity / coil difficulty we can rapidly compute from a near-axis solution?
- If we strive for QS to $O(r^2)$, can we compute the symmetry-breaking error at $O(r^3)$? (So much algebra!!)
- Is there a more robust way to compute the minimum aspect ratio?
- Does this singularity measure reflect the equilibrium $\beta$ limit?
- What else can we compute in < a few ms?