Stellarator figures of merit near the magnetic axis

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Previously we demonstrated that you can generate new quasisymmetric & omnigenous configurations & survey parameter space. [Landreman & Sengupta, JPP (2019)]

Takes only $\sim 1$-$2$ ms to compute & diagnose an equilibrium. Here: what other properties can we diagnose at this speed?
Figures of merit we can now compute near the magnetic axis

• Magnetic well

• Mercier & Glasser-Greene-Johnson stability criteria

• $\nabla B$ and $\nabla^2 B$ tensors

• Departure from quasisymmetry

• Aspect ratio at which surfaces become singular.

• Geometry quantities for gyrokinetic stability/turbulence.
Here we demonstrate these figures of merit using 5 configurations

[Landreman & Sengupta, JPP (2019)]

Section 5.1: QA

Section 5.2: QA

Section 5.3: QA

Section 5.4: QH

Section 5.5: QH

$\imath \sim 0.4$

$\imath \sim 0.4$

$\imath \sim 1.0, \imath_{\text{vac}} \sim 0.2$

$\imath \sim 1.1$

$\imath \sim 0.8$
Magnetic well

- Related to MHD interchange stability.
- Dominant term in Mercier’s criterion near the axis at low $\beta$.
- Usually included in stellarator design (W7-X, HSX, LHD, etc)
- Various definitions out there:

$$V'' = \frac{d^2V}{d\psi^2}, \text{ want } < 0.$$  

$$\hat{W} = \frac{V}{\langle B^2 \rangle} \frac{\langle B^2 \rangle}{dV}, \text{ want } > 0.$$  

$$W = \frac{V}{\langle B^2 \rangle} \frac{d}{dV} \langle 2\mu_0 p + B^2 \rangle, \text{ want } > 0.$$  

$V$ = Volume inside flux surface  
$2\pi\psi$ = Toroidal flux
Magnetic well can be computed directly from the near-axis expansion

For quasisymmetry:

\[ V'' = \frac{16\pi^2}{B_0^3} G_0 \left[ \frac{3}{4} \eta^2 - \frac{B_{20}}{B_0} - \frac{\mu_0 p_2}{2B_0^2} \right] + O(\varepsilon^2) \]

where

\[ B(r,\theta,\phi) = B_0 + r\eta B_0 \cos\theta \]

\[ + r^2 \left[ B_{20} + B_{2s} \sin 2\theta + B_{2c} \cos 2\theta \right] + O(\varepsilon^3) \]

\[ B = \beta \nabla \psi + I(\psi) \nabla \theta + G(\psi) \nabla \phi \]

\[ p(r) = p_0 + r^2 p_2 + O(\varepsilon^4) \]

[Landreman & Jorge, JPP (2020)]
Outline

- Magnetic well
- Mercier & Glasser-Greene-Johnson stability criteria
- $\nabla B$ and $\nabla \nabla B$ tensors
- Departure from quasisymmetry
- Aspect ratio at which surfaces become singular.
Mercier criterion

Ideal MHD stability to radially localized perturbations (basically interchanges).

Mercier (1964): \[ M_G = \left[ \frac{s_G}{2} \frac{d(1/|l|)}{d\Phi} + \int \frac{\mathbf{B} \cdot \Xi}{|\nabla \Phi|^3} dS \right]^2 + \left[ \frac{s_i s_\psi}{l^2} \frac{dp}{d\Phi} \frac{d^2 V}{d\Psi^2} - \int \frac{\Xi^2}{|\nabla \Phi|^3} dS \right] \int \frac{B^2 dS}{|\nabla \Phi|^3} > 0 \]

\( \Phi = \) poloidal flux, \( \Psi = \) toroidal flux, \( \Xi = \mathbf{J} - \mathbf{B} \frac{dI_{tor}}{d\Psi} \), \( s_G = \text{sgn}(G) \), \( s_\psi = \text{sgn}(\Psi) \), \( s_i = \text{sgn}(l) \)

Equivalent expression in Bauer, Betancourt, & Garabedian (1984):

\[ M_B = \frac{1}{4} \frac{dt}{d\Psi} \left[ \frac{dt}{d\Psi} \right]^2 - s_G \frac{dt}{d\Psi} \int \frac{d\theta d\phi}{|\nabla \Psi|^2} \sqrt{g} \left( \mathbf{B} \cdot \Xi \right) + \frac{dp}{d\Psi} \left[ s_\psi \frac{d^2 V}{d\Psi^2} - \frac{dp}{d\Psi} \right] \int \frac{d\theta d\phi}{B^2} \sqrt{g} \left( \mathbf{B} \cdot \Xi \right) \int \frac{d\theta d\phi}{|\nabla \Psi|^2} \sqrt{g} B^2 \]

\[ + \left[ \int \frac{d\theta d\phi}{|\nabla \Psi|^2} \sqrt{g} \left( \mathbf{B} \cdot \mathbf{J} \right)^2 \right]^2 - \left[ \int \frac{d\theta d\phi}{|\nabla \Psi|^2} \sqrt{g} B^2 \right] \int \frac{d\theta d\phi}{|\nabla \Psi|^2 B^2} \right] > 0 \]
Mercier stability can now be computed directly from the near-axis expansion

\[
D_{\text{Merc}} = \frac{|G_0| \mu_0 p_2}{8 \pi^4 r^2 B_0^3} \left[ \frac{d^2 V}{d\psi^2} - \frac{8 \pi^2 |G_0| \mu_0 p_2}{B_0^5} - \frac{16 \pi |G_0|^3 \mu_0 p_2 \eta^2}{B_0^7 \nu_{N0}^2} \int_0^{2\pi} d\varphi \frac{\eta^4 + \kappa^4 \sigma^2 + \eta^2 \kappa^2}{\eta^4 + \kappa^4(1 + \sigma^2) + 2\eta^2 \kappa^2} \right]
\]

Section 5.4 configuration

Need to go higher than \(O(r^2)\) in expansion to get small \(D_{\text{shear}}\) and \(D_{\text{curr}}\) terms.

[Landreman & Jorge, JPP (2020)]
Section 5.5 configuration:

The criterion for resistive stability by Glasser, Greene, & Johnson [1975] turns out to be identical to Mercier’s to the accuracy of our expansion.

[Landreman & Jorge, JPP (2020)]
• Magnetic well

• Mercier & Glasser-Greene-Johnson stability criteria

• $\nabla B$ and $\nabla \nabla B$ tensors

• Departure from quasisymmetry

• Aspect ratio at which surfaces become singular.
**∇B and ∇∇B tensors**


- These tensors contain all possible scale lengths in the 1\textsuperscript{st} and 2\textsuperscript{nd} derivatives of the field. These lengths should probably be large in order to make this $B$ with distant coils.

\[ L_{∇B} = B \sqrt{\frac{2}{∇B : ∇B}} \quad L_{∇∇B} = \sqrt[3]{\frac{4B}{\sum_{i,j,k=1}^{3} (∇∇B)^2_{i,j,k}}} \]

At a distance $R$ from an infinite straight wire, $L_{∇B} = L_{∇∇B} = R$. 
Result for $\nabla B$ near the magnetic axis

\[
\nabla B = \frac{B_0}{\ell'} \left[ \left( X'_1 Y_{1s} + i X'_1 Y_{1c} \right) nn + \left( -\ell' \tau - i X^2 \right) bn \right.
\]

\[
+ \left( Y'_1 Y_{1s} - Y'_1 Y_{1c} + \ell' \tau + i Y^2_{1s} + i Y^2_{1c} \right) nb + \left( X_{1c} Y'_1 - i X_{1s} Y_{1c} \right) bb \bigg] + \kappa B_0 (tn + nt)
\]

Frenet frame: $(t, n, b)$

\[
\ell' = (\text{axis length}) / (2\pi)
\]

\[
Y_{1s} = dY_{1s} / d\varphi
\]

where the position vector is

\[
x(r, \theta, \varphi) = x_0(\varphi) + r X_{1c}(\varphi) \cos \theta n + r \left[ Y_{1c}(\varphi) \cos \theta + Y_{1s}(\varphi) \sin \theta \right] b + O(r^2)
\]
These tensor norms seem correlated with intuition for how hard these configurations are to shape.

\[ L_{\nabla B} = B \sqrt{\frac{2}{\nabla B : \nabla B}} \]

\[ L_{\nabla \nabla B} = \sqrt{\frac{4B}{\sum_{i,j,k=1}^{3} (\nabla \nabla B)_{i,j,k}^2}} \]

Section 5.5 has very strong shaping.

Is there anything else useful we can do with these tensors? Are there other good measures of \( B \)-field complexity?
• Magnetic well
• Mercier & Glasser-Greene-Johnson stability criteria
• \( \nabla B \) and \( \nabla \nabla B \) tensors
• Departure from quasisymmetry
• Aspect ratio at which surfaces become singular.
If we strive for QS to $O(r^1)$, we can compute the symmetry-breaking error at $O(r^2)$. 

|B| [T] from VMEC + BOOZ_XFORM

|B| [T] from near-axis expansion
If we strive for QS to $O(r^1)$, we can compute the symmetry-breaking error at $O(r^2)$. 

$$\propto r^2$$

$$\propto r^2 \sin 2\theta$$

Next step: If we strive for QS to $O(r^2)$, can we compute the symmetry-breaking error at $O(r^3)$?

$$\propto r^2 \cos 2\theta$$

$$\propto a^2$$
• Magnetic well

• Mercier & Glasser-Greene-Johnson stability criteria

• $\nabla B$ and $\nabla^2 B$ tensors

• Departure from quasisymmetry

• Aspect ratio at which surfaces become singular.
Limiting factor for the aspect ratio: above some $r$, surfaces are no longer smooth & nested.

Minimize $r$ subject to $\sqrt{g} = 0$.

$$L = r + \lambda \sqrt{g}$$

$$\frac{\partial L}{\partial \lambda} = 0 \implies \sqrt{g} = 0$$

$$\frac{\partial L}{\partial \theta} = 0 \implies \frac{\partial \sqrt{g}}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \phi} = 0 \implies \frac{\partial \sqrt{g}}{\partial \phi} = 0$$

Uninteresting: $\frac{\partial L}{\partial r} = 0 \implies 1 + \lambda \frac{\partial \sqrt{g}}{\partial r} = 0$

\[ \sqrt{g} = \frac{\partial x}{\partial r} \cdot \frac{\partial x}{\partial \theta} \times \frac{\partial x}{\partial \phi} = 0 \]
How can we compute the aspect ratio at which surfaces are no longer smooth & nested?

System of equations to solve:
\[ \sqrt{g} = 0 \quad \frac{\partial \sqrt{g}}{\partial \theta} = 0 \quad \frac{\partial \sqrt{g}}{\partial \phi} = 0 \]

Form of Jacobian for \( O(r^2) \) construction:
\[ \sqrt{g} = r \left[ g_0(\phi) + r g_1(\theta, \phi) + r^2 g_2(\theta, \phi) + r^3 g_3(\theta, \phi) + r^4 g_4(\theta, \phi) \right] \]

where
\[ g_1(\theta, \phi) = g_{1s}(\phi) \sin \theta + g_{1c}(\phi) \cos \theta \]
\[ g_2(\theta, \phi) = g_{20}(\phi) + g_{2s}(\phi) \sin 2\theta + g_{2c}(\phi) \cos 2\theta \]
\[ g_3(\theta, \phi) = g_{3s1}(\phi) \sin \theta + g_{3s3}(\phi) \sin 3\theta + g_{3c1}(\phi) \cos \theta + g_{3c3}(\phi) \cos 3\theta \]

• Could solve with Newton method, but need good initial guess or else not robust.
• Worried most about small-\( r \) solutions, so may be reasonable to set \( g_3=g_4=0? \)
• Then system has analytic solution. Can use as initial guess for Newton with \( g_3 \) & \( g_4 \).
This approach of generating initial guesses for Newton iteration works sometimes but not always.

Section 5.5: Works well

Section 5.2: Not so well

Newton converges to machine precision

Newton fails to converge
Questions for future work

• Is there anything else useful we can do with these $\nabla \mathbf{B}$ and $\nabla \nabla \mathbf{B}$ tensors?

• Are there other measures of $\mathbf{B}$ field complexity / coil difficulty we can rapidly compute from a near-axis solution?

• If we strive for QS to $O(r^2)$, can we compute the symmetry-breaking error at $O(r^3)$? (So much algebra!!)

• Is there a more robust way to compute the minimum aspect ratio?

• Does this singularity measure reflect the equilibrium $\beta$ limit?

• What else can we compute in $< a$ few ms?