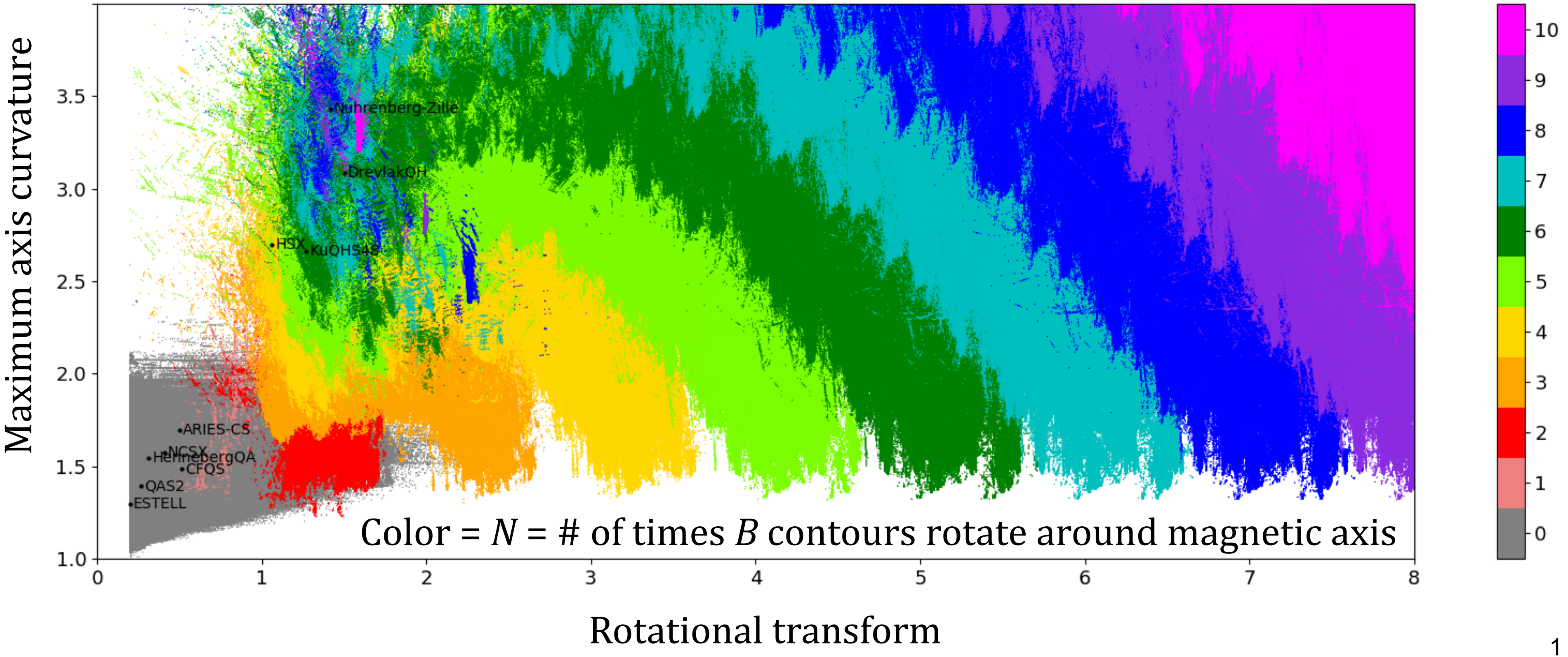
What other quantities can we compute in < 1ms from the near-axis expansion?

Goal: Filter out points from this database that are unacceptable for some reason.



Outline

- Magnetic well
- Mercier stability criterion
- $\nabla \mathbf{B}$ and $\nabla\nabla \mathbf{B}$ tensors
- Departure from quasisymmetry
- Aspect ratio at which surfaces become singular.

Magnetic well

- Related to MHD interchange stability.
- Dominant term in Mercier's criterion near the axis at low β.
- Usually included in stellarator design (W7-X, HSX, LHD, etc)
- Various definitions out there:

$$V'' = \frac{d^2 V}{d\psi^2}, \text{ want} < 0.$$

$$\hat{W} = \frac{V}{\langle B^2 \rangle} \frac{d\langle B^2 \rangle}{dV}, \quad \text{want} > 0.$$

$$W = \frac{V}{\langle B^2 \rangle} \frac{d}{dV} \langle 2\mu_0 p + B^2 \rangle, \quad \text{want} > 0.$$

$V =$ Volume inside flux surface

$2\pi\psi =$ Toroidal flux

Magnetic well can be computed directly from the near-axis expansion

For quasisymmetry:

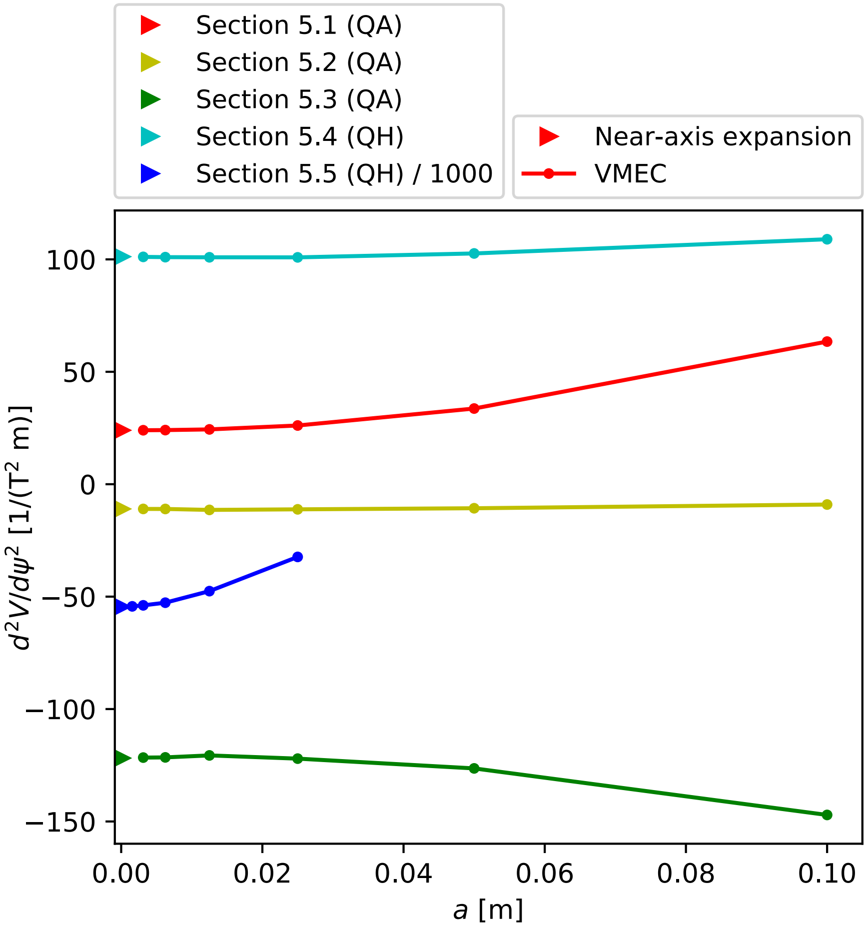
$$V'' = \frac{16\pi^2 |G_0|}{B_0^3}\left[\frac{3}{4}\bar{\eta}^2 - \frac{B_{20}}{B_0} - \frac{\mu_0 p_2}{2B_0^2}\right] + O\left(\varepsilon^2\right)$$

where

$$B\left(r,\theta,\varphi\right) = B_0 + r\bar{\eta}B_0\cos\theta + r^2\left[B_{20} + B_{2s}\sin 2\theta + B_{2c}\cos 2\theta\right] + O\left(\varepsilon^3\right)$$

$$\mathbf{B} = \beta\nabla\psi + I\left(\psi\right)\nabla\theta + G\left(\psi\right)\nabla\varphi$$

$$p\left(r\right) = p_0 + r^2 p_2 + O\left(\varepsilon^4\right)$$



Outline

- Magnetic well
- Mercier stability criterion
- $\nabla \mathbf{B}$ and $\nabla\nabla \mathbf{B}$ tensors
- Departure from quasisymmetry
- Aspect ratio at which surfaces become singular.

Mercier criterion

Ideal MHD stability to radially localized perturbations (basically interchanges).

Mercier (1964):

$$M_G = \left[\frac{s_G}{2} \frac{d\left(1/\left|\iota\right|\right)}{d\Phi} + \int \frac{\mathbf{B}\cdot\Xi \, dS}{\left|\nabla\Phi\right|^3} \right]^2 + \left[\frac{s_\iota s_\psi}{\iota^2} \frac{dp}{d\Phi} \frac{d^2V}{d\Psi^2} - \int \frac{\left|\Xi\right|^2 dS}{\left|\nabla\Phi\right|^3} \right] \int \frac{B^2 dS}{\left|\nabla\Phi\right|^3} > 0$$

$\Phi = \text{poloidal flux}, \quad \Psi = \text{toroidal flux}, \quad \Xi = \mathbf{J} - \mathbf{B}\dfrac{dI_{tor}}{d\Psi}, \quad s_G = \text{sgn}(G), \quad s_\psi = \text{sgn}(\Psi), \quad s_\iota = \text{sgn}(\iota)$

Bauer, Betancourt, & Garabedian (1984):

$$M_B = \frac{1}{4}\left(\frac{d\iota}{d\Psi}\right)^2 - s_G \frac{d\iota}{d\Psi} \iint \frac{d\theta d\varphi \left|\sqrt{g}\right| \mathbf{B}\cdot\Xi}{\left|\nabla\Psi\right|^2} + \frac{dp}{d\Psi}\left[s_\psi \frac{d^2V}{d\Psi^2} - \frac{dp}{d\Psi} \iint \frac{d\theta d\varphi \left|\sqrt{g}\right|}{B^2} \right] \iint \frac{d\theta d\varphi \left|\sqrt{g}\right| B^2}{\left|\nabla\Psi\right|^2}$$
$$+ \left[\iint \frac{d\theta d\varphi \left|\sqrt{g}\right| \mathbf{B}\cdot\mathbf{J}}{\left|\nabla\Psi\right|^2} \right]^2 - \left[\iint \frac{d\theta d\varphi \left|\sqrt{g}\right| B^2}{\left|\nabla\Psi\right|^2} \right]\left[\iint \frac{d\theta d\varphi \left|\sqrt{g}\right| \left(\mathbf{B}\cdot\mathbf{J}\right)^2}{\left|\nabla\Psi\right|^2 B^2} \right] > 0$$

All statements of Mercier stability Rogerio & I can find do not respect parity transformations

$$\mathbf{B}=\frac{1}{2\pi}\left(\nabla\Psi\times\nabla\theta+\nabla\varphi\times\nabla\Phi\right)=\beta\nabla\psi+I\nabla\theta+G\nabla\varphi$$

Parity transformation 1: Flip signs of Ψ, θ, β, I, ι. Unchanged: φ, G, Φ.

Parity transformation 2: Flip signs of φ, G, Φ, ι. Unchanged: Ψ, θ, β, I.

Mercier (1964): $$M_G=\left[\frac{1}{2}\frac{d(1/\iota)}{d\Phi}+\int\frac{\mathbf{B}\cdot\Xi\,dS}{\left|\nabla\Phi\right|^3}\right]^2+\left[\frac{1}{\iota^2}\frac{dp}{d\Phi}\frac{d^2V}{d\Psi^2}-\int\frac{\left|\Xi\right|^2 dS}{\left|\nabla\Phi\right|^3}\right]\int\frac{B^2 dS}{\left|\nabla\Phi\right|^3}>0$$

$\Phi=\text{poloidal flux},\quad \Psi=\text{toroidal flux},\quad \Xi=\mathbf{J}-\mathbf{B}\dfrac{dI_{tor}}{d\Psi},\quad s_G=\operatorname{sgn}(G),\quad s_\psi=\operatorname{sgn}(\Psi),\quad s_\iota=\operatorname{sgn}(\iota)$

Invariant: $$M_G=\left[\frac{s_G}{2}\frac{d(1/|\iota|)}{d\Phi}+\int\frac{\mathbf{B}\cdot\Xi\,dS}{\left|\nabla\Phi\right|^3}\right]^2+\left[\frac{s_\iota s_\psi}{\iota^2}\frac{dp}{d\Phi}\frac{d^2V}{d\Psi^2}-\int\frac{\left|\Xi\right|^2 dS}{\left|\nabla\Phi\right|^3}\right]\int\frac{B^2 dS}{\left|\nabla\Phi\right|^3}>0$$

All statements of Mercier stability Rogerio & I can find do not respect parity transformations

$$\mathbf{B} = \frac{1}{2\pi}\left(\nabla\Psi\times\nabla\theta + \nabla\varphi\times\nabla\Phi\right) = \beta\nabla\psi + I\nabla\theta + G\nabla\varphi$$

Parity transformation 1: Flip signs of Ψ, θ, β, I, ι. Unchanged: φ, G, Φ.

Parity transformation 2: Flip signs of φ, G, Φ, ι. Unchanged: Ψ, θ, β, I.

Is there a slick way to get a parity-transformation-invariant form of Mercier's criterion?

Mercier (1

$$\Phi = \text{poloidal flux}, \quad \Psi = \text{toroidal flux}, \quad \Xi = \mathbf{J} - \mathbf{B}\frac{dI_{tor}}{d\Psi}, \quad s_G = \mathrm{sgn}(G), \quad s_\psi = \mathrm{sgn}(\Psi), \quad s_\iota = \mathrm{sgn}(\iota)$$

Invariant: $$M_G = \left[\frac{s_G}{2}\frac{d(1/|\iota|)}{d\Phi} + \int\frac{\mathbf{B}\cdot\Xi\, dS}{|\nabla\Phi|^3}\right]^2 + \left[\frac{s_\iota s_\psi}{\iota^2}\frac{dp}{d\Phi}\frac{d^2V}{d\Psi^2} - \int\frac{|\Xi|^2\, dS}{|\nabla\Phi|^3}\right]\int\frac{B^2 dS}{|\nabla\Phi|^3} > 0$$

Outline

- Magnetic well
- Mercier stability criterion
- $\nabla \mathbf{B}$ and $\nabla\nabla \mathbf{B}$ tensors
- Departure from quasisymmetry
- Aspect ratio at which surfaces become singular.

∇B and $\nabla\nabla$B tensors

- Andrew Giuliani targets $\nabla\mathbf{B}$ in his direct coil optimization for QS.

- These tensors contain all possible scale lengths in the 1st and 2nd derivatives of the field. These should probably be long in order to make this **B** with distant coils.

$$L_{\nabla B} = B\sqrt{\frac{2}{\nabla\mathbf{B}:\nabla\mathbf{B}}} \qquad L_{\nabla\nabla B} = \sqrt{\frac{4B}{\sqrt{\sum_{i,j,k=1}^{3}\left(\nabla\nabla\mathbf{B}\right)_{i,j,k}^{2}}}}$$

At a distance R from an infinite straight wire, $L_{\nabla B} = L_{\nabla\nabla B} = R$.

Garren-Boozer ∇B

$$\nabla \mathbf{B} = \frac{B_0}{\ell'}\Big[\left(X_{1c}' Y_{1s} + \iota X_{1c} Y_{1c}\right)\mathbf{nn} + \left(-\ell' \tau - \iota X_{1c}^2\right)\mathbf{bn}$$

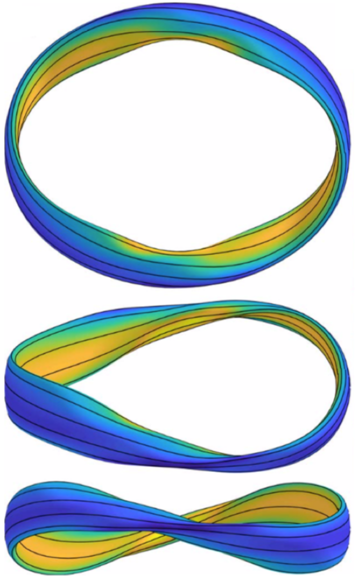
$$+\left(Y_{1c}' Y_{1s} - Y_{1s}' Y_{1c} + \ell' \tau + \iota Y_{1s}^2 + \iota Y_{1c}^2\right)\mathbf{nb} + \left(X_{1c} Y_{1s}' - \iota X_{1c} Y_{1c}\right)\mathbf{bb}\Big] + \kappa B_0 \left(\mathbf{tn} + \mathbf{nt}\right)$$

Frenet frame: $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ $\qquad$ $\ell' = (\text{axis length})/(2\pi)$ $\qquad$ $Y_{1s}' = dY_{1s} / d\varphi$

$$\mathbf{x}(r,\theta,\varphi) = \mathbf{x}_0(\varphi) + r X_{1c}(\varphi)\cos\theta\,\mathbf{n} + r\left[Y_{1c}(\varphi)\cos\theta + Y_{1s}(\varphi)\sin\theta\right]\mathbf{b} + O(r^2)$$

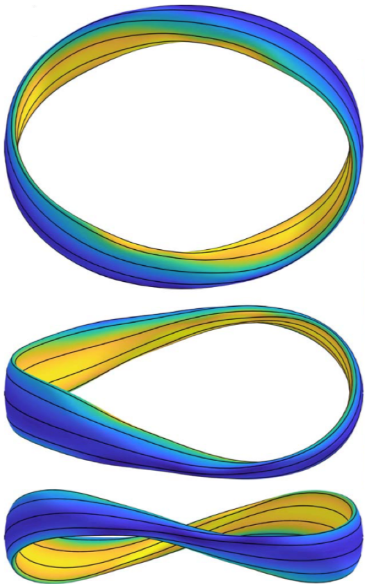
5 configurations to compare

Section 5.1



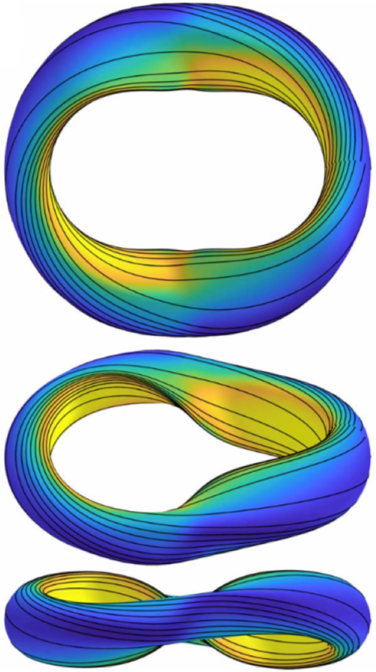
$\iota \sim 0.4$

Section 5.2



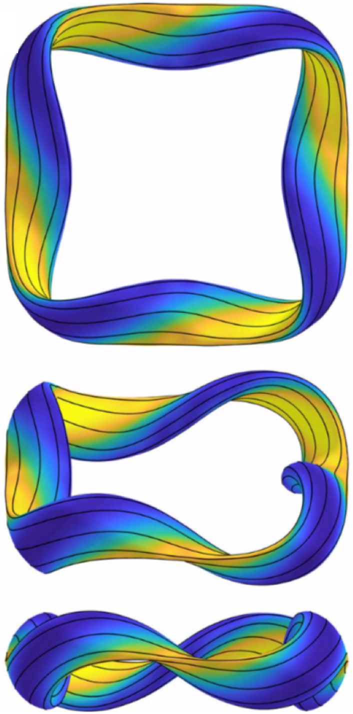
$\iota \sim 0.4$

Section 5.3



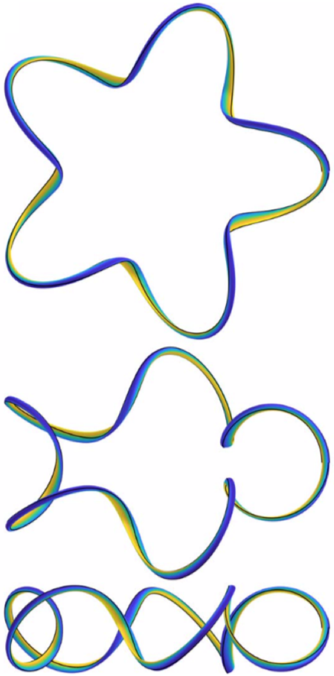
$\iota \sim 1.0$,
$\iota_{vac} \sim 0.2$

Section 5.4



$\iota \sim 1.1$

Section 5.5

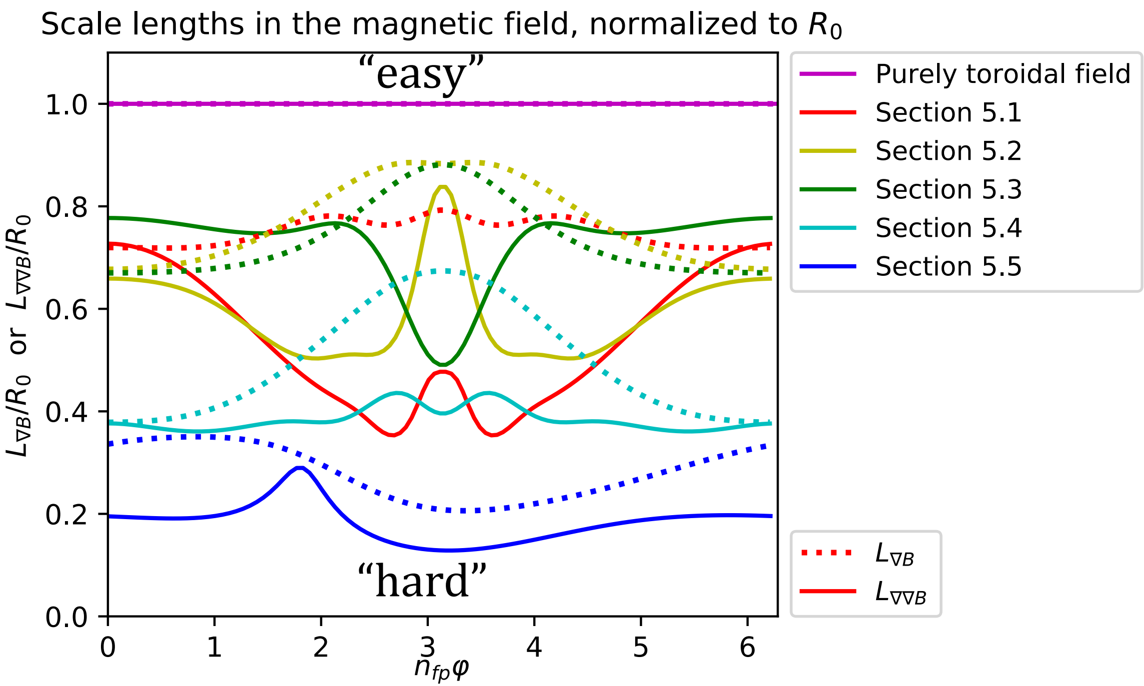


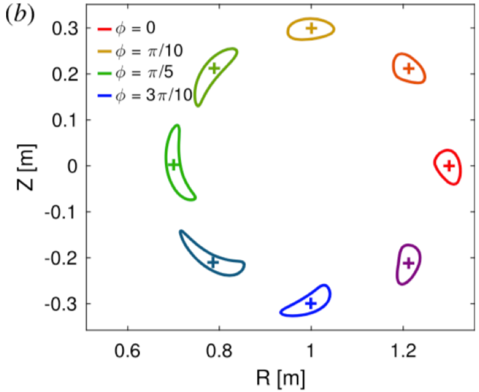
$\iota \sim 0.8$

These tensor norms seem correlated with intuition for how hard these configurations are to shape

$$L_{\nabla B} = B\sqrt{\frac{2}{\nabla\mathbf{B}:\nabla\mathbf{B}}}$$

$$L_{\nabla\nabla B} = \sqrt{\frac{4B}{\sqrt{\sum_{i,j,k=1}^{3}\left(\nabla\nabla\mathbf{B}\right)_{i,j,k}^{2}}}}$$





Section 5.5
has very strong shaping

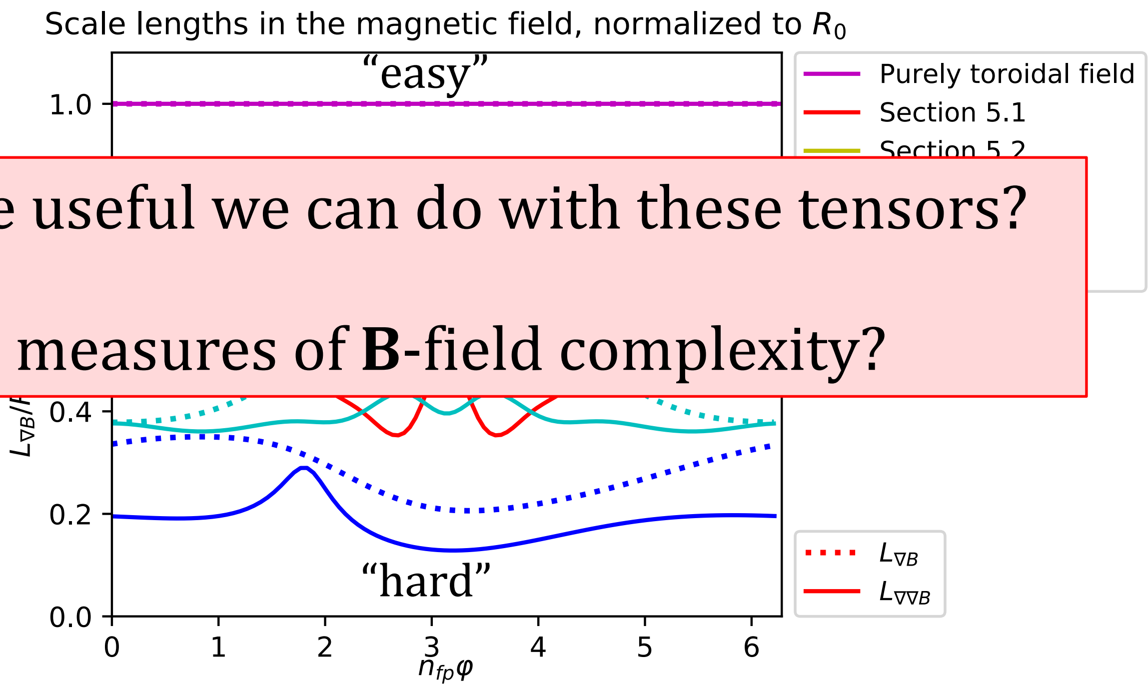
These tensor norms seem correlated with intuition for how hard these configurations are to shape

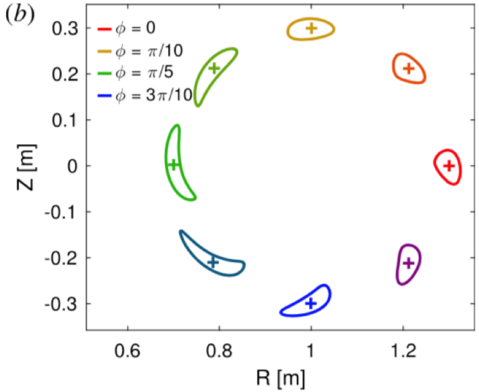
$$L_{\nabla B} = B\sqrt{\frac{2}{\nabla \mathbf{B} : \nabla \mathbf{B}}}$$

$L_{\nabla\nabla B} =$

Is there anything else useful we can do with these tensors?

Are there other good measures of **B**-field complexity?



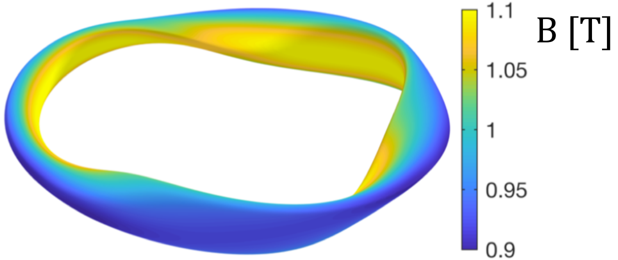


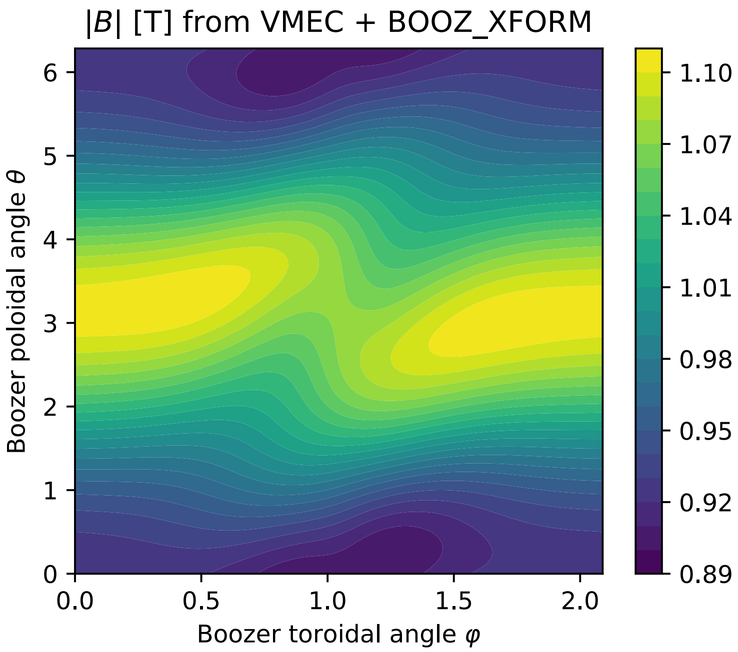
Section 5.5
has very strong shaping

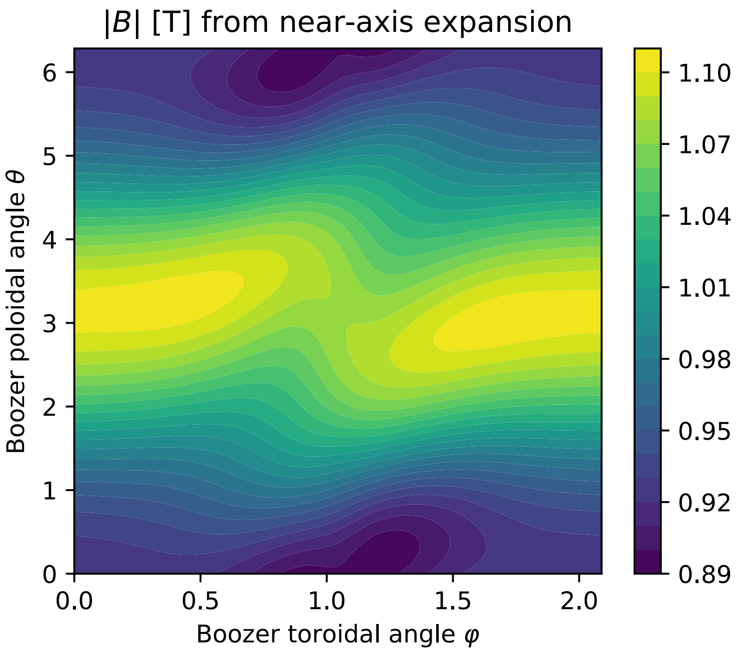
Outline

- Magnetic well
- Mercier stability criterion
- $\nabla \mathbf{B}$ and $\nabla\nabla \mathbf{B}$ tensors
- Departure from quasisymmetry
- Aspect ratio at which surfaces become singular.

If we strive for QS to $O(r^1)$, we can compute the symmetry-breaking error at $O(r^2)$.

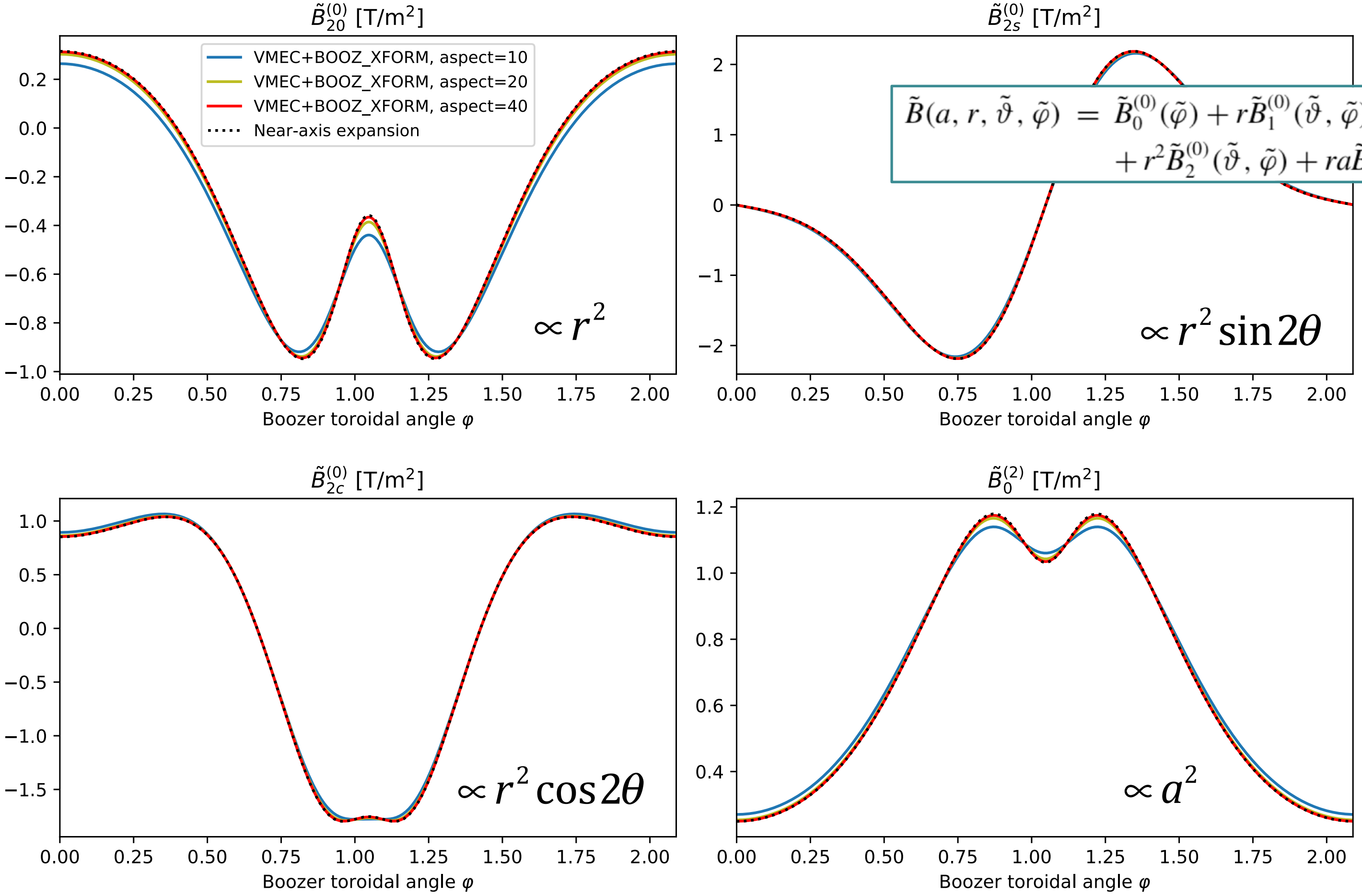




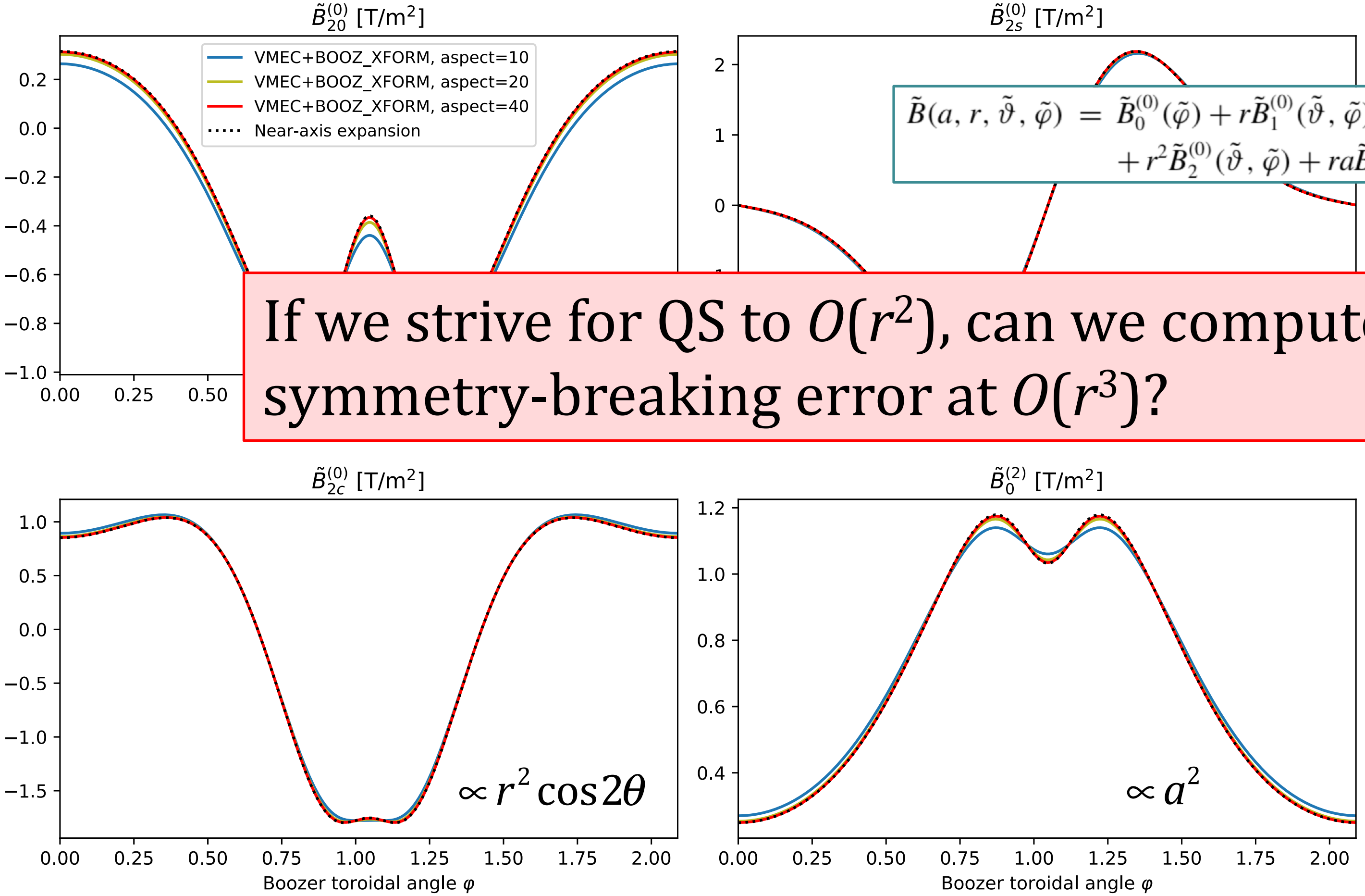


If we strive for QS to O(r^1), we can compute the symmetry-breaking error at O(r^2).

$$\tilde{B}(a, r, \tilde{\vartheta}, \tilde{\varphi}) = \tilde{B}_0^{(0)}(\tilde{\varphi}) + r\tilde{B}_1^{(0)}(\tilde{\vartheta}, \tilde{\varphi}) + a\tilde{B}_0^{(1)}(\tilde{\varphi}) + r^2\tilde{B}_2^{(0)}(\tilde{\vartheta}, \tilde{\varphi}) + ra\tilde{B}_1^{(1)}(\tilde{\vartheta}, \tilde{\varphi}) + a^2\tilde{B}_0^{(2)}(\tilde{\varphi}) + O((r/\mathcal{R})^3).$$



If we strive for QS to O(r¹), we can compute the symmetry-breaking error at O(r²).



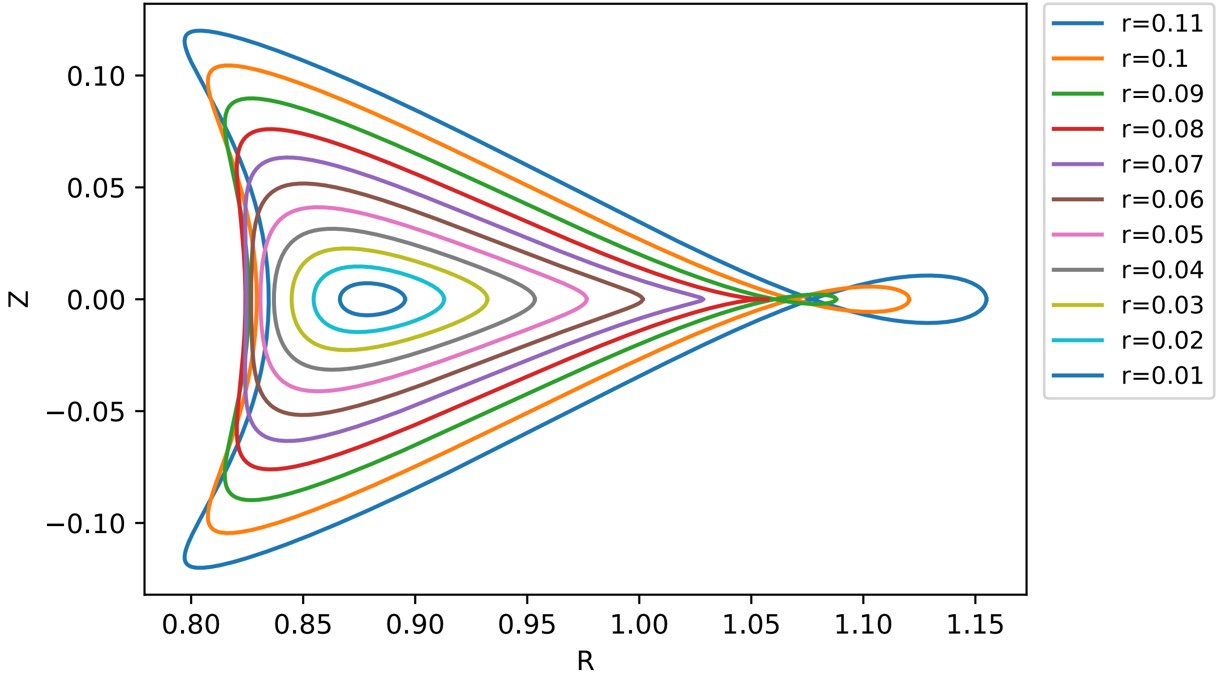
$$
\begin{aligned}
\tilde{B}(a, r, \tilde{\vartheta}, \tilde{\varphi}) \;=\; & \tilde{B}_0^{(0)}(\tilde{\varphi}) + r\tilde{B}_1^{(0)}(\tilde{\vartheta}, \tilde{\varphi}) + a\tilde{B}_0^{(1)}(\tilde{\varphi}) \\
& + r^2\tilde{B}_2^{(0)}(\tilde{\vartheta}, \tilde{\varphi}) + ra\tilde{B}_1^{(1)}(\tilde{\vartheta}, \tilde{\varphi}) + a^2\tilde{B}_0^{(2)}(\tilde{\varphi}) + O((r/\mathcal{R})^3).
\end{aligned}
$$

If we strive for QS to $O(r^2)$, can we compute the symmetry-breaking error at $O(r^3)$?

Outline

- Magnetic well
- Mercier stability criterion
- $\nabla\mathbf{B}$ and $\nabla\nabla\mathbf{B}$ tensors
- Departure from quasisymmetry
- Aspect ratio at which surfaces become singular.

How can we compute the aspect ratio at which surfaces are no longer smooth & nested?



$$\sqrt{g} = \frac{\partial \mathbf{x}}{\partial r} \cdot \frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \varphi} = 0$$

Minimize r subject to $\sqrt{g} = 0$.

$$L = r + \lambda \sqrt{g}$$

$$\frac{\partial L}{\partial \lambda} = 0 \;\Rightarrow\; \sqrt{g} = 0$$

$$\frac{\partial L}{\partial \theta} = 0 \;\Rightarrow\; \frac{\partial \sqrt{g}}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \varphi} = 0 \;\Rightarrow\; \frac{\partial \sqrt{g}}{\partial \varphi} = 0$$

Uninteresting I think: $\frac{\partial L}{\partial r} = 0 \;\Rightarrow\; 1 + \lambda \frac{\partial \sqrt{g}}{\partial r} = 0$

How can we compute the aspect ratio at which surfaces are no longer smooth & nested?

$$\sqrt{g}=0 \qquad \frac{\partial\sqrt{g}}{\partial\theta}=0 \qquad \frac{\partial\sqrt{g}}{\partial\varphi}=0 \qquad \sqrt{g}=\frac{\partial\mathbf{x}}{\partial r}\cdot\frac{\partial\mathbf{x}}{\partial\theta}\times\frac{\partial\mathbf{x}}{\partial\varphi}$$

$$\mathbf{x}(r,\theta,\varphi)=\mathbf{x}_0(\varphi)+X(r,\theta,\varphi)\mathbf{n}(\varphi)+Y(r,\theta,\varphi)\mathbf{b}(\varphi)+Z(r,\theta,\varphi)\mathbf{t}(\varphi)$$

$$X=r\left[X_{1s}(\varphi)\sin\theta+X_{1c}(\varphi)\cos\theta\right]+r^2\left[X_{20}(\varphi)+X_{2s}(\varphi)\sin 2\theta+X_{2c}(\varphi)\cos 2\theta\right]$$

$$\sqrt{g}=r\left[g_0(\varphi)+rg_1(\theta,\varphi)+r^2g_2(\theta,\varphi)+r^3g_3(\theta,\varphi)+r^4g_4(\theta,\varphi)\right]$$

$$g_1(\theta,\varphi)=g_{1s}(\varphi)\sin\theta+g_{1c}(\varphi)\cos\theta$$

$$g_2(\theta,\varphi)=g_{20}(\varphi)+g_{2s}(\varphi)\sin 2\theta+g_{2c}(\varphi)\cos 2\theta$$

$$g_3(\theta,\varphi)=g_{3s1}(\varphi)\sin\theta+g_{3s3}(\varphi)\sin 3\theta+g_{3c1}(\varphi)\cos\theta+g_{3c3}(\varphi)\cos 3\theta$$

How can we compute the aspect ratio at which surfaces are no longer smooth & nested?

$$\sqrt{g}=0 \qquad \frac{\partial\sqrt{g}}{\partial\theta}=0 \qquad \frac{\partial\sqrt{g}}{\partial\varphi}=0$$ 🡐Can replace last equation with min over *φ*.

- Could solve with Newton method, but need good initial guess or else not robust.
- Worried most about small-*r* solutions, so may be reasonable to set $g_3=g_4=0$.
- Then system has analytic solution. Can use as initial guess for Newton with g_3 & g_4.

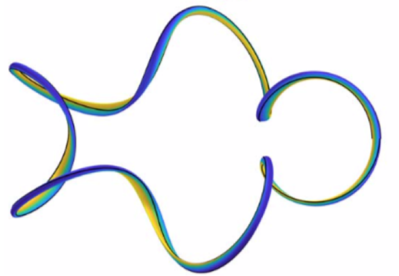
$$\sqrt{g}=r\left[g_0(\varphi)+rg_1(\theta,\varphi)+r^2g_2(\theta,\varphi)+r^3g_3(\theta,\varphi)+r^4g_4(\theta,\varphi)\right]$$

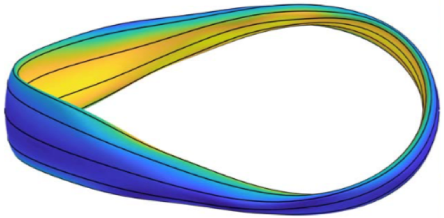
$$g_1(\theta,\varphi)=g_{1s}(\varphi)\sin\theta+g_{1c}(\varphi)\cos\theta$$

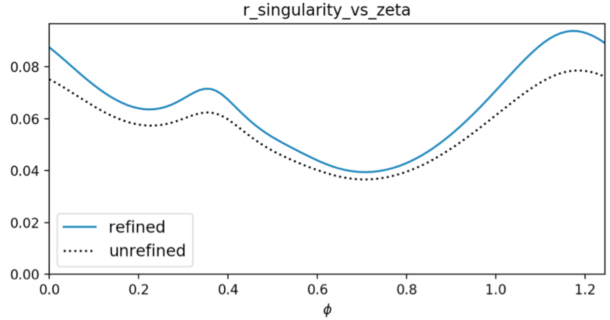
$$g_2(\theta,\varphi)=g_{20}(\varphi)+g_{2s}(\varphi)\sin 2\theta+g_{2c}(\varphi)\cos 2\theta$$

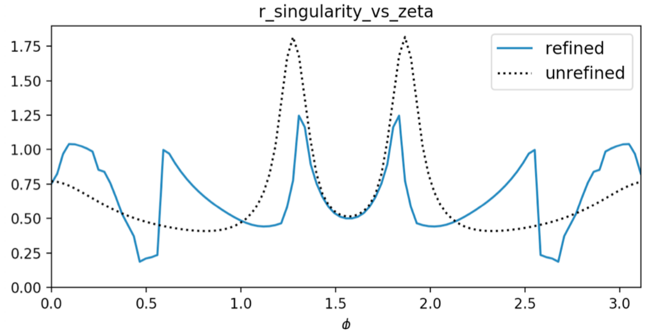
$$g_3(\theta,\varphi)=g_{3s1}(\varphi)\sin\theta+g_{3s3}(\varphi)\sin 3\theta+g_{3c1}(\varphi)\cos\theta+g_{3c3}(\varphi)\cos 3\theta$$

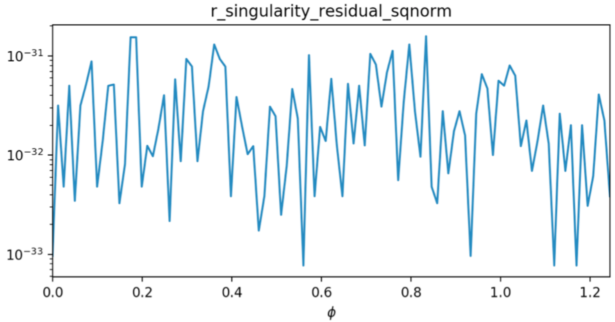
This approach of generating initial guesses for Newton iteration works sometimes but not always

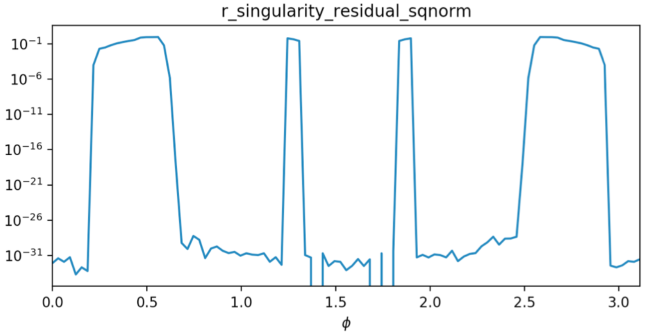




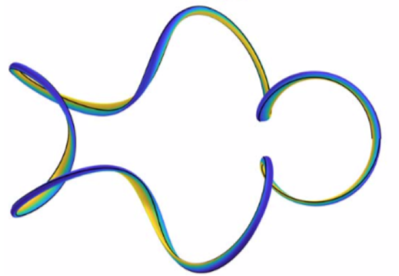


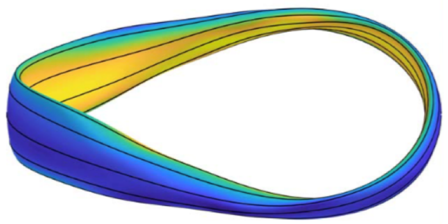


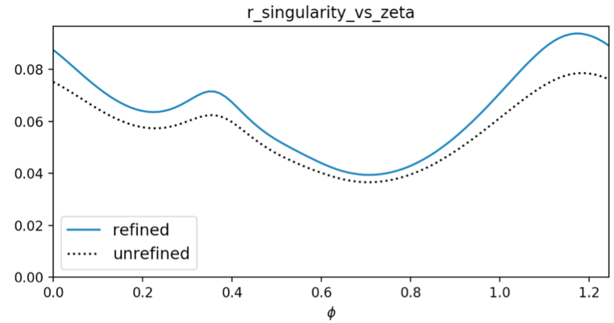


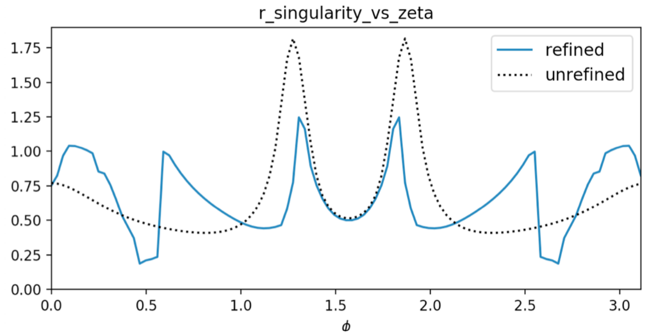


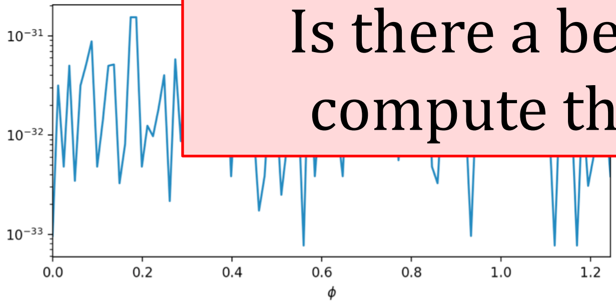
This approach of generating initial guesses for Newton iteration works sometimes but not always

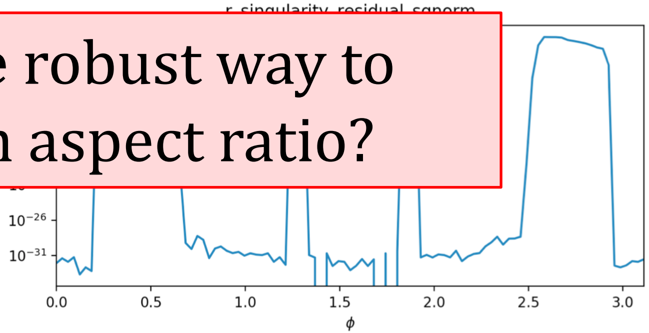












Is there a better / more robust way to compute the minimum aspect ratio?

Closing questions

- Is there a slicker way to get a parity-transformation-invariant form of Mercier's criterion?
- Is there anything else useful we can do with these $\nabla\mathbf{B}$ and $\nabla\nabla\mathbf{B}$ tensors?
- Are there other measures of **B** field complexity / coil difficulty?
- If we strive for QS to $O(r^2)$, can we compute the symmetry-breaking error at $O(r^3)$? (So much algebra!!)
- Is there a better / more robust way to compute the minimum aspect ratio?
- What else can we compute in < a few ms?