A new & faster method to generate transport-optimized stellarators

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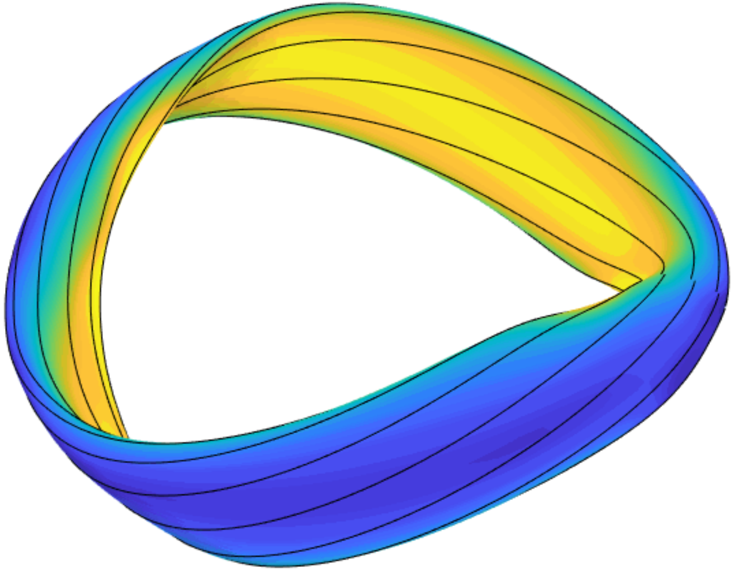
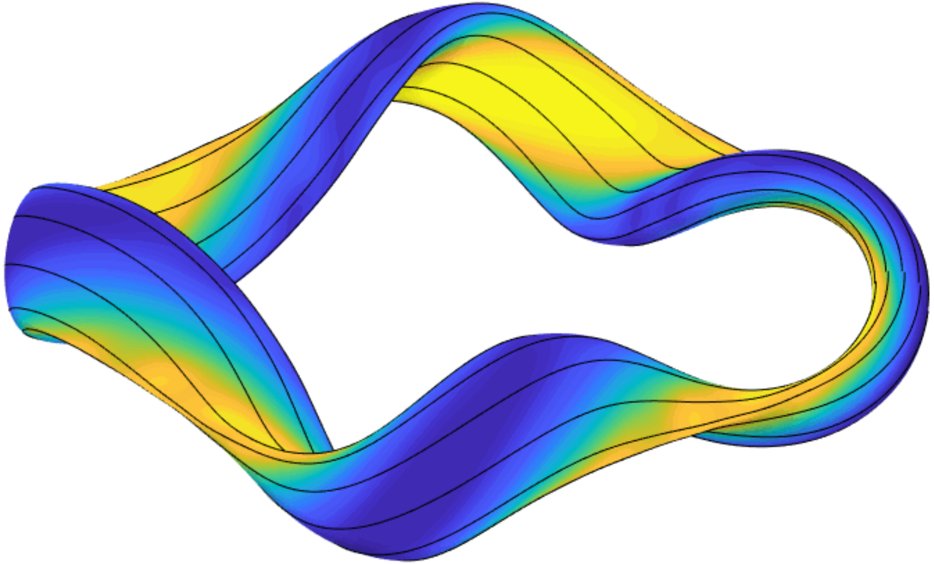
Wrick Sengupta[2], Georg Stadler[2], 1. *University of Maryland* 2. *NYU* 3. *IPP*





J Plasma Phys 84, 905840616 (2018) J Plasma Phys 85, 905850103 (2019) PPCF 61, 075001 (2019)
arXiv:1909.08919 arXiv:1908.10253 github.com/landreman/quasisymmetry

The conventional approach to finding quasisymmetric fields works but has shortcomings

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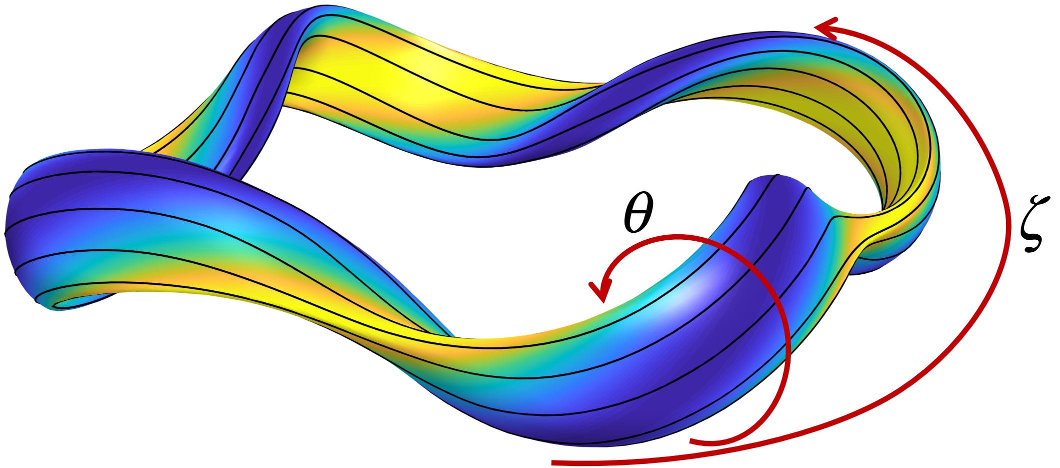
$$B = B\left(r,\ \theta - N\zeta\right)$$



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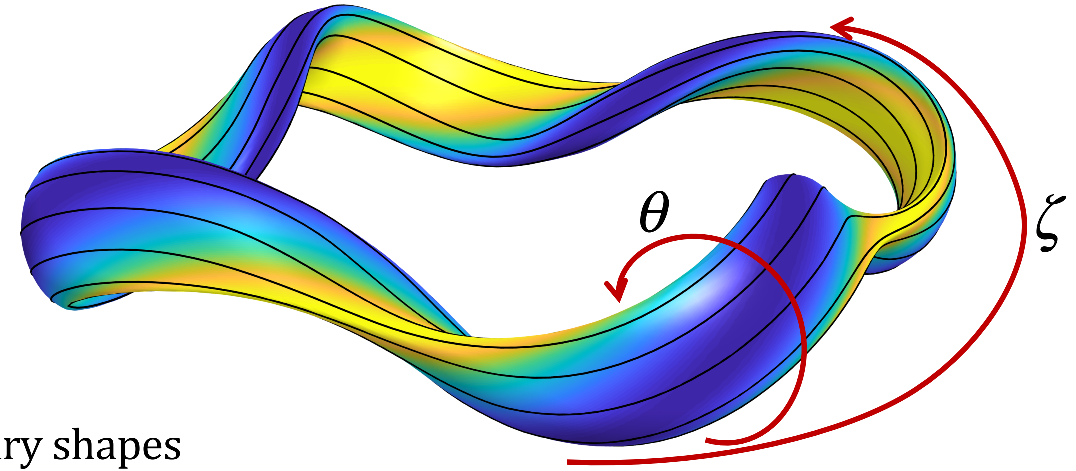

$$\min_X f(X)$$

Parameter space: $X =$ toroidal boundary shapes

Objective: $f = \sum_{m,n \neq Nm} B_{m,n}^2(r_0)$ where $B(r,\theta,\zeta) = \sum_{m,n} B_{m,n}(r)\exp(im\theta - in\zeta)$

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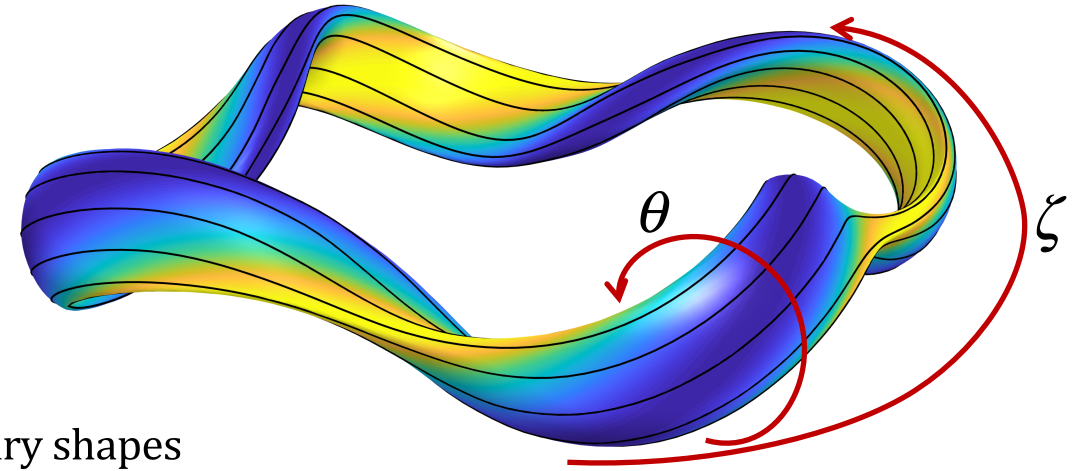

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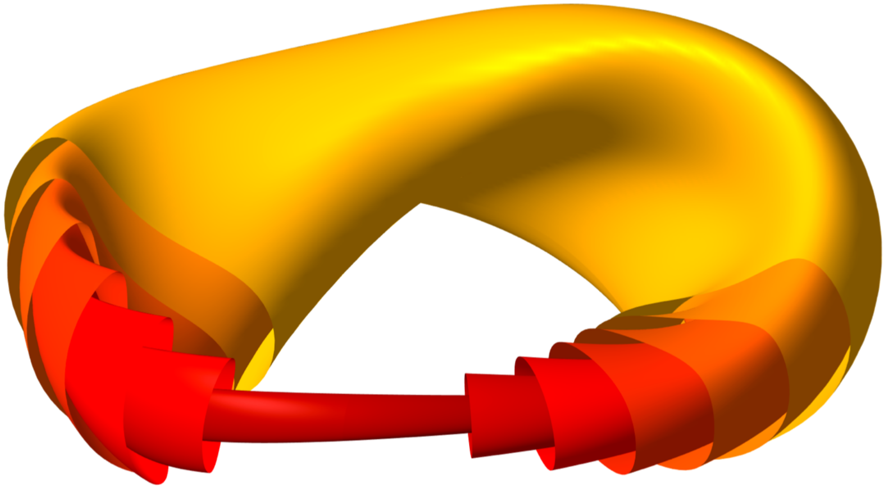
- Computationally expensive.
- What is the size & character of the solution space?
- Result depends on initial condition, so cannot be sure you've found all solutions.

Expansion about the magnetic axis can be a powerful practical tool for generating quasisymmetric & omnigenous stellarators



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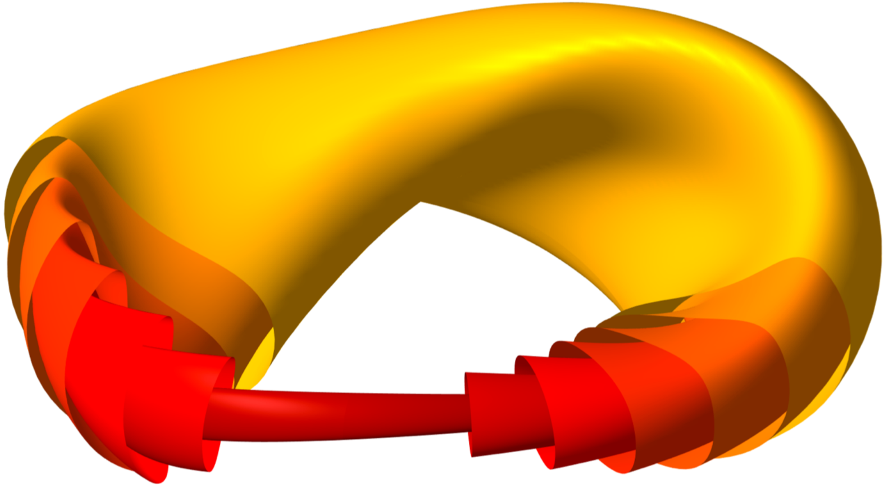

- Accurate at least in the core of *any* configuration.
- Hasn't been considered much since numerical optimization began in ~1980s.
- Complements the traditional optimization approach:
  - Many orders of magnitude faster.
  - You can parameterize all possible solutions.
  - Can generate new initial conditions that can be refined by optimization.

We have translated analytic work by Garren & Boozer (1991) into practical algorithms

<u>Fundamental equations:</u>

$$\mathbf{x}(r,\theta,\zeta)=\mathbf{x}_0(\zeta)+\text{Taylor series in } r=\sqrt{2\psi/B_0},\qquad (\nabla\times\mathbf{B})\times\mathbf{B}=\mu_0\nabla p,$$

$$\mathbf{B}=\nabla\psi\times\nabla\theta+\iota\nabla\zeta\times\nabla\psi=\beta\nabla\psi+I(\psi)\nabla\theta+G(\psi)\nabla\zeta,\quad B(r,\theta,\zeta)=B_0\left[1+r\bar{\eta}\cos(\theta-N\zeta)+O(r^2)\right]$$

Mercier's inverse expansion of $\psi(\mathbf{x})$ gives equivalent results.

Rogerio Jorge & Wrick Sengupta.

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Algorithm Inputs:

ML, Sengupta, & Plunk, *J Plasma Phys* (2019)

- Shape of the magnetic axis, with $\kappa \neq 0$. (Determines QA vs QH.)
- 3 numbers:
  - I_2: Current density on the axis. (Usually 0).
  - Rotation of the elliptical flux surfaces at ζ=0. (Usually 0).
  - $\bar{\eta}$, which controls elongation and field strength.

Theorem: Given this data, a unique $O(r)$ quasisymmetric solution exists.

$\Longrightarrow$ The space of configurations that are quasisymmetric to $O(r)$ is precisely understood.

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Outputs:

- Shape of the surfaces around the axis. (Elongation & rotation of ellipses.)
- Rotational transform on axis.

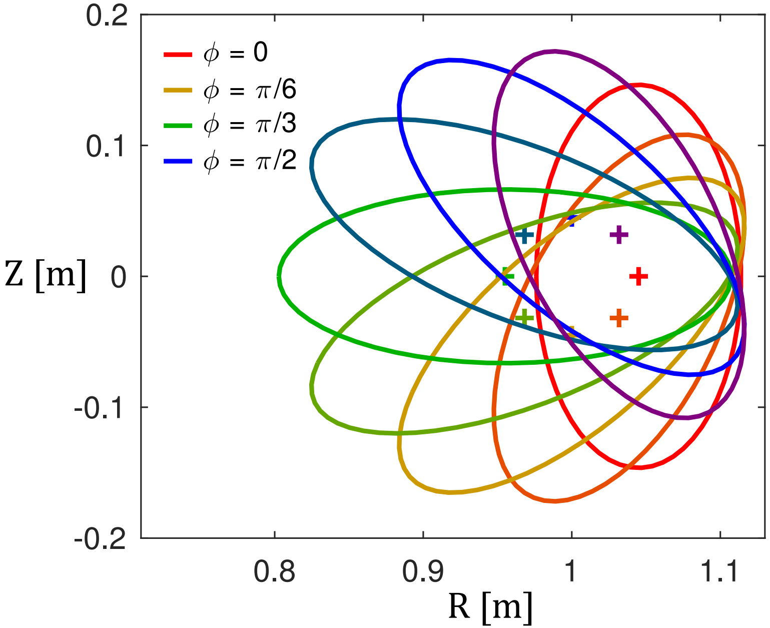
Example $O(r)$ construction: quasi-axisymmetry

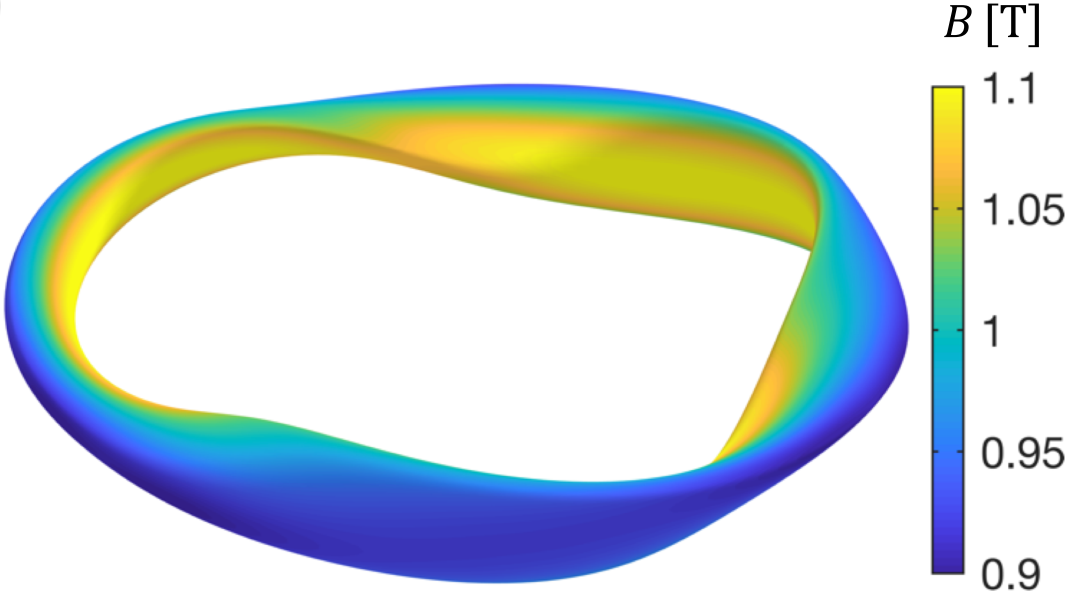
Inputs: axis shape $R_0(\phi) = 1 + 0.045\cos(3\phi)\ \left[\mathrm{m}\right]$, $I_2 = 0$, $\bar{\eta} = -0.9$.

$$Z_0(\phi) = -0.045\sin(3\phi)\ \left[\mathrm{m}\right],\qquad \sigma(0) = 0,$$

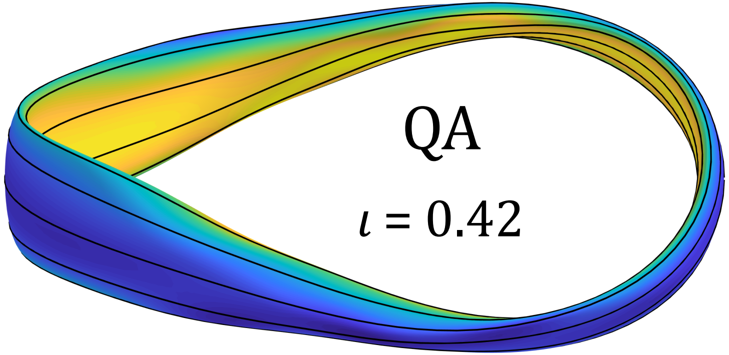
Plug in r = 0.1 m.

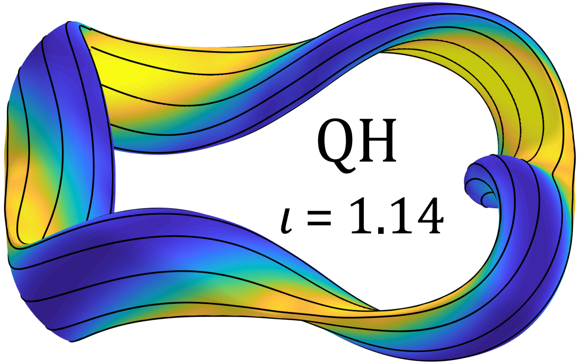
Results:

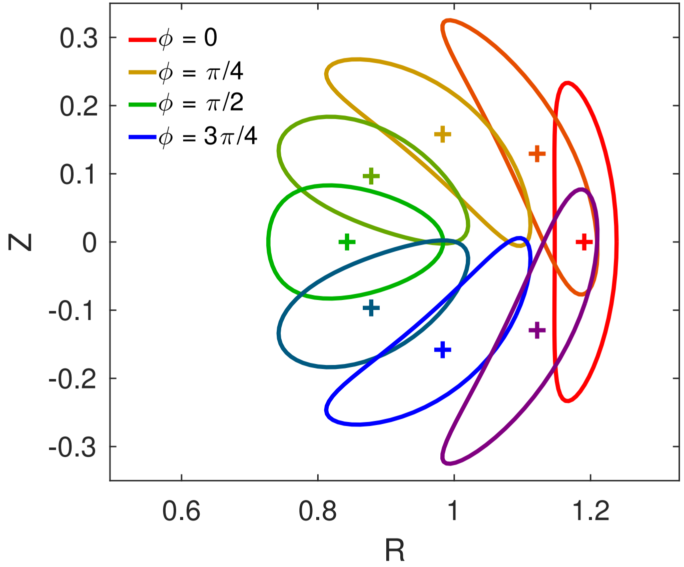


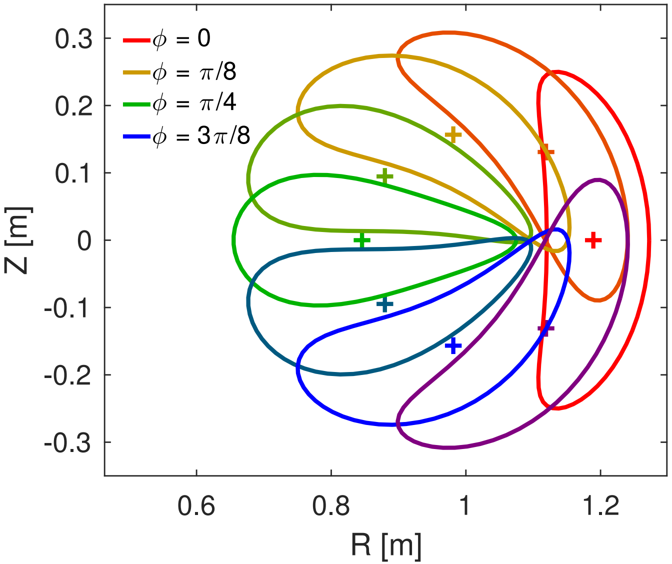


Extending the construction to O(r^2), you get triangularity and better quasisymmetry

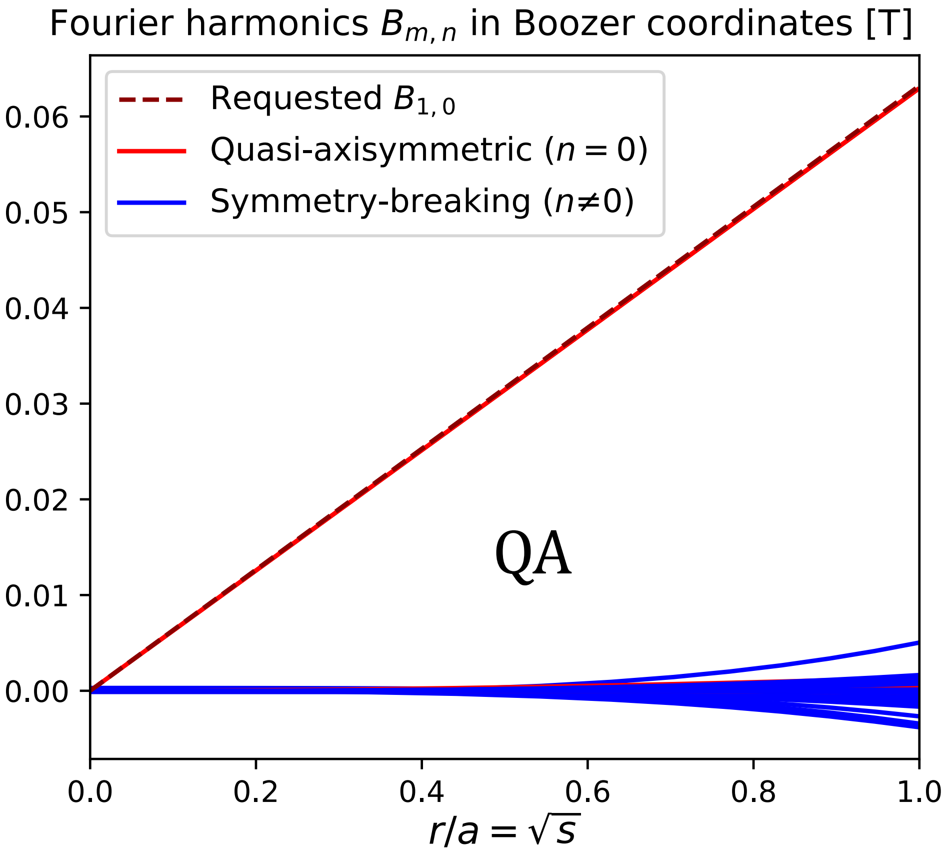


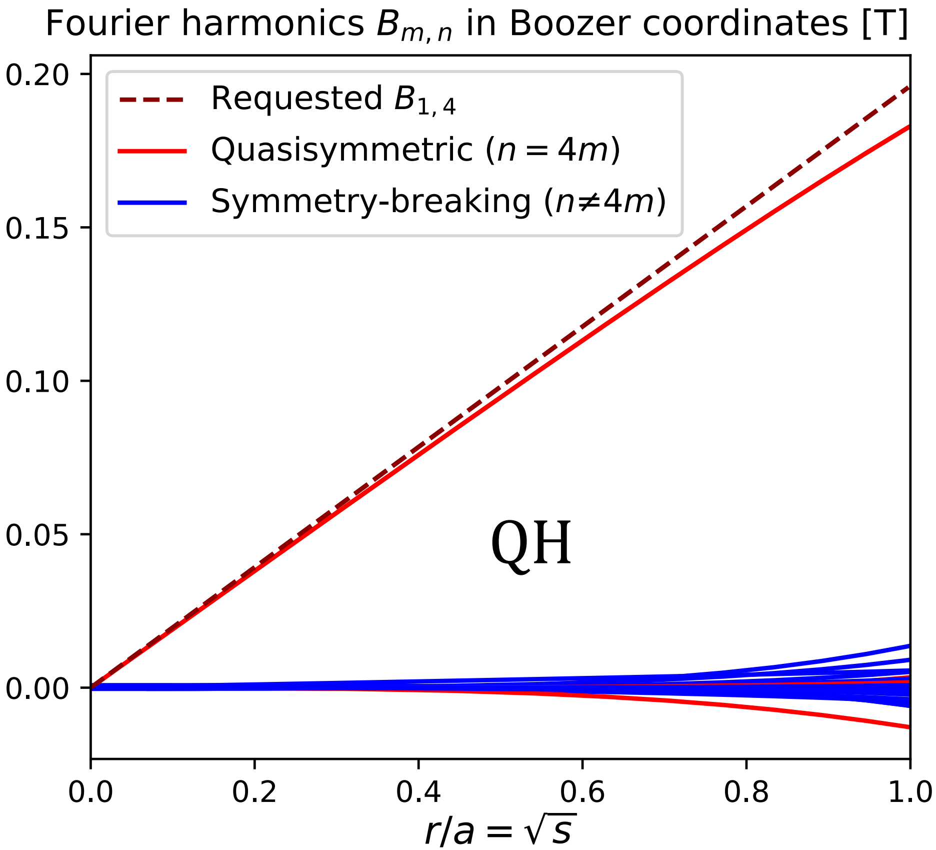






The construction can be verified by running an MHD equilibrium code (VMEC) which does not make the expansion.



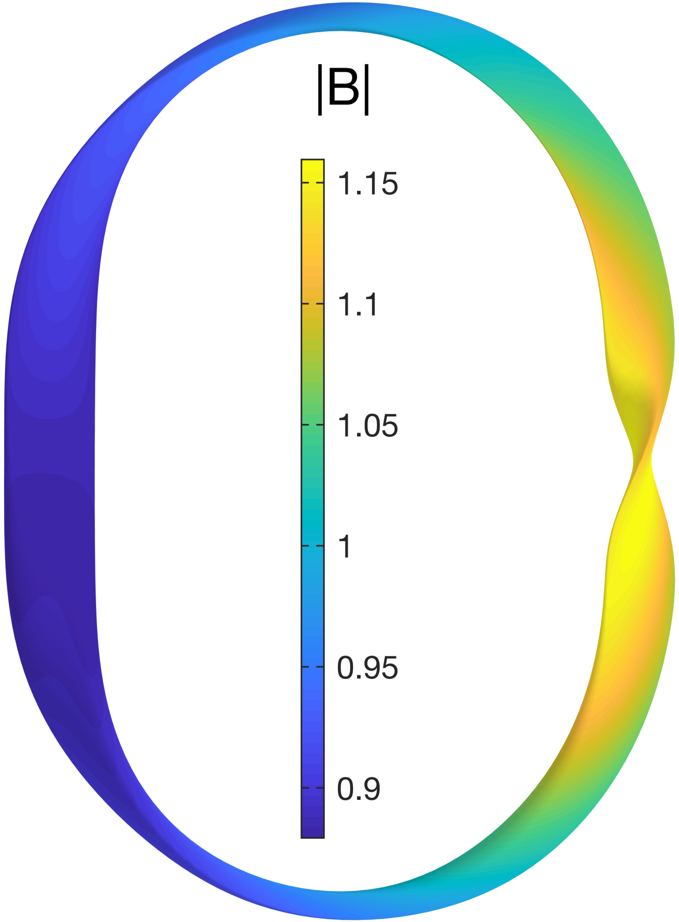


The near-axis analysis can be generalized to construct omnigenous configurations

G G Plunk, ML, and P Helander, arXiv:1909.08919, Accepted in J Plasma Phys

Omnigenity: $\oint \left(\mathbf{v}_d \cdot \nabla \psi \right) dt = 0$

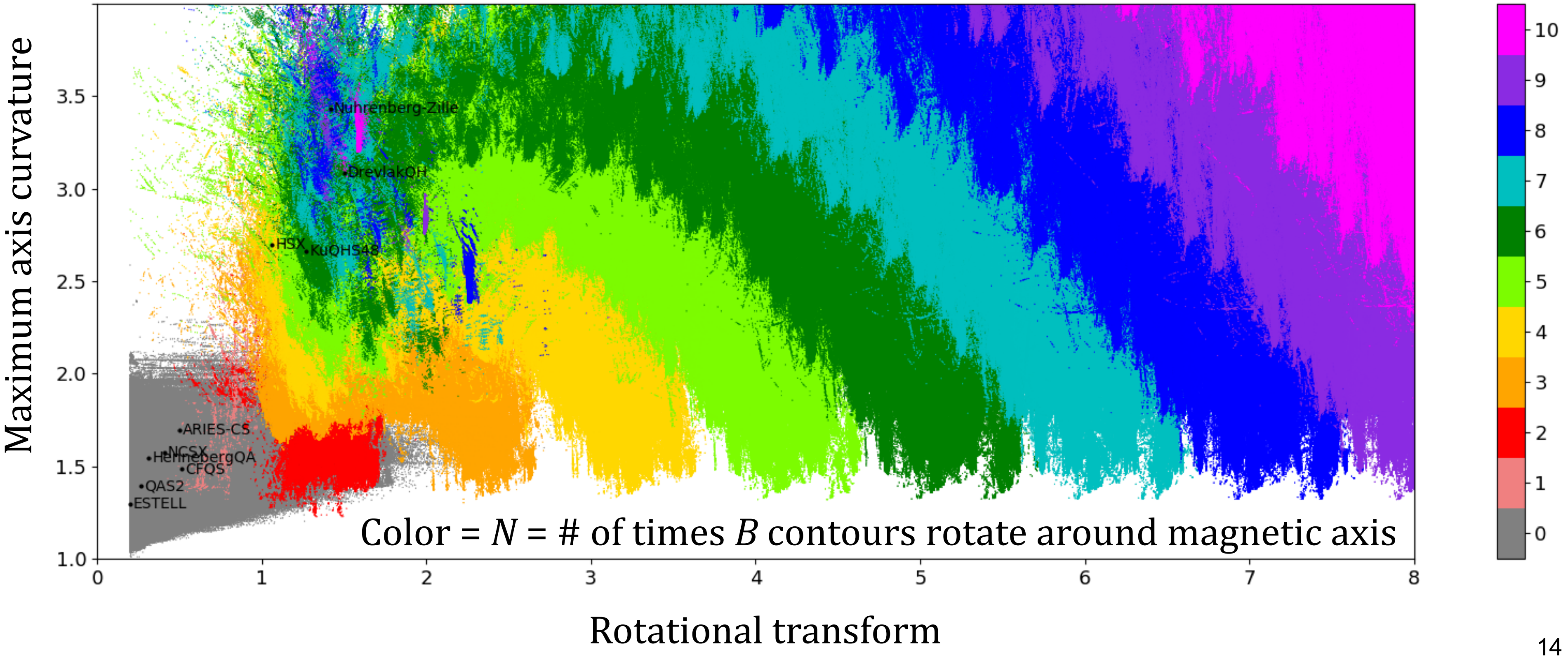
$\forall$ magnetic moments & energies.



The fast construction enables brute-force surveys of "all" quasisymmetric fields

Axis shape: $R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(j n_{fp} \phi), \quad Z_0(\phi) = \sum_{j=1}^{3} Z_j \sin(j n_{fp} \phi)$

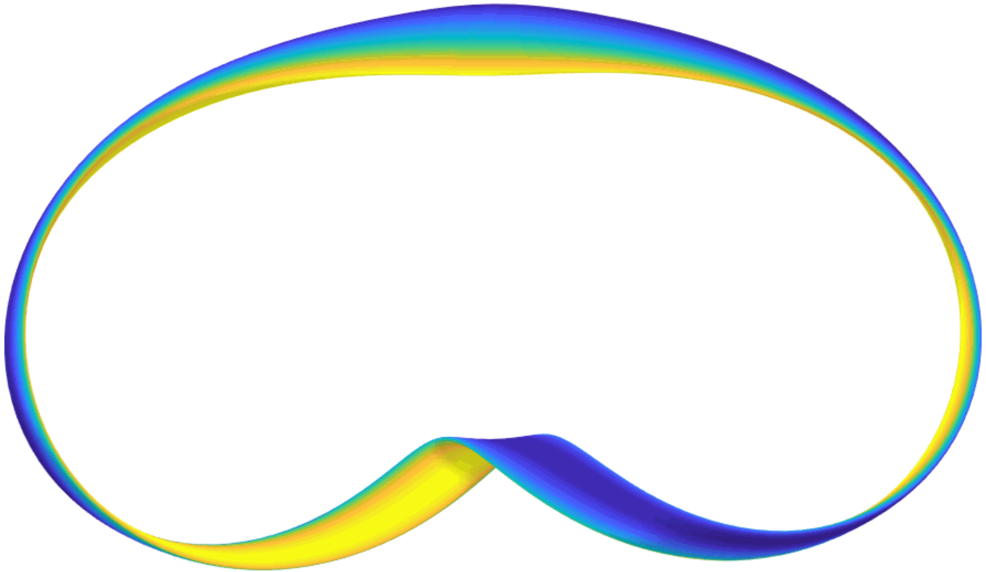
$2.4\text{x}10^8$ configurations



Brute-force searching is already yielding some new configurations

Quasi-helical symmetry with

1 field period



2 field periods

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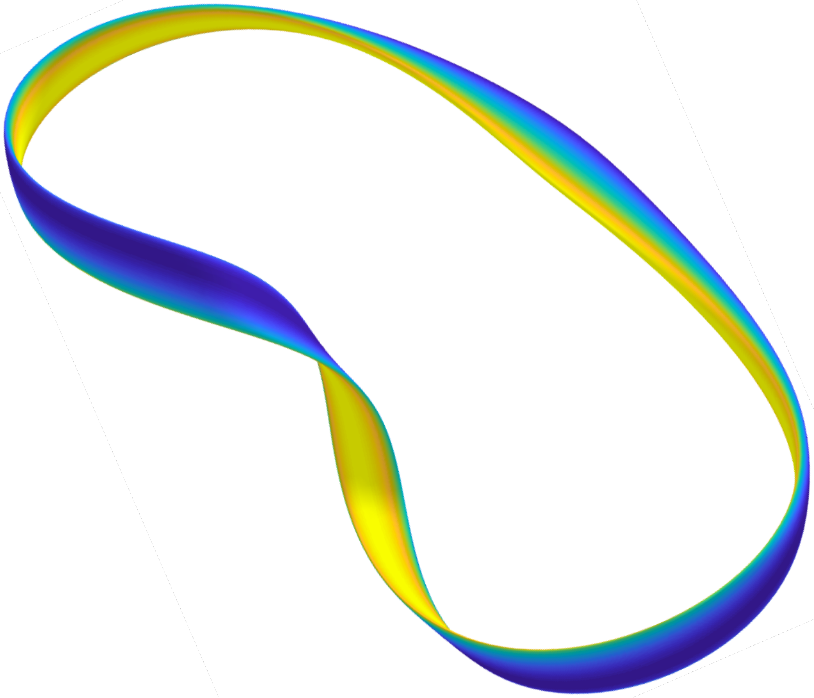

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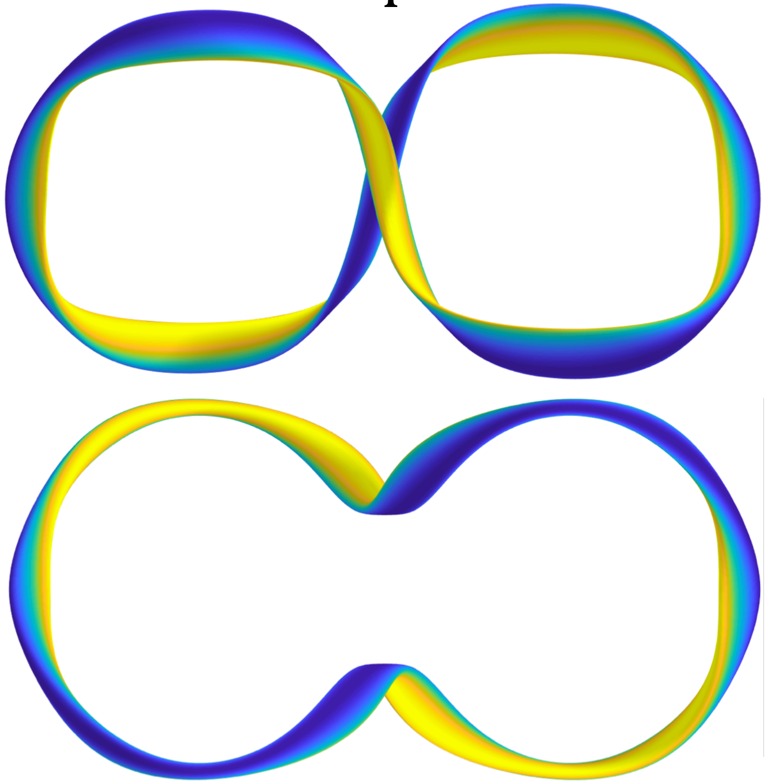
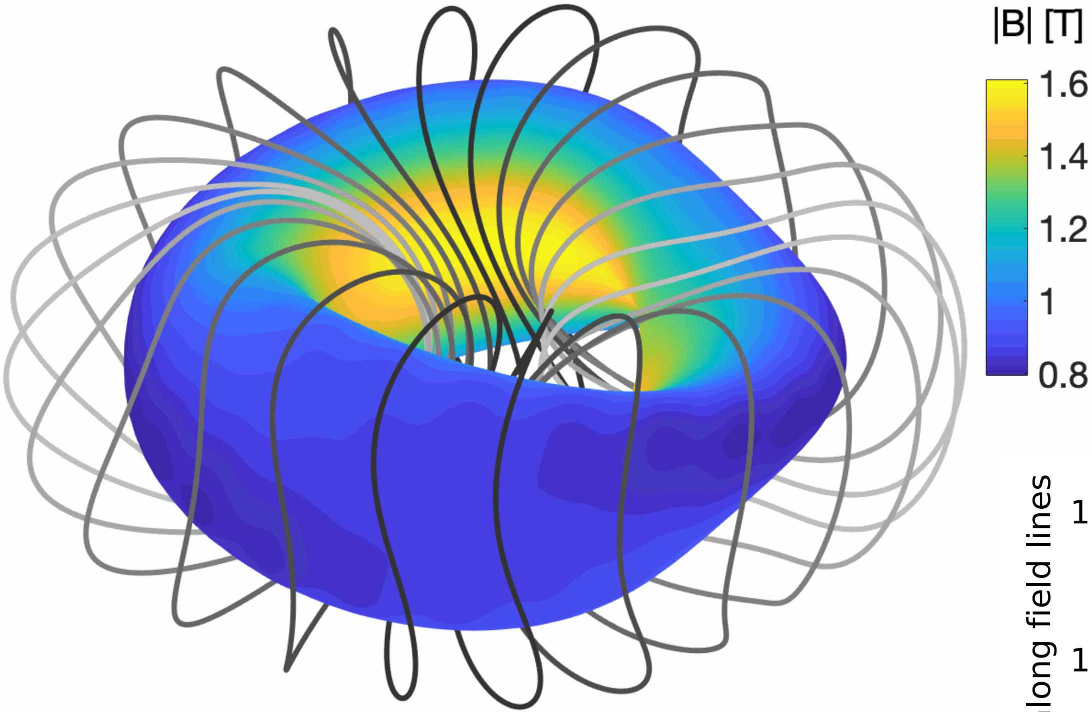


The axis expansion enables a combined (1-stage) coil + quasisymmetry optimization using analytic derivatives

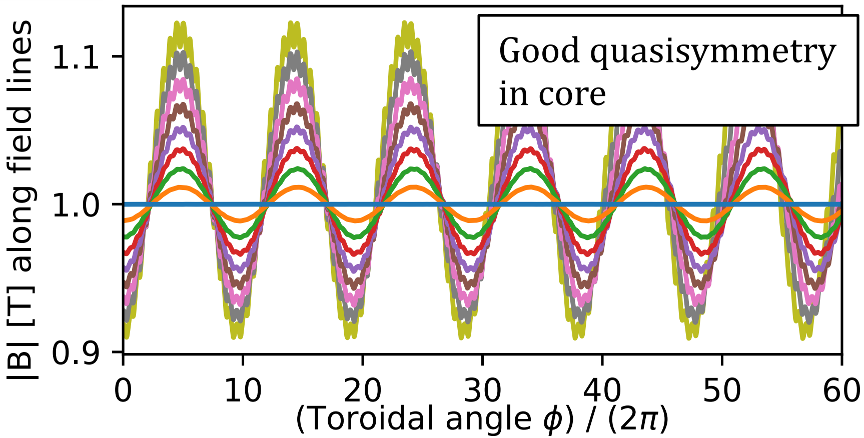


With Andrew Giuliani, Georg Stadler, Antoine Cerfon (NYU)

$$\min_X f(X)$$

$$X = \{\text{Coil shapes, axis shape, } \bar{\eta}\}$$

$$f = \oint_{\text{axis}} d\ell \left| \mathbf{B}_{\substack{\text{Biot-}\\\text{Savart}}} - \mathbf{B}_{\substack{\text{Near-axis}\\\text{quasisymmetry}}} \right|^2 + \left(\frac{\iota_{\text{axis}} - \iota_{\text{axis}}^{\text{target}}}{\iota_{\text{axis}}^{\text{target}}} \right)^2 + \left(\begin{array}{c}\text{other}\\\text{terms}\end{array} \right)$$



Conclusions

- The near-axis expansion enables transport-optimized stellarator configurations to be generated orders of magnitude faster than before.
- We now precisely understand the space of quasisymmetric fields to $O(r)$.
- There is hope of definitively identifying all regions of parameter space with practical quasisymmetric & omnigenous fields (near the axis).
- Much more can be done, e.g. gyrokinetic & MHD analysis near axis.

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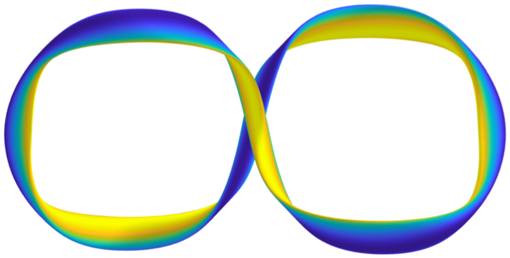
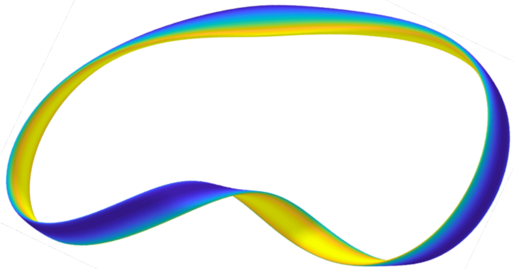
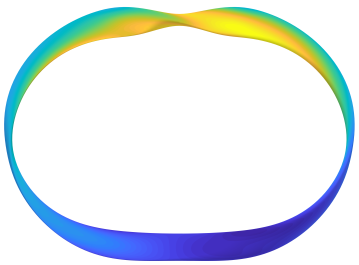
- We might discover qualitatively new magnetic configurations for fusion?







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Extra slides

- Advantages of stellarators: steady-state, no disruptions, no power recirculated for current drive, no Greenwald limit, don’t rely on plasma for confinement.

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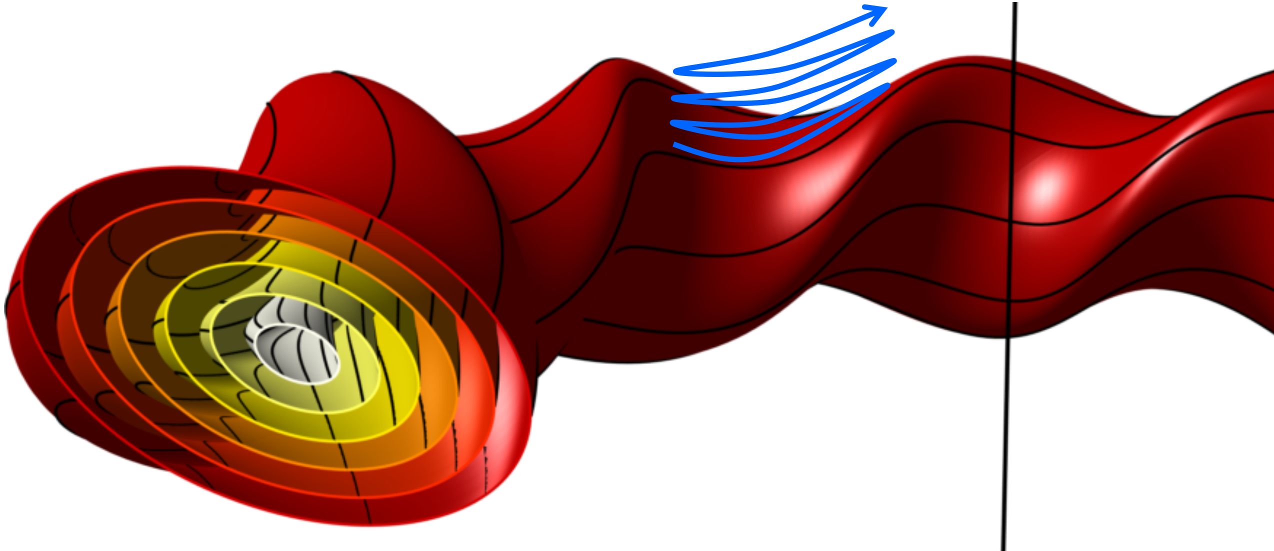
- But, alpha losses & neoclassical transport would be too large unless you carefully choose the geometry.

$$\oint \left(\mathbf{v}_d \cdot \nabla r \right) dt = 0 \text{ in axisymmetry,} \quad \neq 0 \text{ in a general stellarator.}$$



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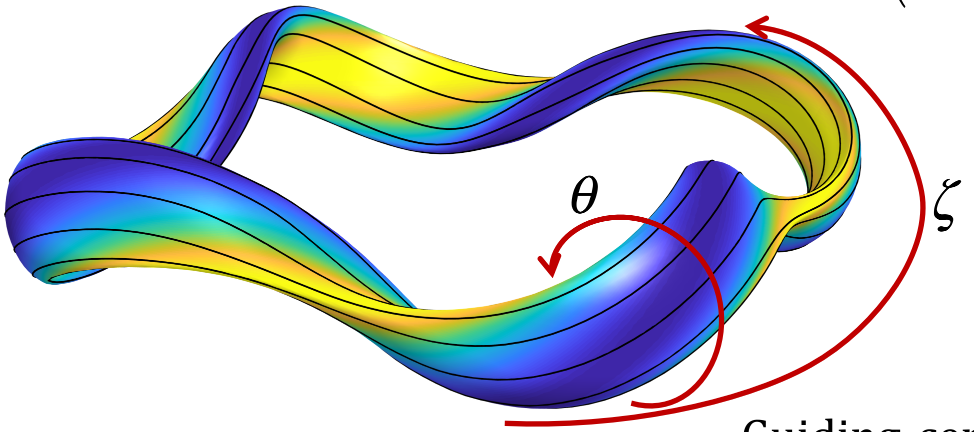
Boozer angles



Guiding-center Lagrangian in Boozer coordinates depends on (θ,ζ) only through B=$|\mathbf{B}|$.

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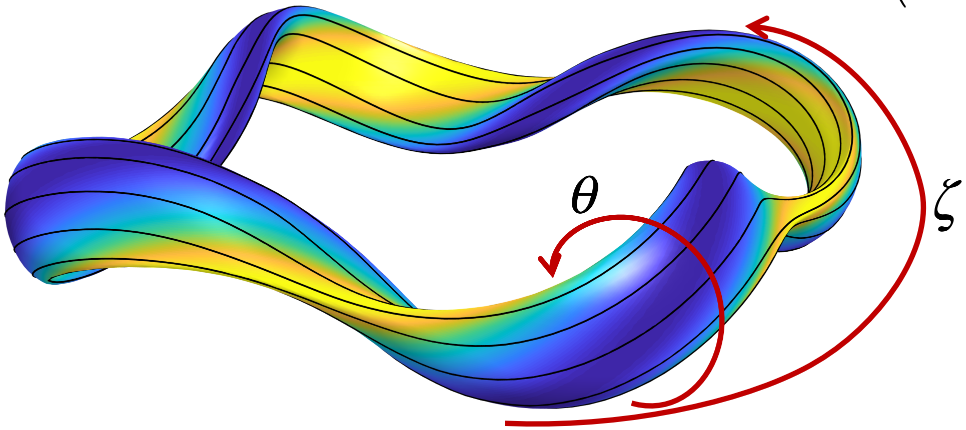

(Or, weaker conditions like omnigenity)

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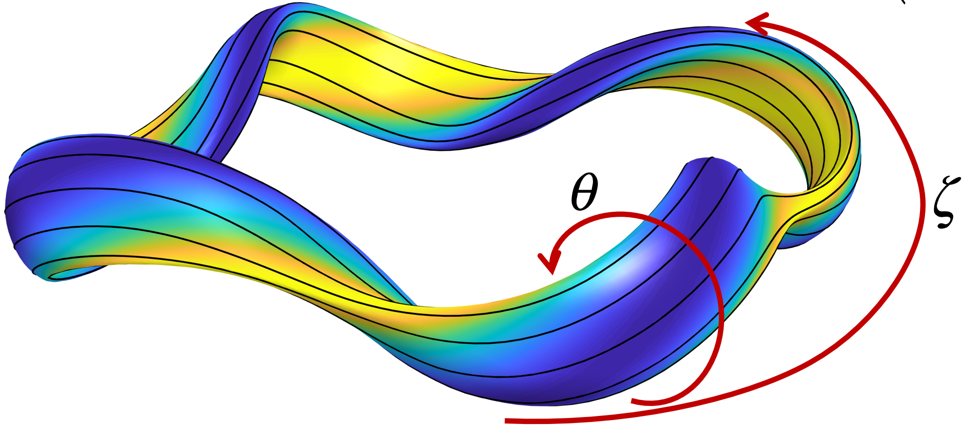

(Or, weaker conditions like omnigenity)

- How do you find MHD equilibria with these properties?

Outline

- Theory for $O(r)$ quasisymmetry
- Comparison to optimized configurations
- The landscape of solutions
- $O(r)$ omnigenity
- $O(r^2)$ quasisymmetry

Future work

- Higher order: Calculate B_3 so symmetry-breaking can be minimized.
- Examine MHD & gyrokinetic stability using the expansion.
- Can anything be proved about the number or character of $O(r^2)$ solutions?
- Is there an analogous construction to give quasisymmetry at an off-axis surface?
- Check coil feasibility for newly discovered configurations.

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The near-axis analysis can be generalized to construct omnigenous configurations

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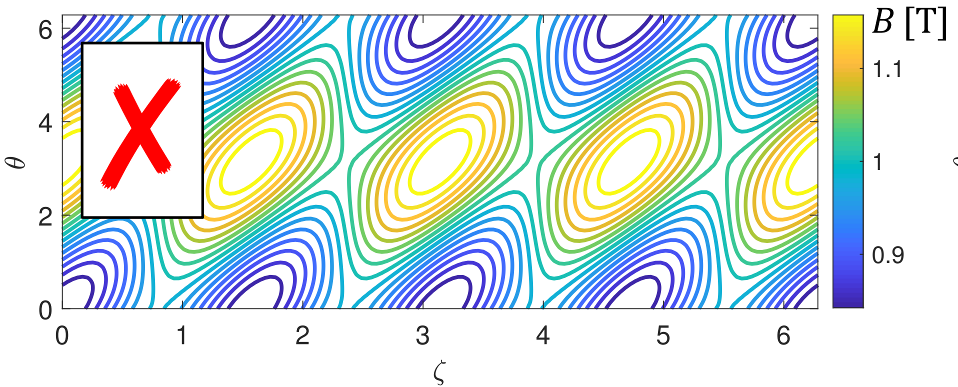
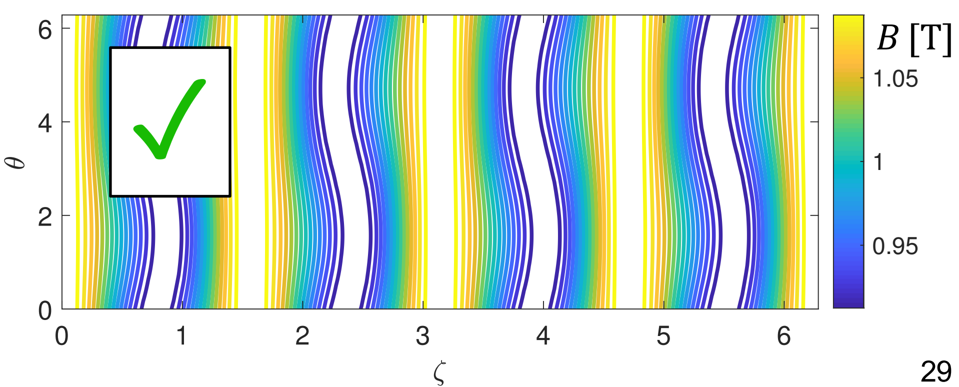
$$\text{Omnigenity:} \quad \oint_{\text{bounce}} \left(\mathbf{v}_d \cdot \nabla \psi \right) dt = 0 \quad \forall \text{ magnetic moments \& energies.}$$

Implications for $B(r,\theta,\zeta)$:
[Cary & Shasharina (1997)]

- All B contours close toroidally, helically, or poloidally.
- Distance along **B** between bounce points is the same for every field line on a flux surface.





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Relates $B\left(\psi,\theta,\zeta\right)$ to $\mathbf{x}_0$, X, Y, Z near axis for *any* equilibrium, not just quasisymmetry.

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(Cary & Shasharina 1997) + (Garren & Boozer 1991) = (Plunk et al 2019)

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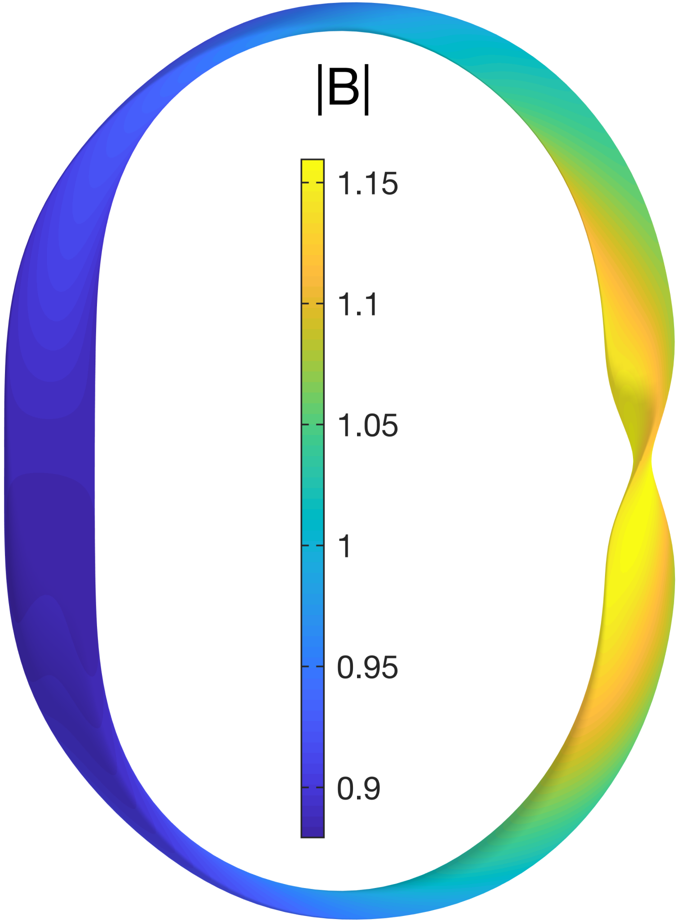
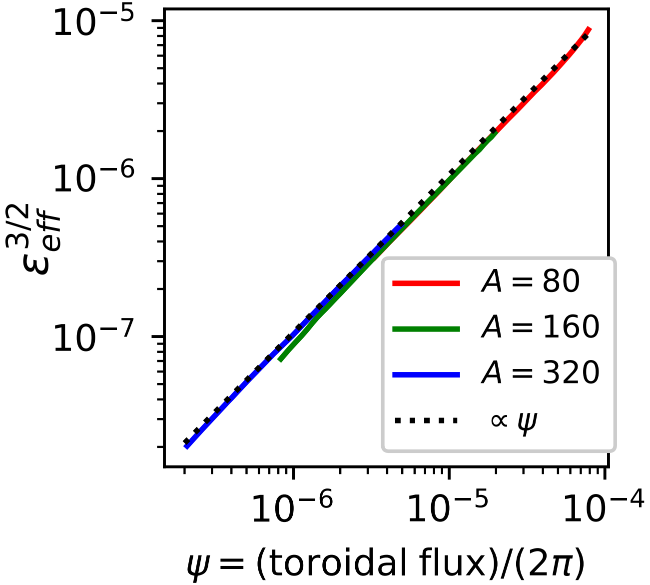


$1/\nu$ transport computed by VMEC + BOOZ_XFORM + NEO codes scales as expected



The $O(r^2)$ construction allows triangularity, Shafranov shift, and more accurate quasisymmetry.

$$\mathbf{x}(r,\vartheta,\zeta) = \mathbf{x}_0(\zeta) + X(r,\vartheta,\zeta)\mathbf{n}(\zeta) + Y(r,\vartheta,\zeta)\mathbf{b}(\zeta) + Z(r,\vartheta,\zeta)\mathbf{t}(\zeta)$$

$$X(r,\vartheta,\zeta) = r\left[X_{1c}\cos\vartheta + X_{1s}\sin\vartheta\right] + r^2\left[X_{20} + X_{2c}\cos 2\vartheta + X_{2s}\sin 2\vartheta\right] + O(r^3)$$

Same for Y & Z.

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$$X(r,\vartheta,\zeta)=r\left[X_{1c}\cos\vartheta+X_{1s}\sin\vartheta\right]+r^2\left[X_{20}+X_{2c}\cos 2\vartheta+X_{2s}\sin 2\vartheta\right]+O(r^3)$$

Same for Y & Z.

- 3 new input parameters: p_2, B_{2c}, B_{2s}.

$$p(r)=p_0+r^2 p_2+O(r^4)$$

$$B(r,\vartheta,\varphi)=B_0+rB_0\bar{\eta}\cos\vartheta+r^2\left[B_{20}+B_{2c}\cos 2\vartheta+B_{2s}\sin 2\vartheta\right]+O(r^3)$$

The $O(r^2)$ construction allows triangularity, Shafranov shift, and more accurate quasisymmetry.

$$\mathbf{x}(r,\vartheta,\zeta)=\mathbf{x}_0(\zeta)+X(r,\vartheta,\zeta)\mathbf{n}(\zeta)+Y(r,\vartheta,\zeta)\mathbf{b}(\zeta)+Z(r,\vartheta,\zeta)\mathbf{t}(\zeta)$$

$$X(r,\vartheta,\zeta)=r\left[X_{1c}\cos\vartheta+X_{1s}\sin\vartheta\right]+r^2\left[X_{20}+X_{2c}\cos 2\vartheta+X_{2s}\sin 2\vartheta\right]+O(r^3)$$

Same for Y & Z.

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- Net loss of 1 degree of freedom. My approach: $B_{20}(\zeta)$ is an output. Need to adjust inputs so $B_{20}(\zeta)\approx$ constant.

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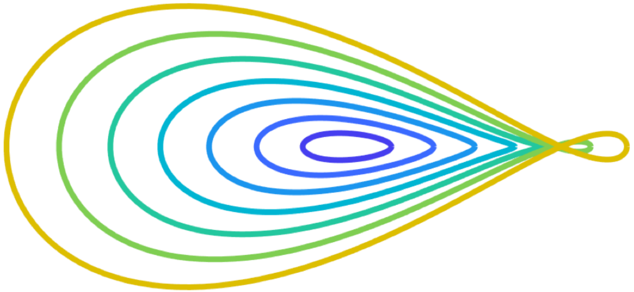
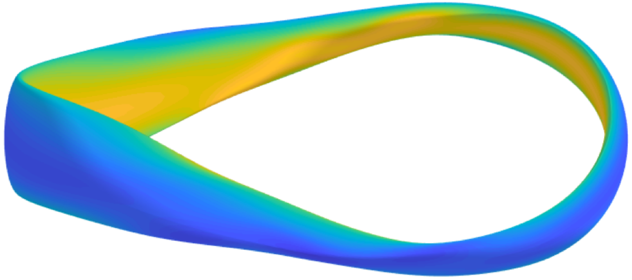
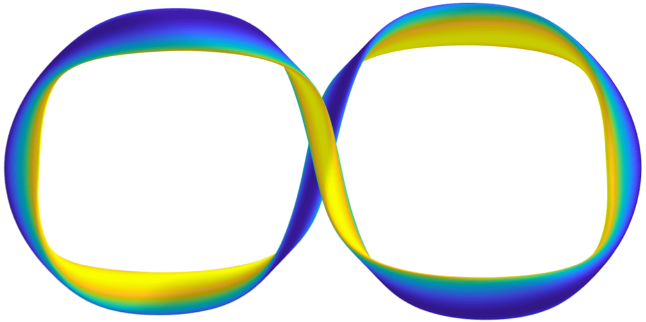
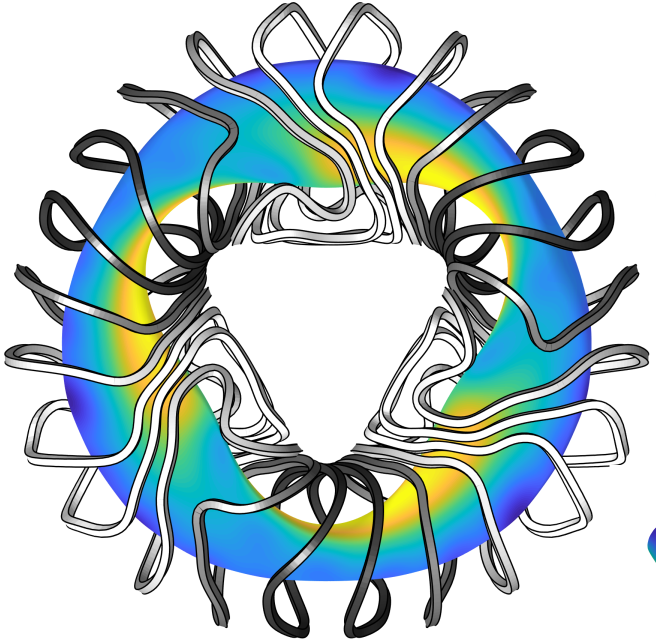
- Minimize X_2 & Y_2, to maximize the r at which surfaces begin to self-intersect.

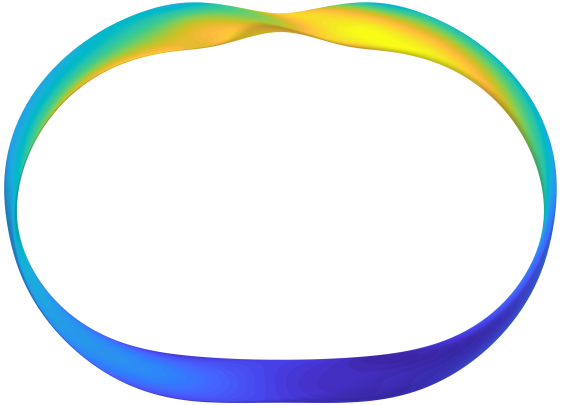


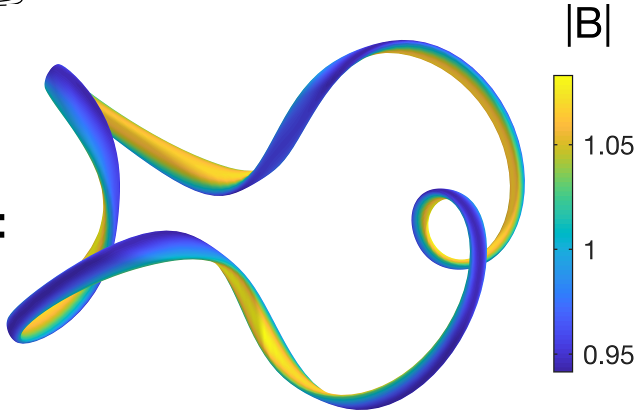
Expansion about the magnetic axis can be a powerful practical tool for generating quasisymmetric & omnigenous stellarators











Complements the traditional optimization approach:

- Many orders of magnitude faster.
- You can parameterize *all* possible solutions.
- Can generate initial conditions that can be refined by optimization.

The construction can be fit to quasisymmetric stellarators designed by optimization

- Adopt the same axis shape.
- Fit $\bar{\eta}$ to minimize difference in the shapes of a near-axis surface.

The direct construction gives an accurate match to the on-axis rotational transform in quasisymmetric stellarators designed by optimization

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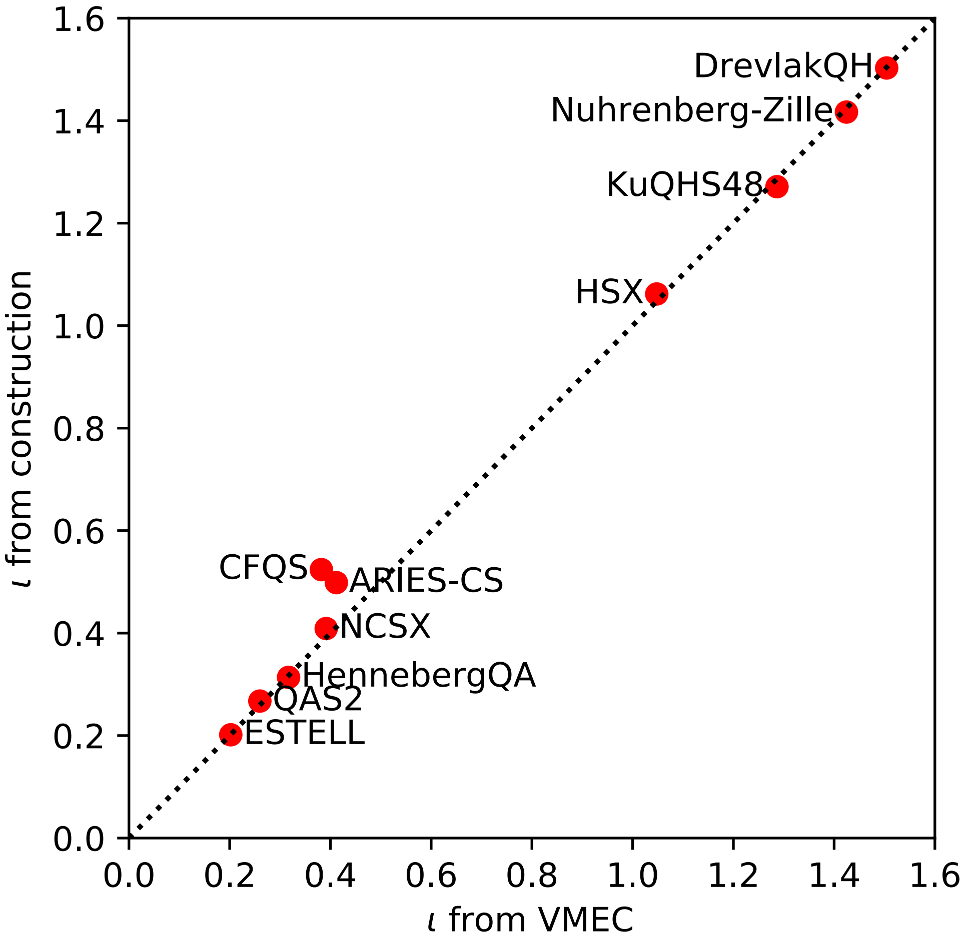
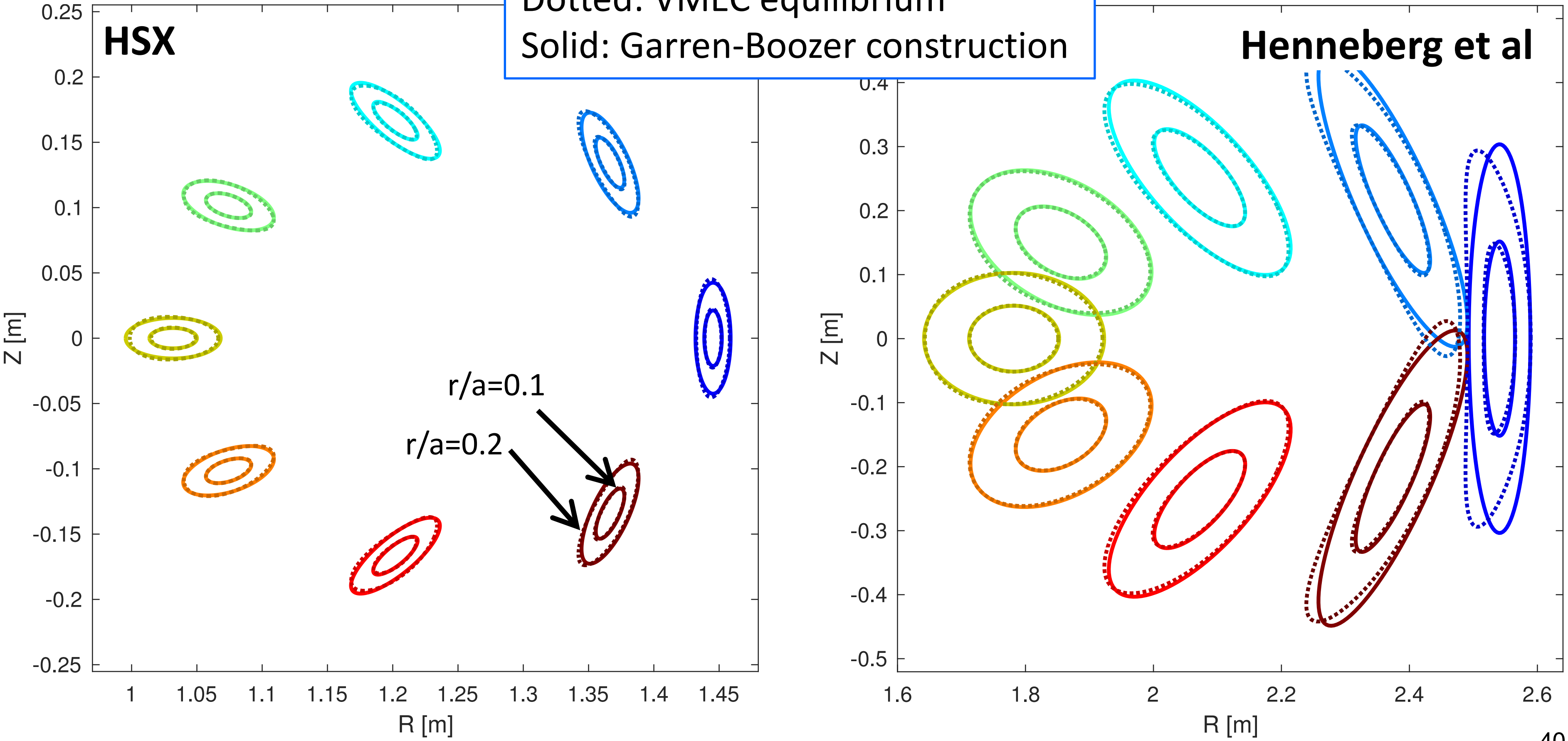


The direct construction gives an accurate match to the near-axis surface shapes of quasisymmetric stellarators designed by optimization

Dotted: VMEC equilibrium
Solid: Garren-Boozer construction



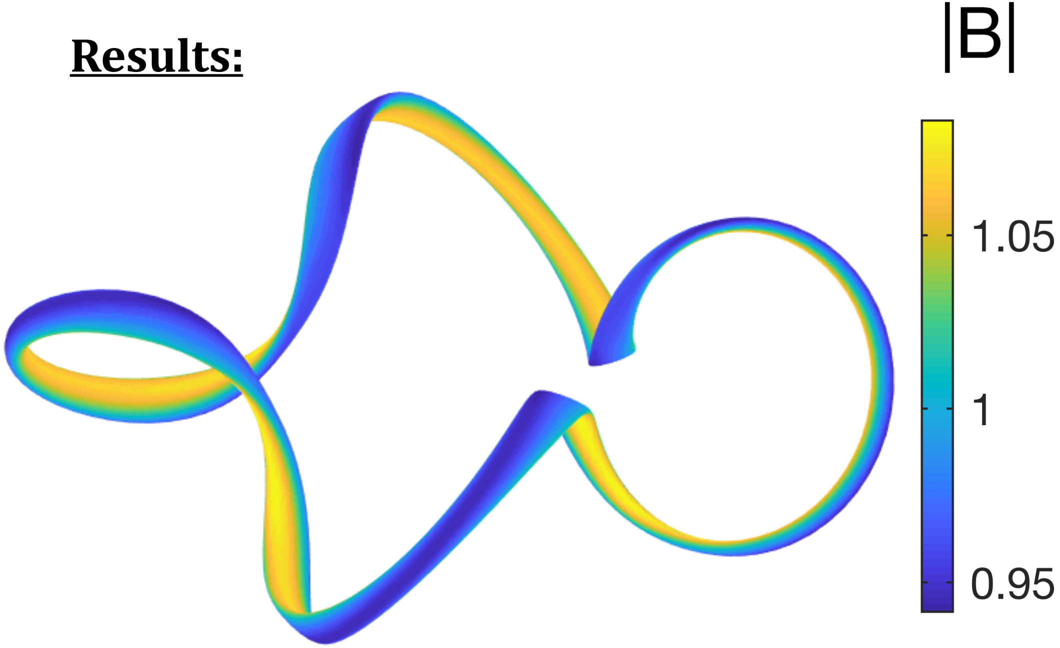
We now have a recipe for generating quasisymmetric VMEC input files: Set *r* to a small finite value *a*.

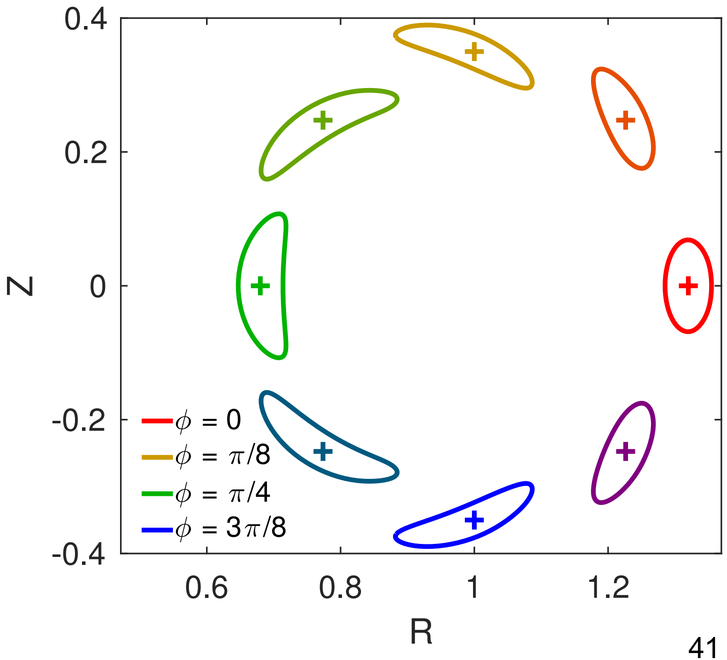
Inputs:

$$\text{axis shape } R_0(\phi) = 1 + 0.32\cos(4\phi), \quad Z_0(\phi) = 0.35\sin(4\phi),$$

$$I_2 = 0, \quad \sigma(0) = 0, \quad \bar{\eta} = 1.5, \quad R/a = 18.$$

Results:





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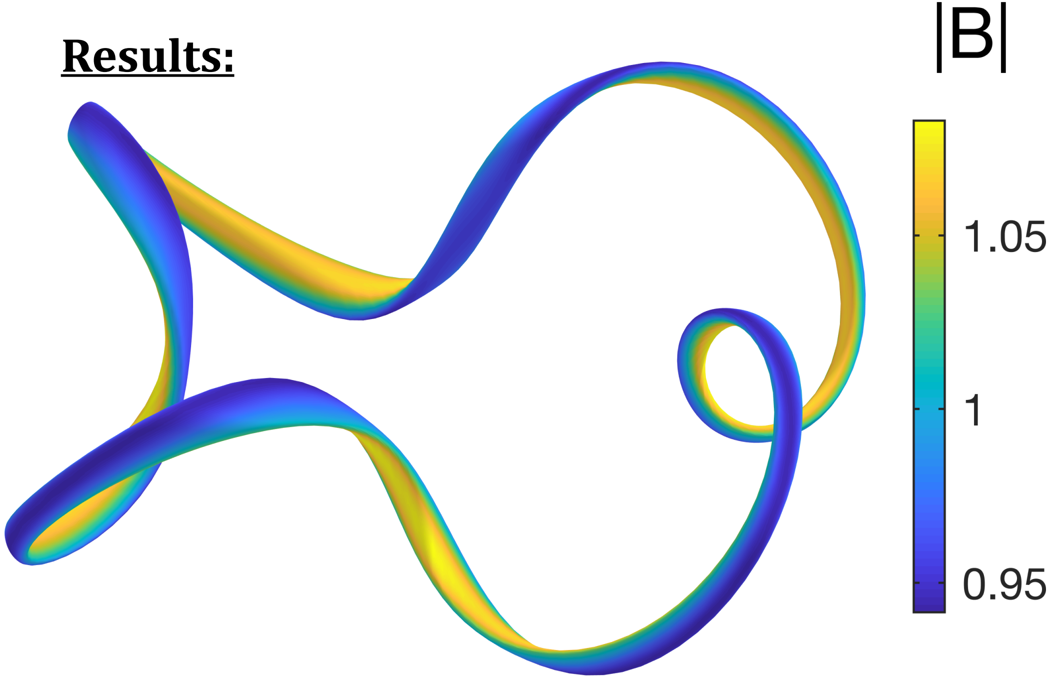
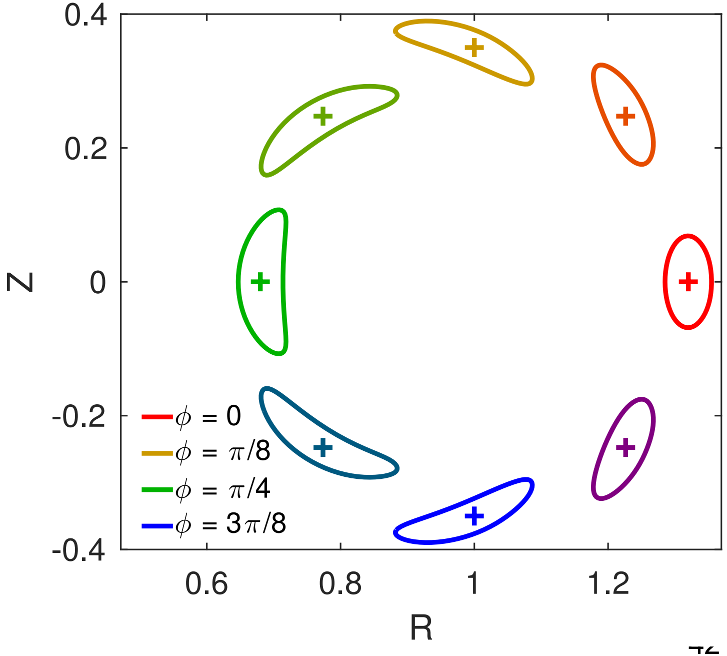
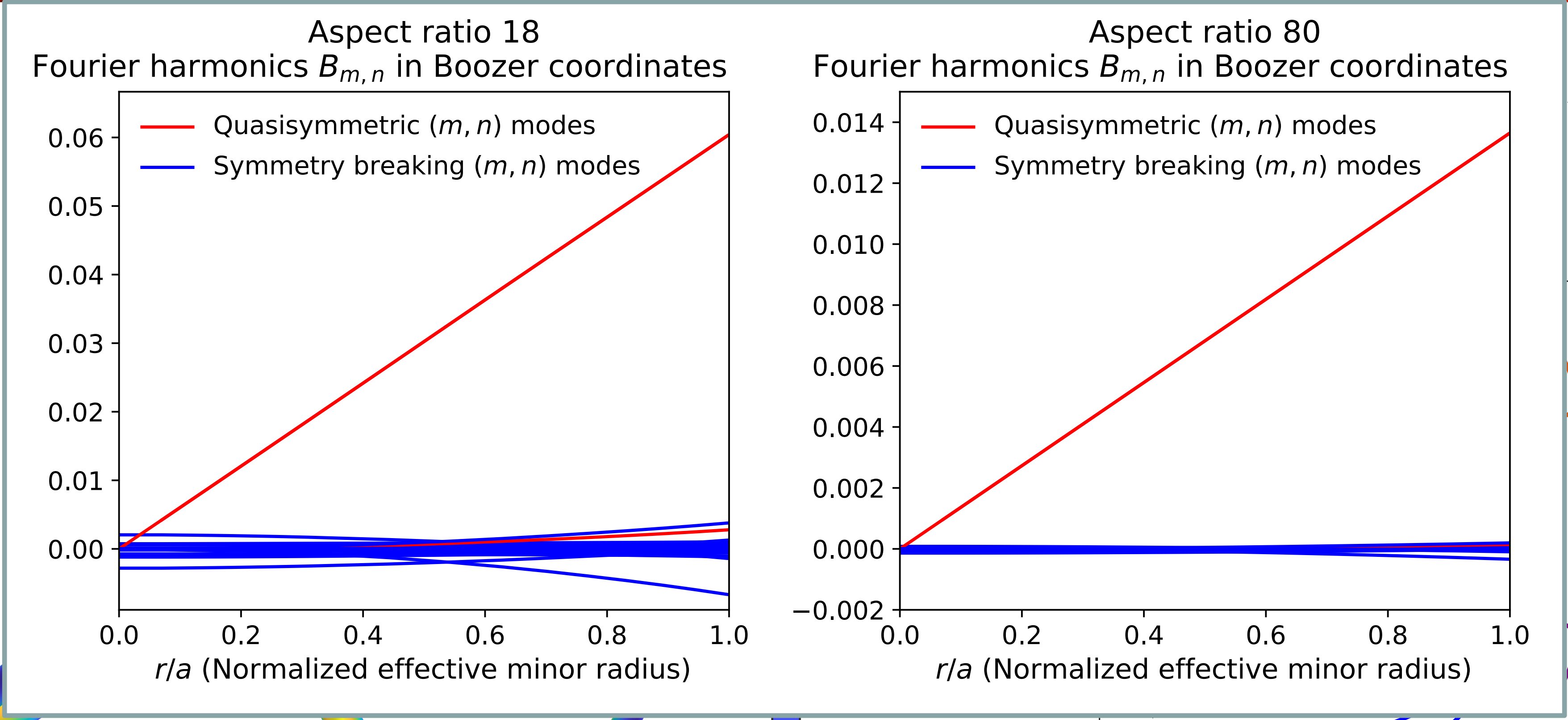
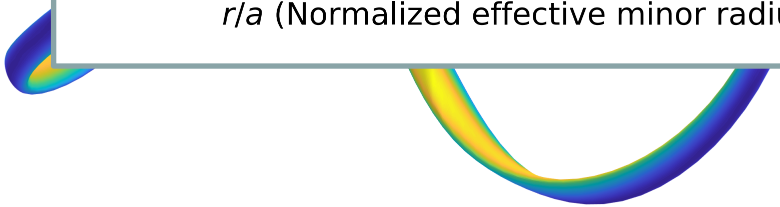


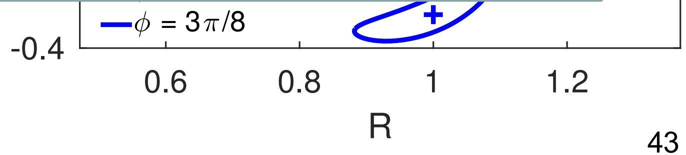


The construction can be verified by comparing to VMEC + BOOZ_XFORM.



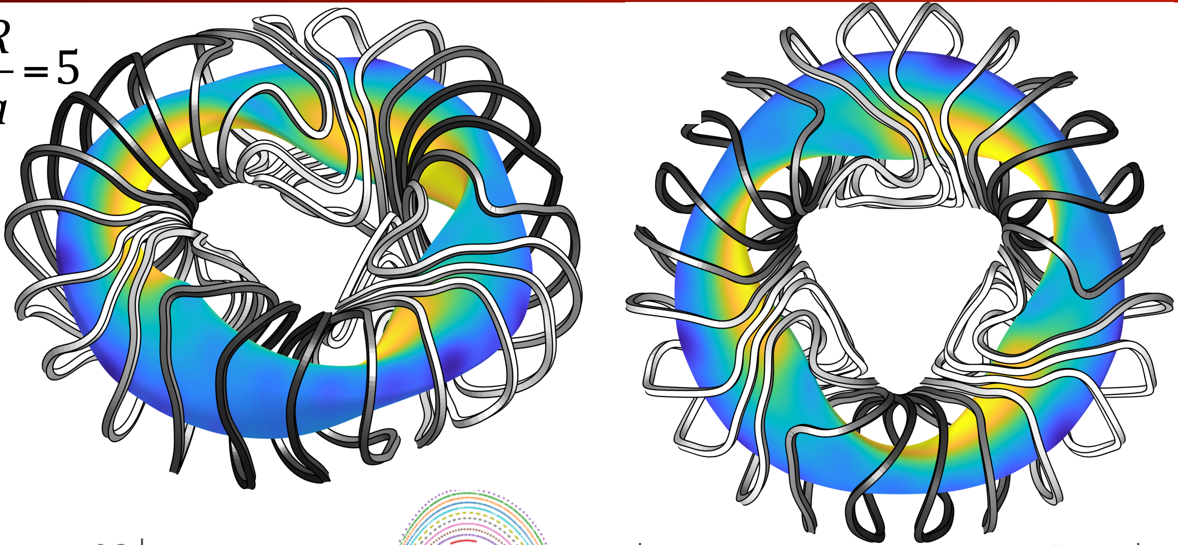
’
18.

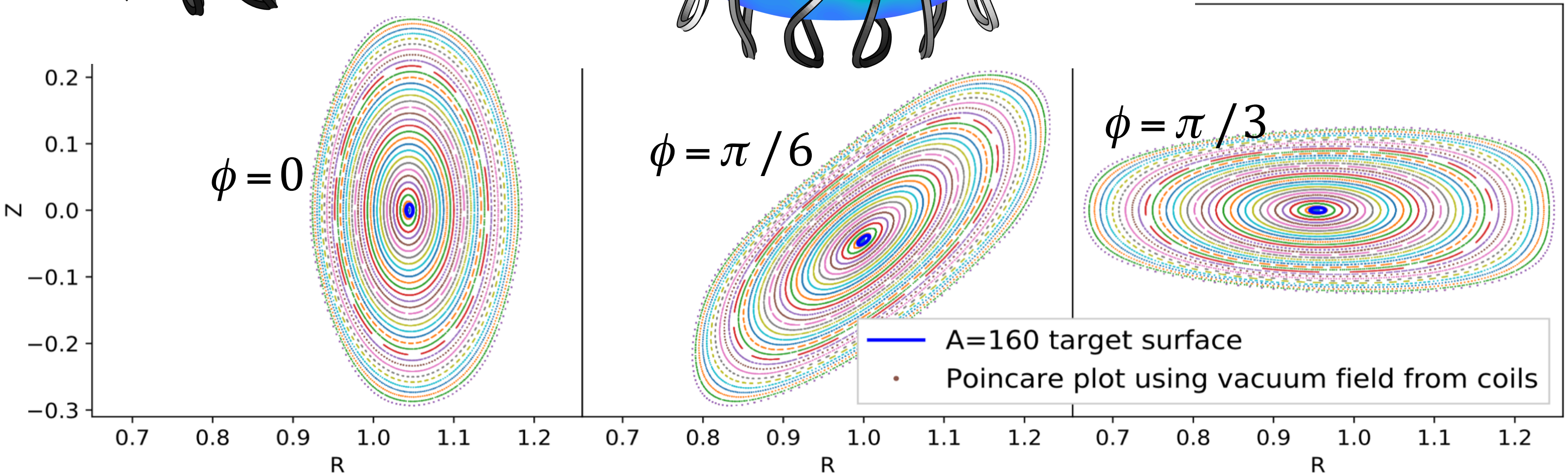




Alternative method to generate a finite-thickness boundary: find coils to make a skinny surface, then see what you get outside.

$\frac{R}{a}=5$





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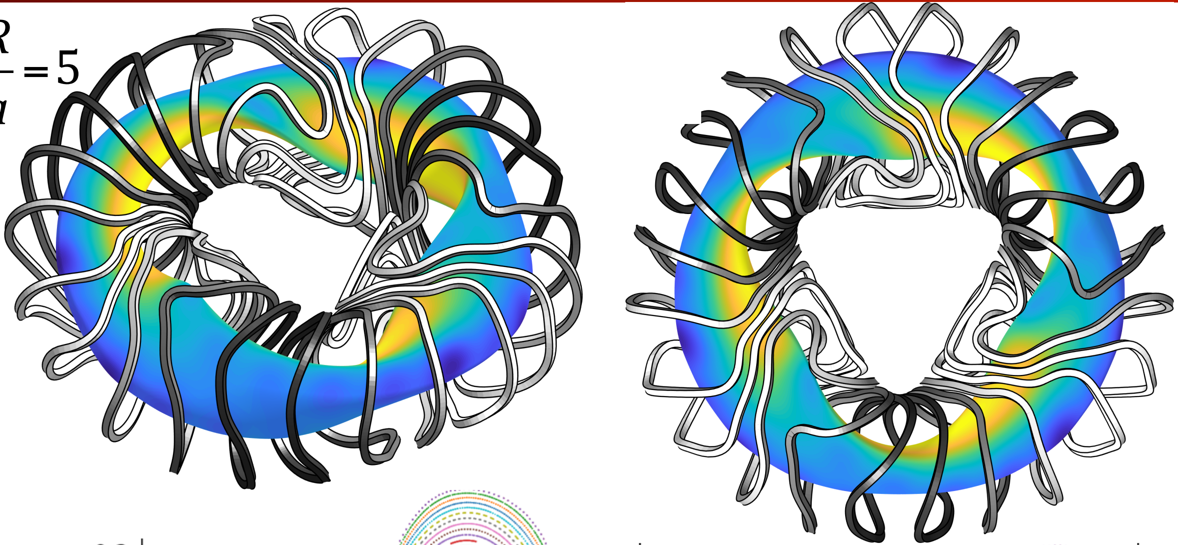

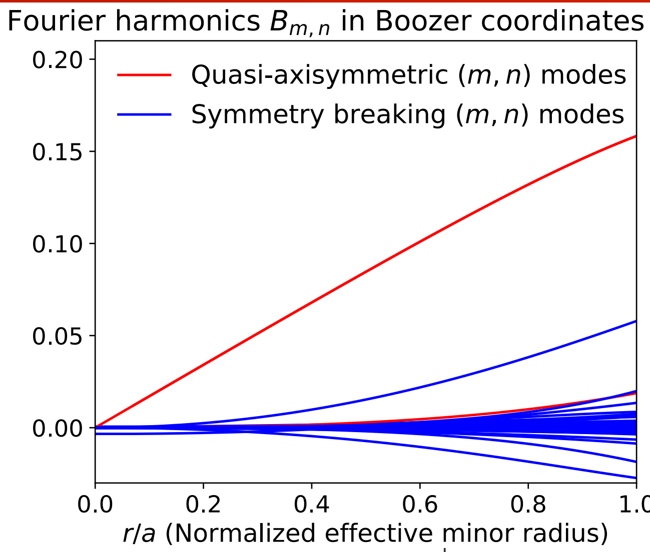

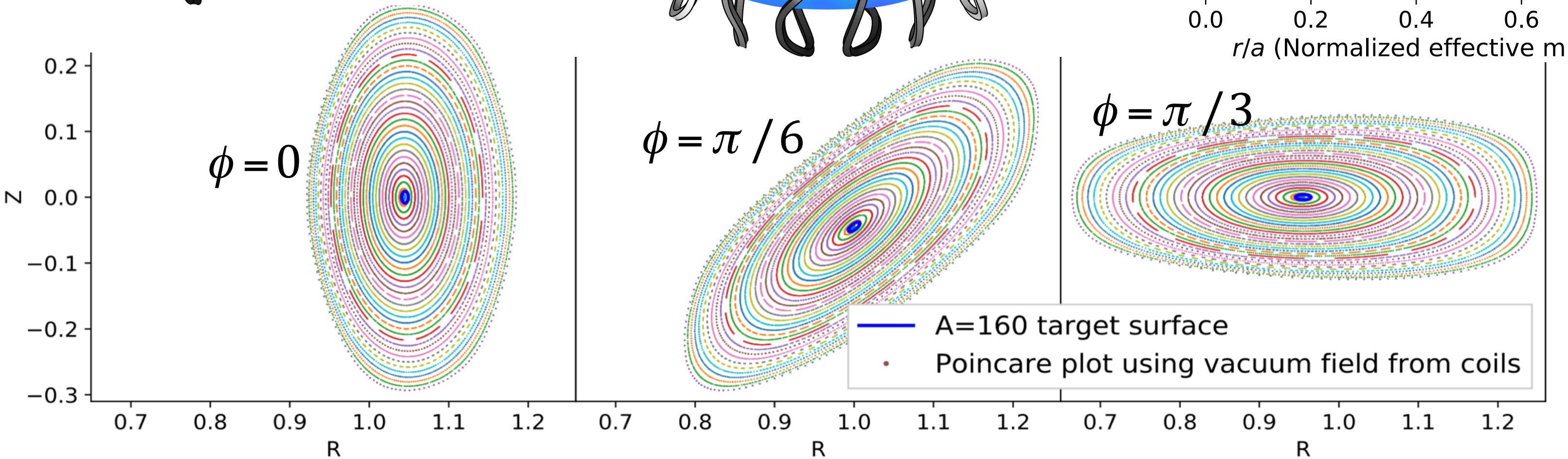

Quasi-axisymmetry vs quasi-helical symmetry is determined purely by the axis shape

$$\mathbf{J}\times\mathbf{B}=\cancel{\nabla p} \qquad \Rightarrow \qquad \boxed{\nabla_{\perp}B=B\kappa\mathbf{n}}$$

So B contours rotate about axis with the same topology as $\mathbf{n}$.

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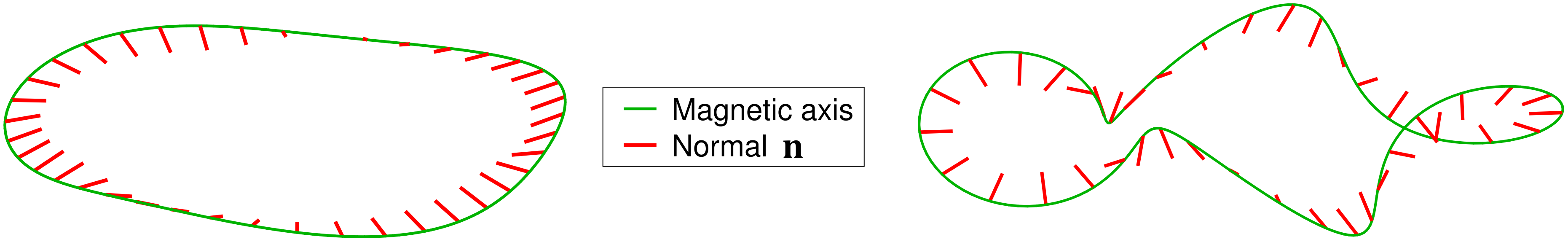


$\mathbf{n}$ does not rotate about the axis as you follow the axis around.

$\Rightarrow$ Quasi-axisymmetry

$$B=B(r,\theta)$$

$\mathbf{n}$ rotates about the axis 4 times as you follow the axis around.

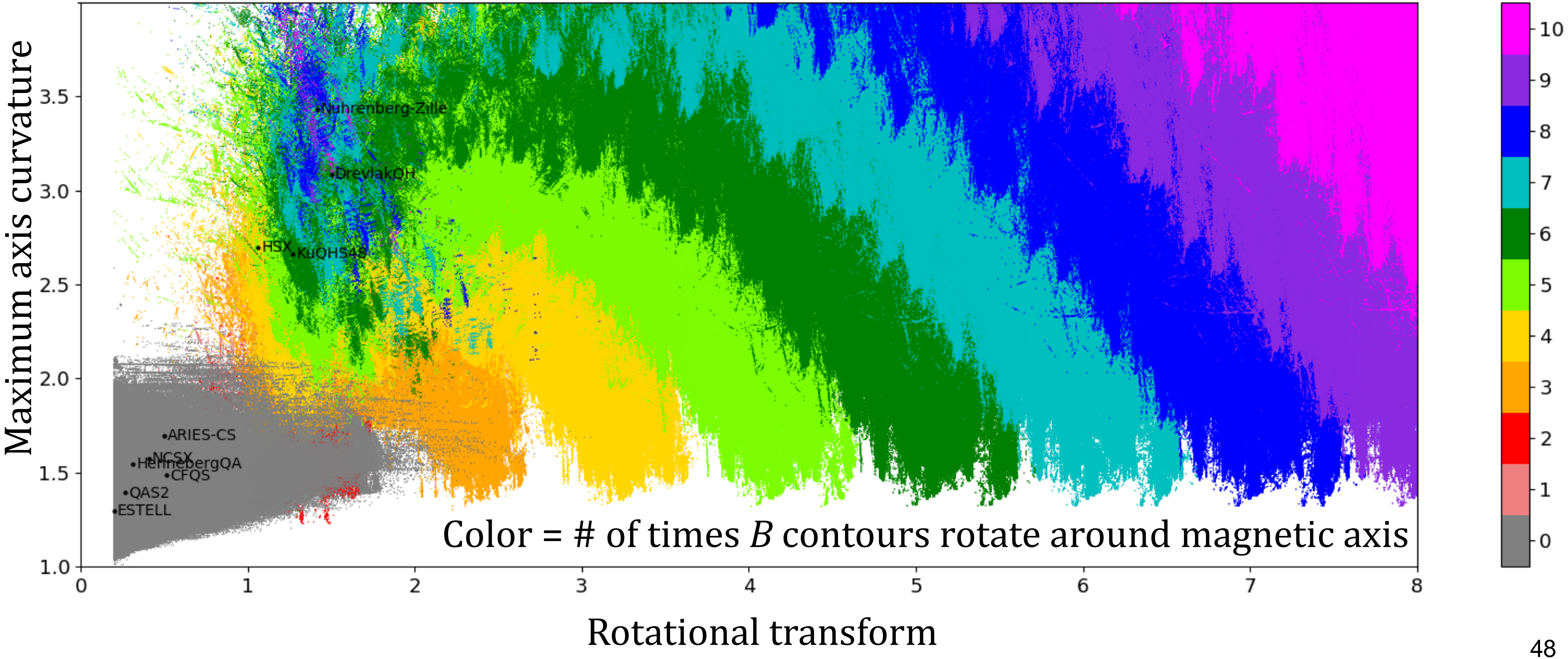
$\Rightarrow$ Quasi-helical symmetry

$$B=B(r,\theta-4\zeta)$$

The fast construction enables brute-force surveys of "all" quasisymmetric fields

Axis shape: $R_0(\phi)=1+\sum_{j=1}^{3} R_j \cos(j n_{fp}\phi), \quad Z_0(\phi)=1+\sum_{j=1}^{3} Z_j \sin(j n_{fp}\phi)$

2.4x10^8 configurations



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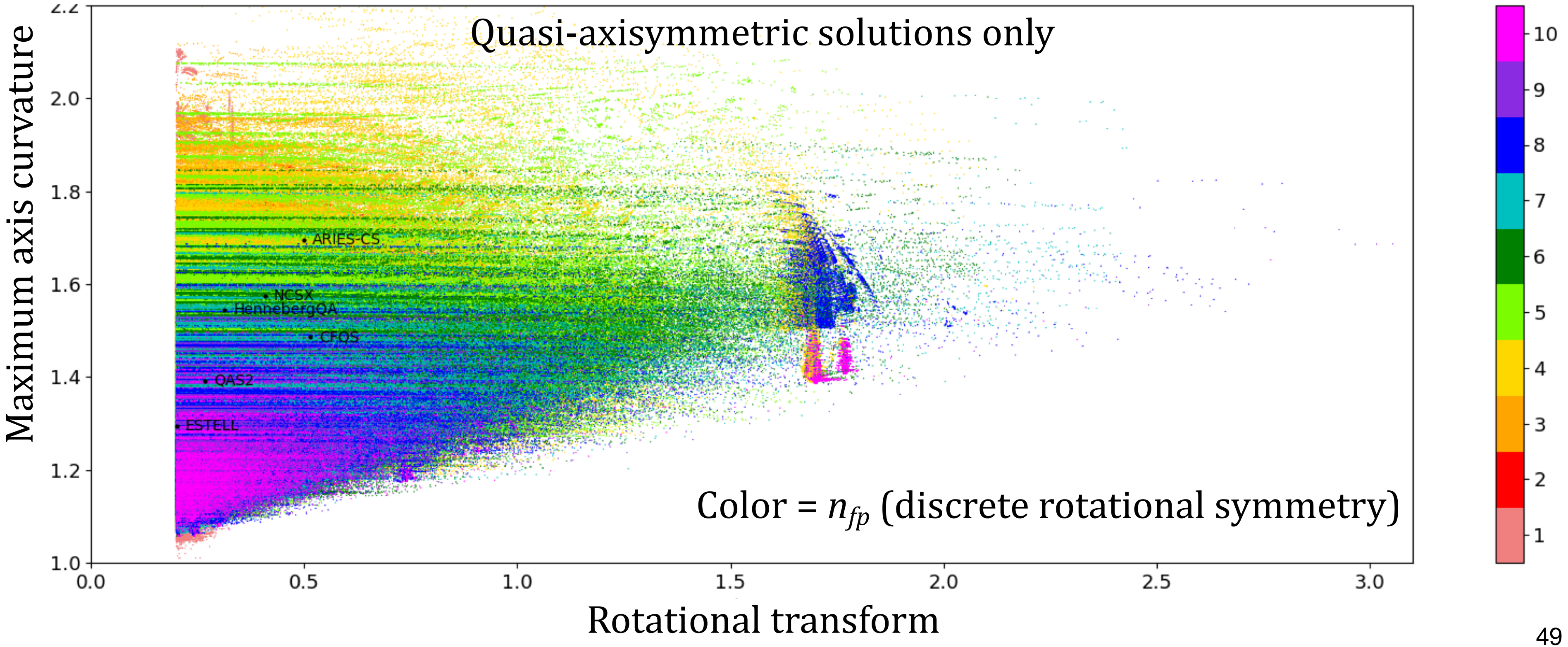
$4\text{x}10^6$ configurations



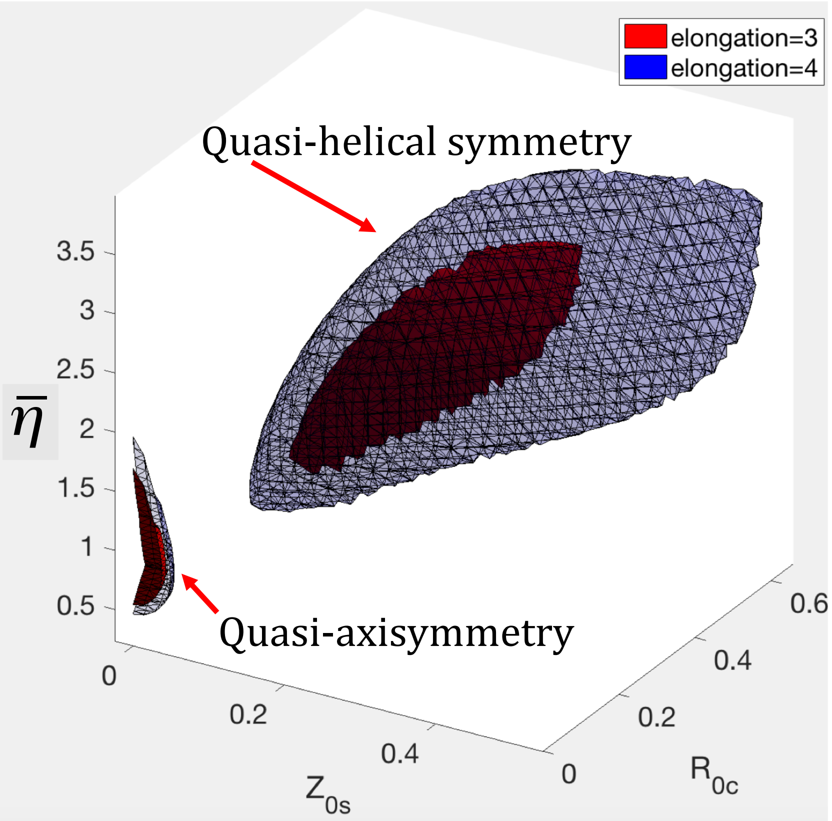
The construction enables fast scans over parameter space.

E.g. Scan over $\left(R_{0c}, Z_{0s}, \bar{\eta}\right)$ where magnetic axis shape is

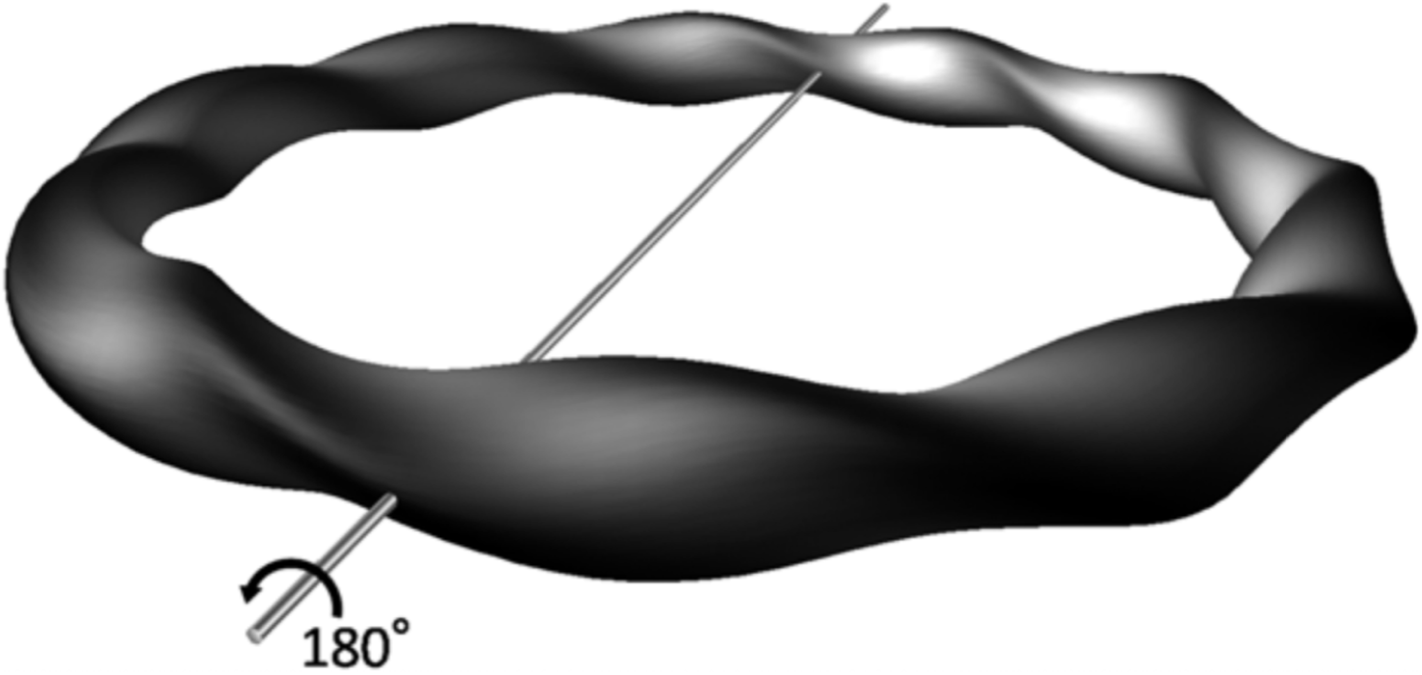
$$R_0\left(\phi\right) = 1 + R_{0c}\cos\left(4\phi\right)$$

$$Z_0\left(\phi\right) = Z_{0s}\sin\left(4\phi\right)$$

274,560 solutions generated in <30s on a laptop.



All stellarators built to date have ‘stellarator symmetry’, which is unrelated to quasisymmetry

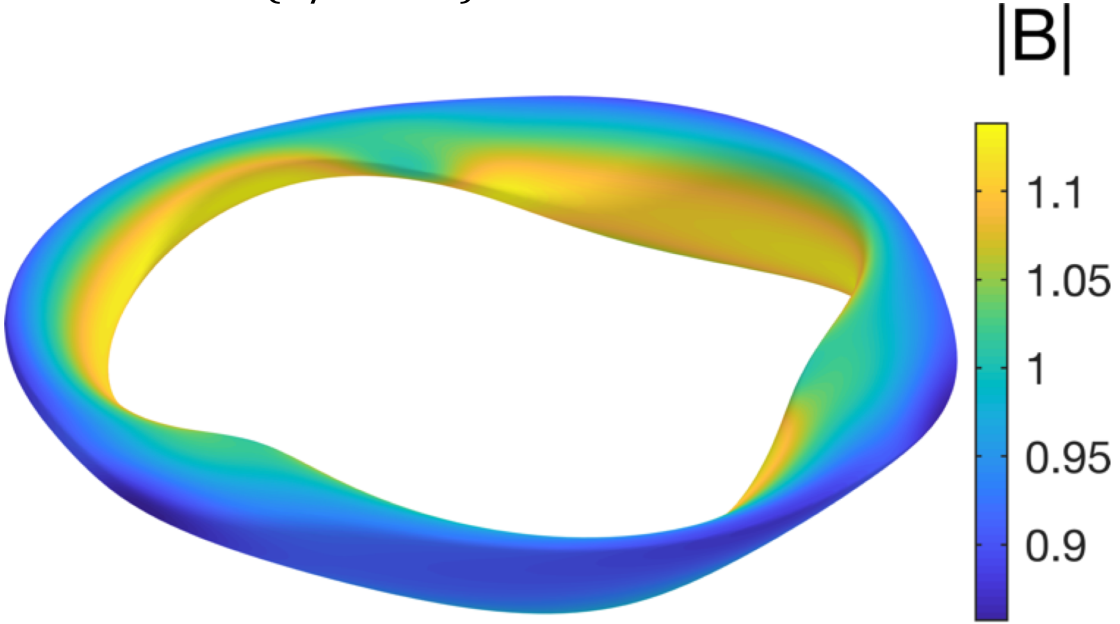


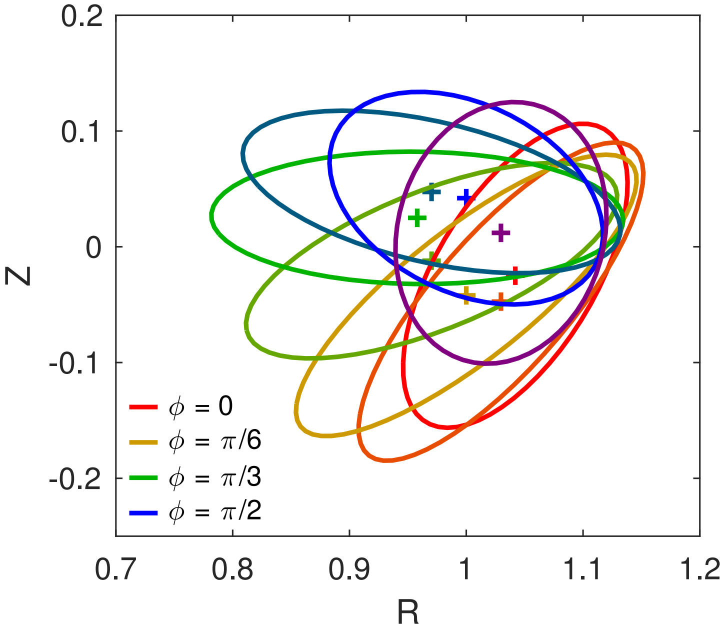
Sugama et al (2011)

You can make a quasi-axisymmetric stellarator without stellarator symmetry

Inputs: axis shape $R_0(\phi) = 1 + 0.042\cos(3\phi)$, $Z_0(\phi) = -0.042\sin(3\phi) - 0.025\cos(3\phi)$, $I_2 = 0$, $\sigma(0) = -0.6$, $\bar{\eta} = -1.1$.

Results: $(R/a = 10)$





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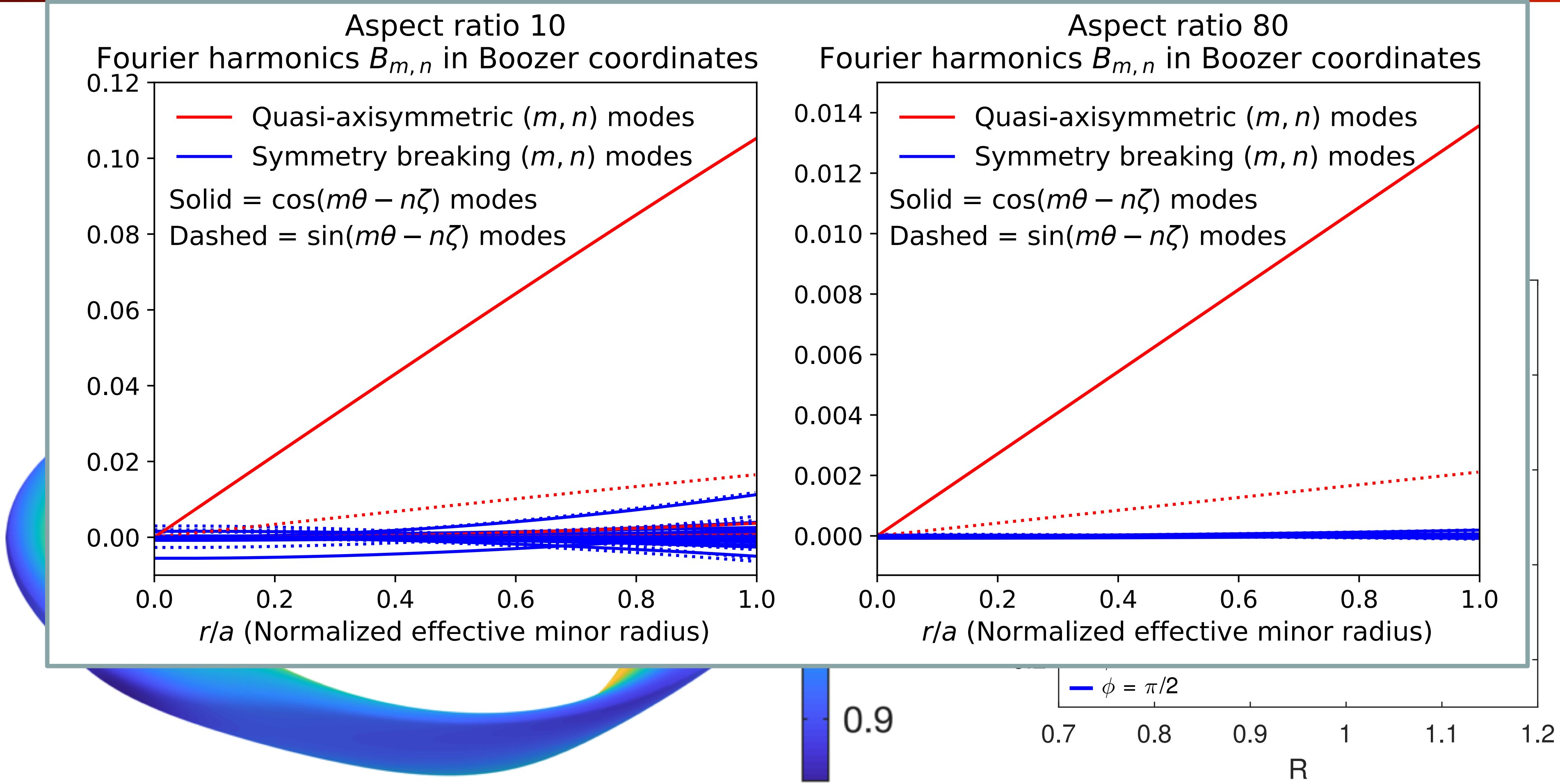
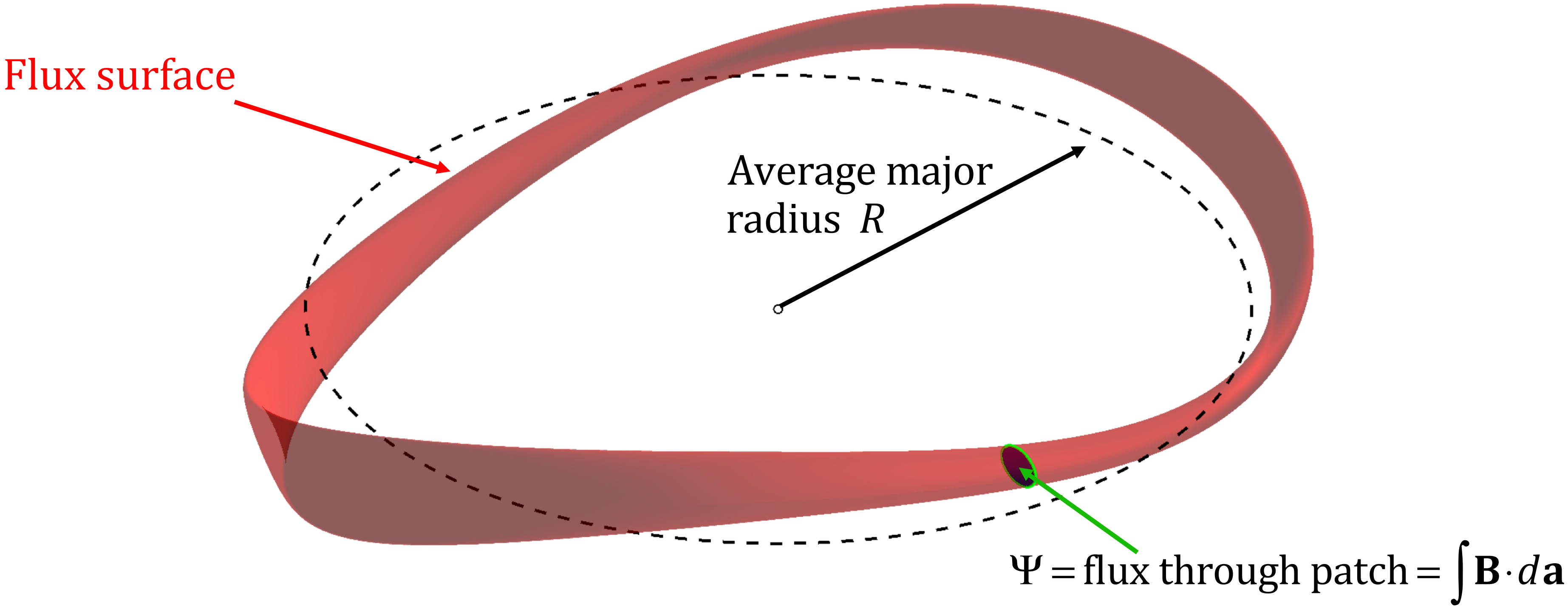


We will expand in the skinniness of the inner flux surfaces



Define effective radius r by $\Psi = \pi r^2 B_{axis}$.

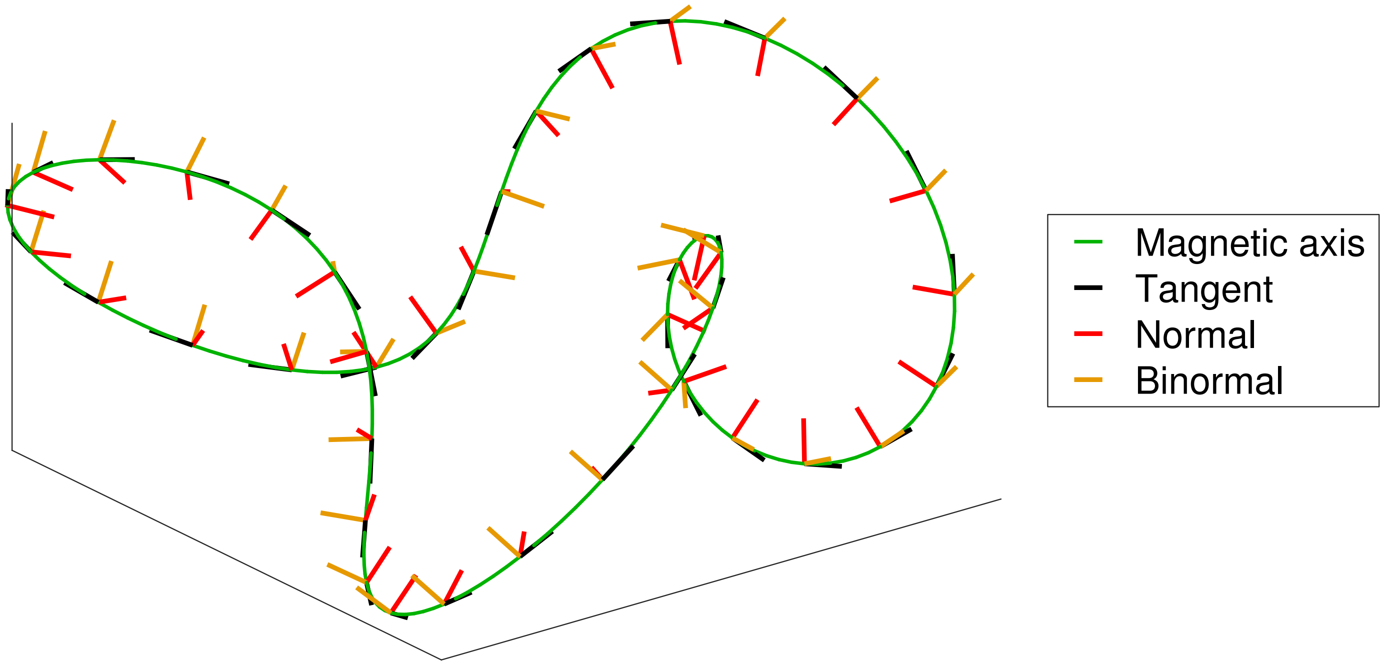
"Aspect ratio" $= \dfrac{R}{r}$ is $\gg 1$

Theory: Expand position vector using Frenet frame, equate 2 representations of B.

Frenet frame $(\mathbf{t},\mathbf{n},\mathbf{b})$: $\dfrac{\partial \mathbf{r}_0}{\partial \ell} = \mathbf{t}, \quad \dfrac{d\mathbf{t}}{d\ell} = \kappa \mathbf{n}, \quad \dfrac{d\mathbf{n}}{d\ell} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \dfrac{d\mathbf{b}}{d\ell} = -\tau \mathbf{n}$

$\mathbf{r}_0 =$ magnetic axis, $\kappa =$ curvature, $\tau =$ torsion

$\mathbf{t} =$ tangent, $\mathbf{n} =$ normal, $\mathbf{b} =$ binormal

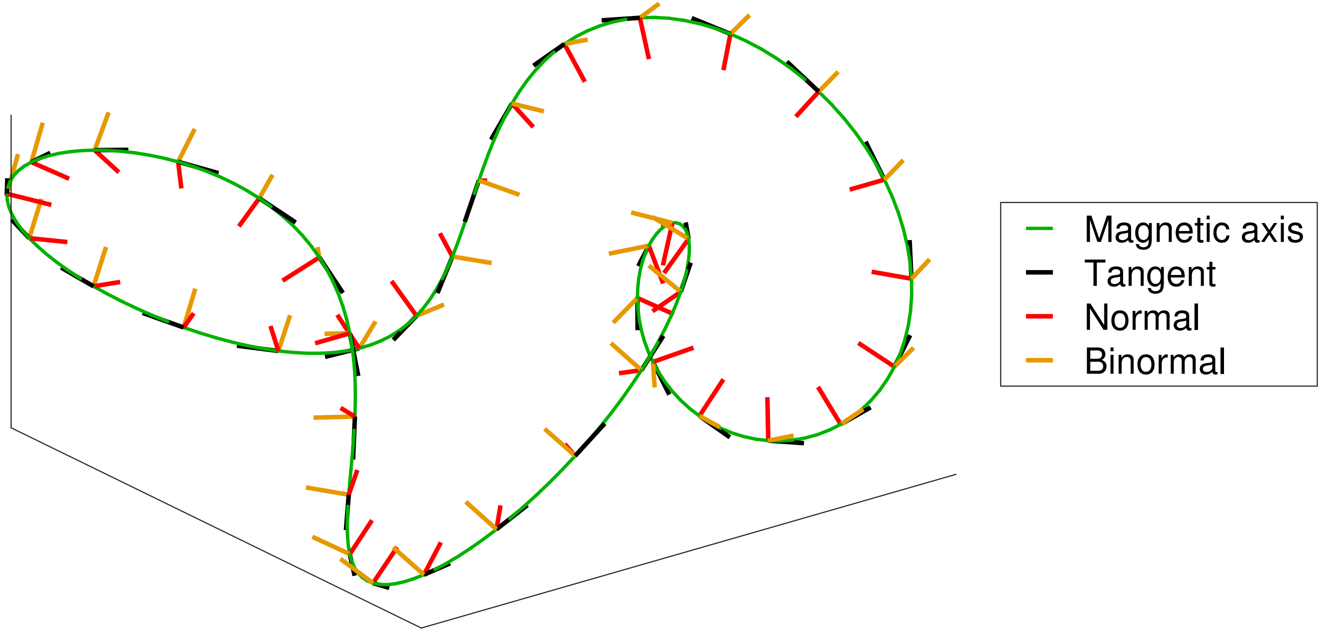


Theory: Write position vector using Frenet frame

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}):\quad \frac{d\mathbf{r}_0}{d\ell}=\mathbf{t},\qquad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n},\qquad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b},\qquad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

$\mathbf{r}_0=$ magnetic axis, $\kappa=$ curvature, $\tau=$ torsion, $\mathbf{t}=$ tangent, $\mathbf{n}=$ normal, $\mathbf{b}=$ binormal

$$\mathbf{r}(r,\theta,\zeta)=\mathbf{r}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta)$$



Theory: Write position vector using Frenet frame, expand in small *r* = (flux)[1/2]

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}):\ \frac{d\mathbf{r}_0}{d\ell}=\mathbf{t},\quad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n},\quad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b},\quad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

$\mathbf{r}_0 = \text{magnetic axis},\quad \kappa = \text{curvature},\quad \tau = \text{torsion},\quad \mathbf{t} = \text{tangent},\quad \mathbf{n} = \text{normal},\quad \mathbf{b} = \text{binormal}$

$$\begin{aligned}\mathbf{r}(r,\theta,\zeta) &= \mathbf{r}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta)\\ &= \mathbf{r}_0(\zeta)+rX_{1c}(\zeta)\cos\theta\,\mathbf{n}(\zeta)+r\left[Y_{1s}(\zeta)\sin\theta+Y_{1c}(\zeta)\cos\theta\right]\mathbf{b}(\zeta)+O(r^2)\end{aligned}$$

Using magnetohydrodynamic equilibrium $(\mathbf{J}\times\mathbf{B}=\nabla p)$

Theory: Write position vector using Frenet frame, expand in small r = (flux)$^{1/2}$

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}):\quad \frac{d\mathbf{r}_0}{d\ell}=\mathbf{t},\qquad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n},\qquad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b},\qquad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

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$$\begin{aligned}\mathbf{r}(r,\theta,\zeta) &= \mathbf{r}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta)\\ &= \mathbf{r}_0(\zeta)+rX_{1c}(\zeta)\cos\theta\,\mathbf{n}(\zeta)+r\left[Y_{1s}(\zeta)\sin\theta+Y_{1c}(\zeta)\cos\theta\right]\mathbf{b}(\zeta)+O(r^2)\end{aligned}$$

$$X_{1c}(\zeta)=\frac{\bar{\eta}}{\kappa(\zeta)},\qquad Y_{1s}(\zeta)=\frac{\kappa(\zeta)}{\bar{\eta}},\qquad Y_{1c}(\zeta)=\frac{\sigma(\zeta)\kappa(\zeta)}{\bar{\eta}}$$

Toroidal angle $\zeta \propto$ arclength, $\bar{\eta}=$ constant: $B=B_0\left[1+r\bar{\eta}\cos(\theta-N\varphi)+O(r^2)\right]$

$$\boxed{\frac{d\sigma}{d\zeta}+\iota\left[\frac{\bar{\eta}^4}{\kappa^4}+1+\sigma^2\right]-2\frac{\bar{\eta}^2}{\kappa^2}\left[I_2-\tau\right]=0}$$

$I_2=$ current density

Theory: Expand position vector using Frenet frame, equate 2 representations of B.

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$$\mathbf{B} = B_r \nabla r + B_\theta \nabla\theta + B_\zeta \nabla\zeta, \qquad \mathbf{B} = \nabla\psi \times \nabla\theta + \iota \nabla\zeta \times \nabla\psi$$

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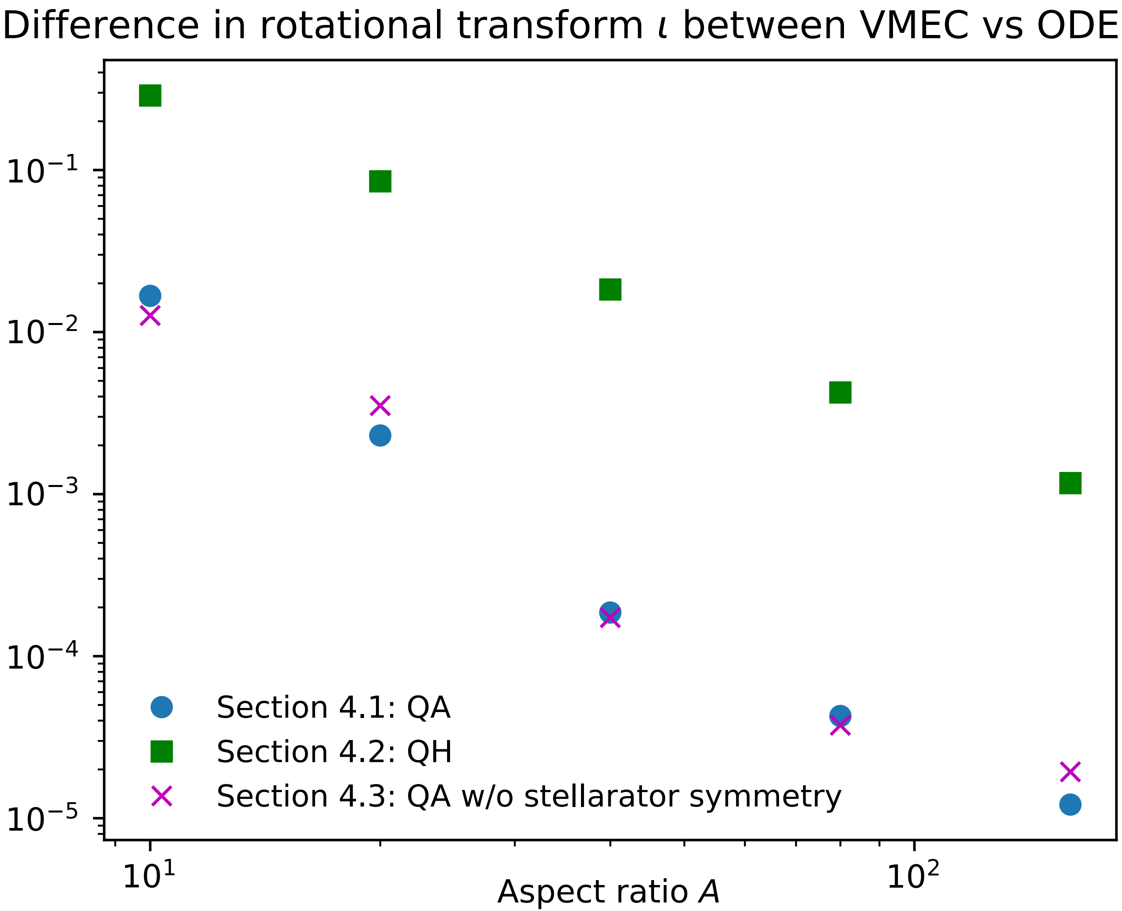
$$X(r,\theta,\zeta) = r\left[X_{1s}(\zeta)\sin\theta + X_{1c}(\zeta)\cos\theta\right] + O(r^2). \quad \text{Same for } Y, Z.$$

$$\mathbf{B} = B_r \nabla r + B_\theta \nabla \theta + B_\zeta \nabla \zeta, \qquad \mathbf{B} = \nabla\psi \times \nabla\theta + \iota \nabla\zeta \times \nabla\psi$$

$$\text{Dual relations:} \quad \nabla r = \left[\frac{\partial \mathbf{r}}{\partial r} \cdot \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \zeta}\right]^{-1} \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \zeta}, \quad \text{cyclic permutations.}$$

The rotational transform computed by VMEC converges to the value computed by the Garren-Boozer approach.



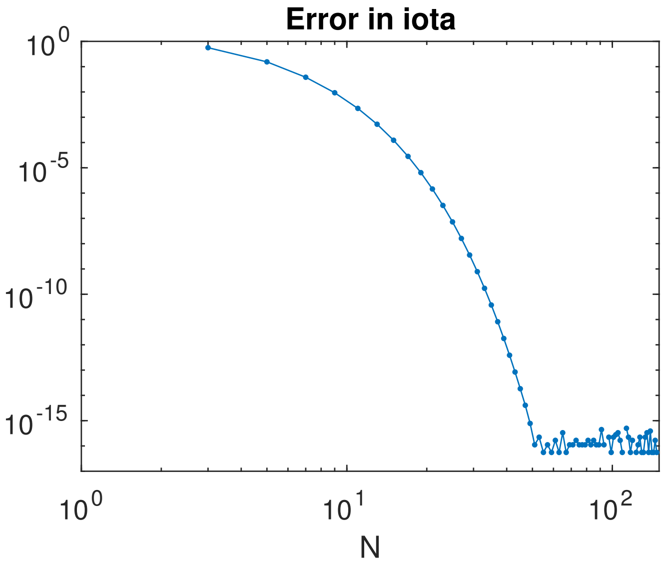
The ODE is solved with spectral accuracy using pseudospectral discretization + Newton iteration

Uniform grid in ϕ with N points: $\phi_1 = 0,\ \phi_2 = 2\pi/\left(Nn_{fp}\right),\ \ldots,\ \phi_N = 2\pi\left(N-1\right)/\left(Nn_{fp}\right)$.

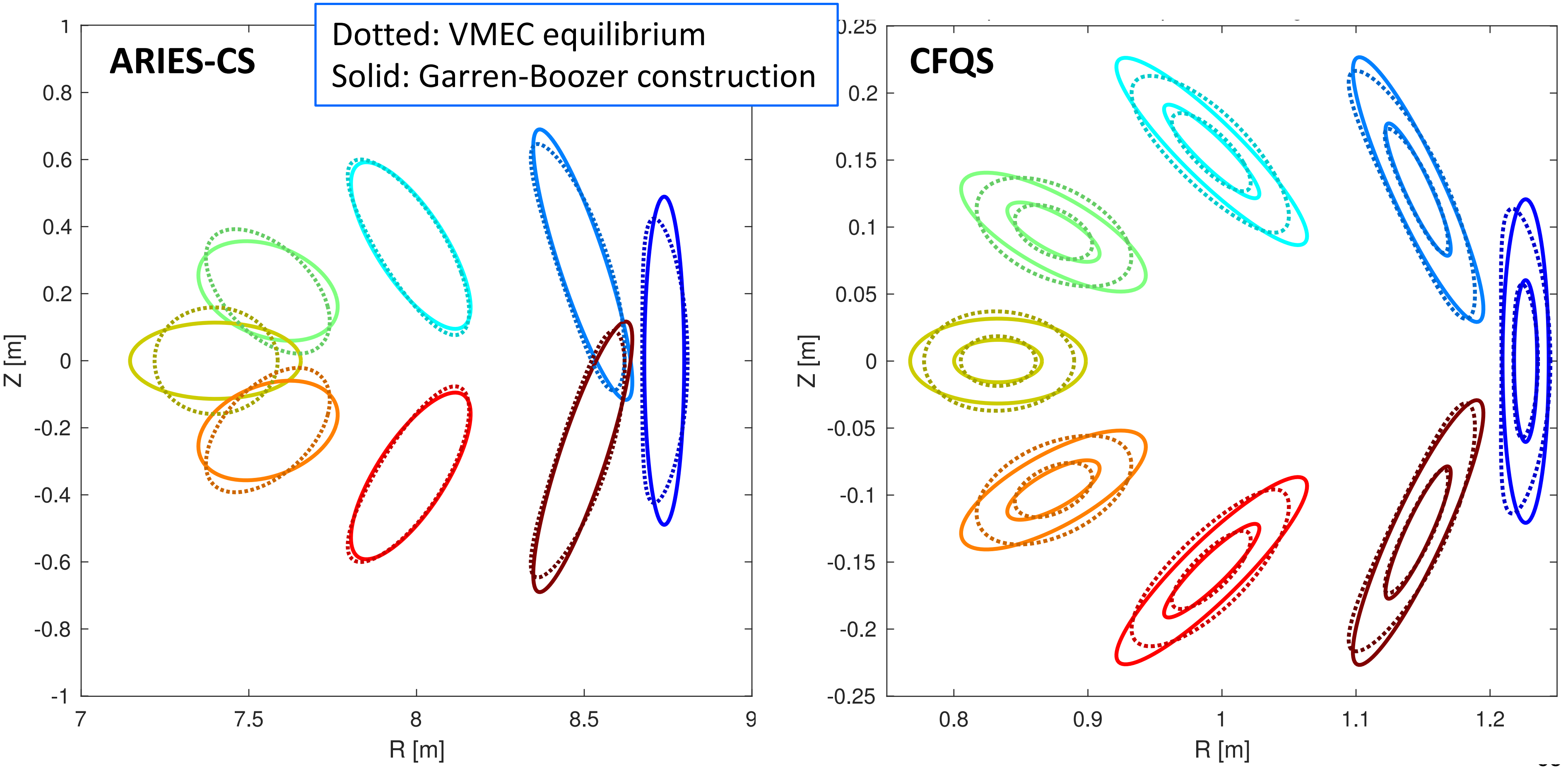
Vector of N unknowns: $\left(\iota,\ \sigma\left(\phi_2\right),\ \sigma\left(\phi_3\right),\ \ldots,\ \sigma\left(\phi_N\right)\right)^T$

N equations: impose ODE at $\phi_1,\ \ldots,\ \phi_N$.

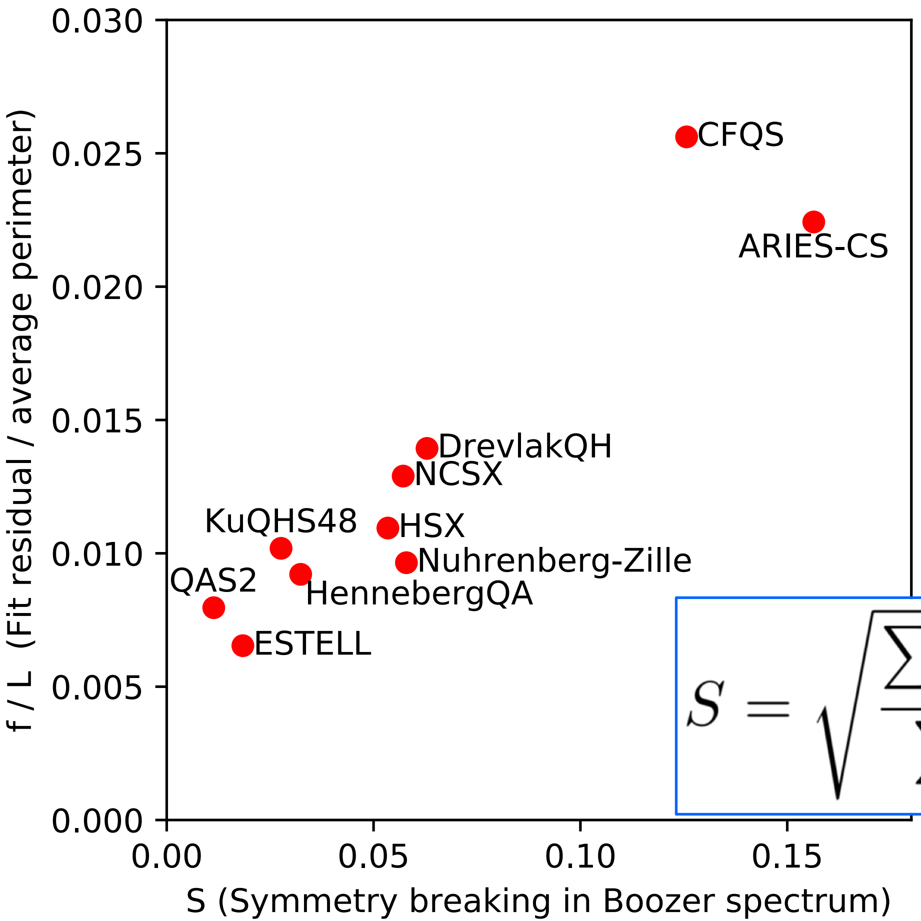
$\dfrac{d\sigma}{d\phi} \rightarrow D\sigma$ where D is a pseudospectral differentiation matrix.



Of 10 configurations examined, the fit is less good for 2



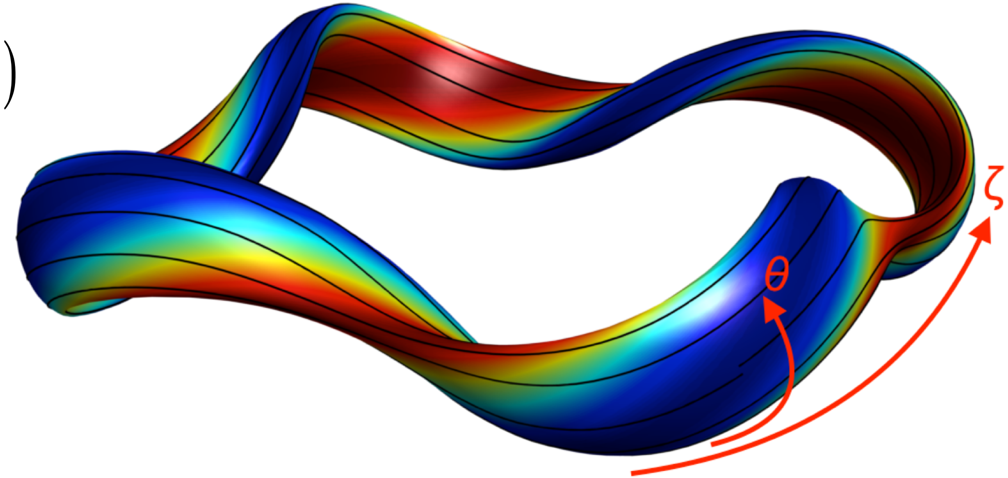
The configurations with relatively poor fits can be explained by their larger symmetry-breaking



$$S = \sqrt{\frac{\sum_{m\neq 0, n\neq mN} B_{m,n}^2}{\sum_{m,n=mN} B_{m,n}^2}}$$

The conventional approach to finding quasisymmetric fields works but has shortcomings

Want magnetic field strength B
to have quasisymmetry: $B = B(r,\ \theta - N\zeta)$



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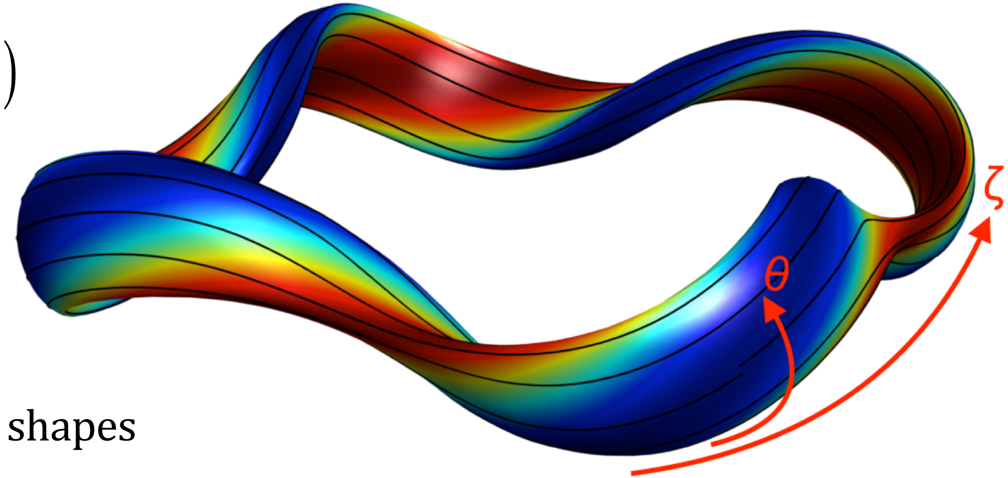

$$\min_X f(X)$$

Parameter space: $X =$ toroidal boundary shapes

Objective: $f = \sum_{m,n \neq Nm} B_{m,n}^2(r_0)$ where $B(r,\theta,\zeta) = \sum_{m,n} B_{m,n}(r) \exp(im\theta - in\zeta)$

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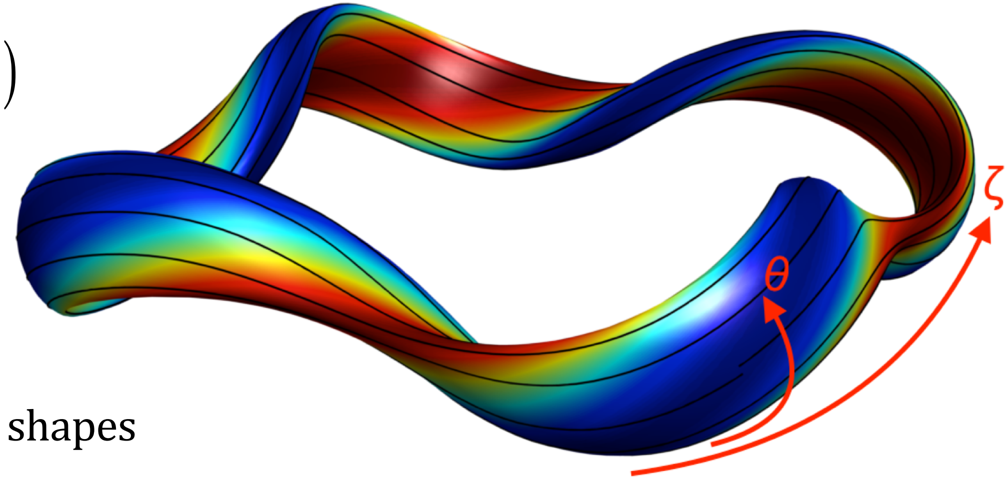

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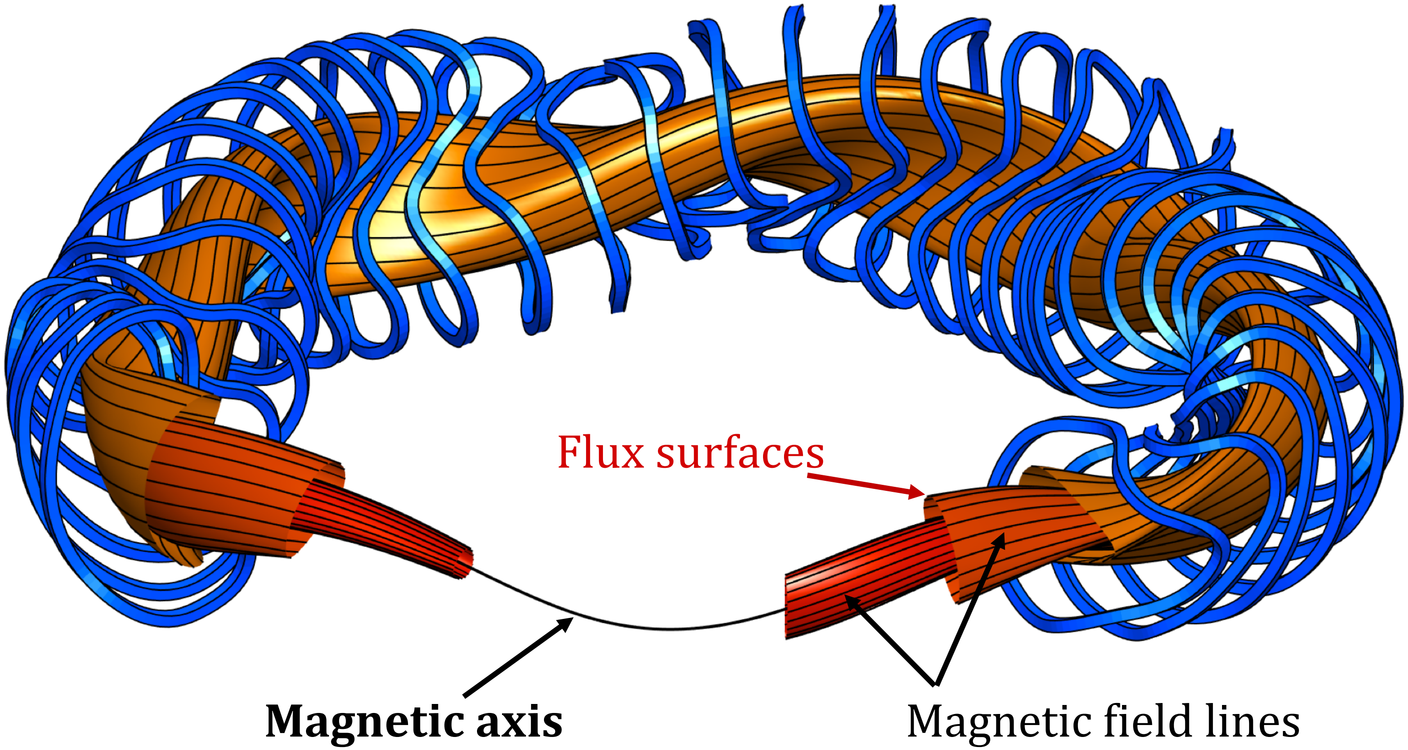
- Computationally expensive.
- What is the size & character of the solution space?
- Result depends on initial condition, so cannot be sure you’ve found all solutions.

Alternative: expand equations near the magnetic axis



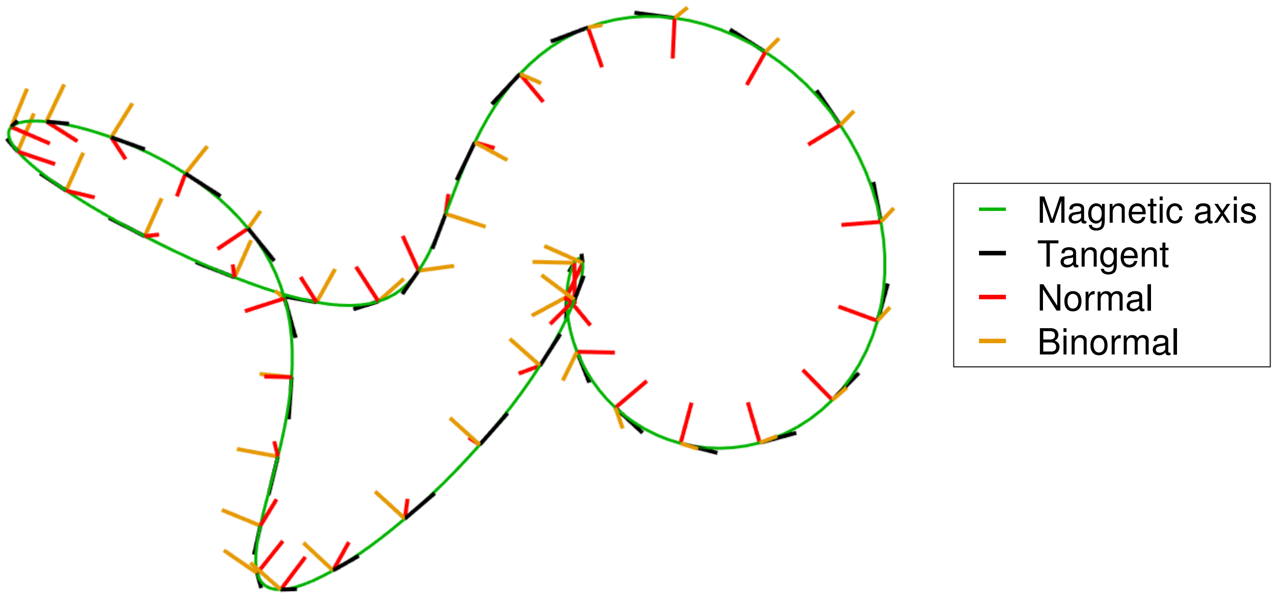
Mercier (1964), Lortz & Nührenberg (1976), Garren & Boozer (1991)

A key ingredient of the theory is the Frenet frame of the magnetic axis

Frenet frame $(\mathbf{t},\mathbf{n},\mathbf{b})$: $\dfrac{d\mathbf{x}_0}{d\ell}=\mathbf{t}, \quad \dfrac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n}, \quad \dfrac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b}, \quad \dfrac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$

$\mathbf{x}_0=$ magnetic axis, $\kappa=$ curvature, $\tau=$ torsion, $\mathbf{t}=$ tangent, $\mathbf{n}=$ normal, $\mathbf{b}=$ binormal

Mercier (1964), Solovev & Shafranov (1970), Lortz & Nührenberg (1976), Garren & Boozer (1991)



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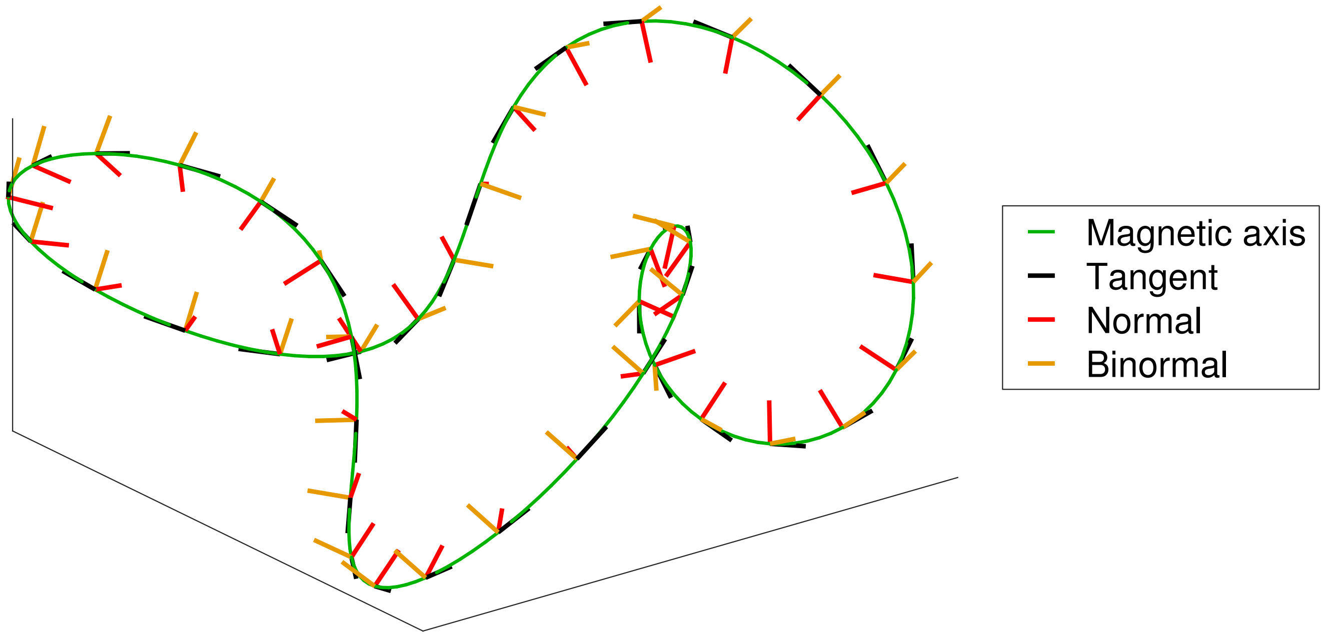
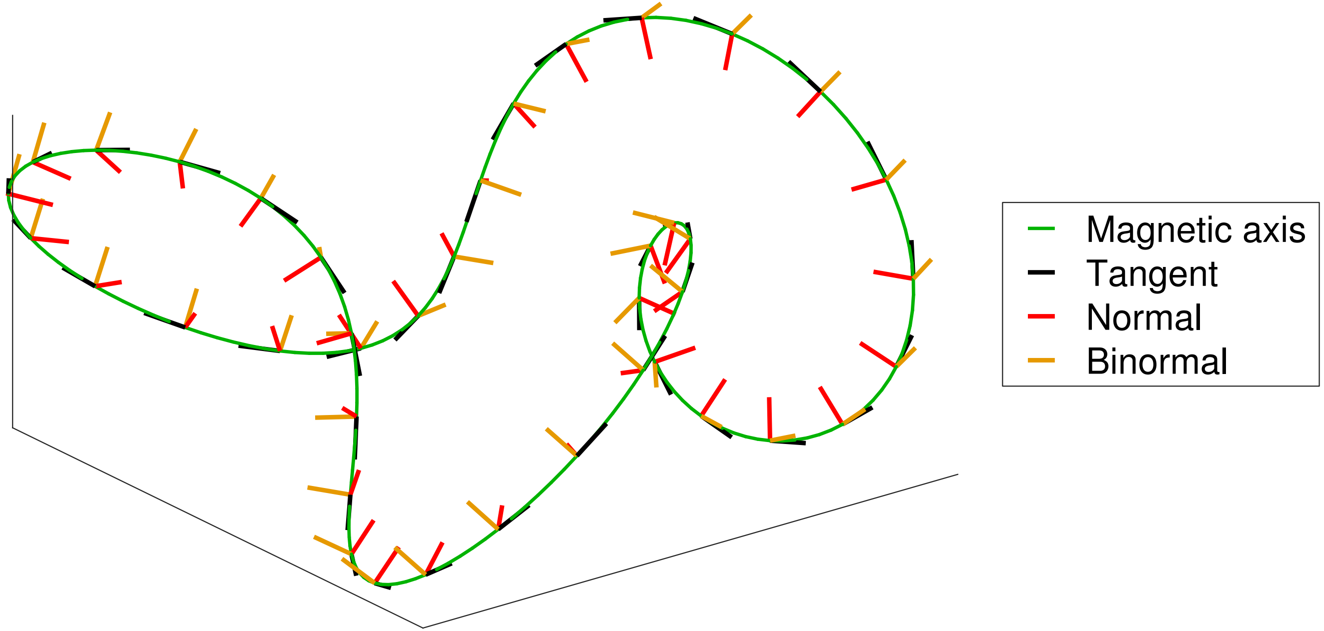


Garren & Boozer (1991): Write position vector **x** using axis's Frenet frame, expand in small *r*

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}):\quad \frac{d\mathbf{x}_0}{d\ell}=\mathbf{t},\qquad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n},\qquad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b},\qquad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

$\mathbf{x}_0=$ magnetic axis, $\kappa=$ curvature, $\tau=$ torsion, $\mathbf{t}=$ tangent, $\mathbf{n}=$ normal, $\mathbf{b}=$ binormal

$$\mathbf{x}(r,\theta,\zeta)=\mathbf{x}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta),\qquad r\propto\sqrt{\psi}$$



The size of the space of fields that are quasisymmetric to O(r) can be precisely understood.

Given $P(\zeta) > 0$, $Q(\zeta)$, and $\sigma(0)$, with $P(\zeta)$ and $Q(\zeta)$

2π-periodic, bounded, and integrable, a solution to

$$\boxed{\frac{d\sigma}{d\zeta} + \iota\left(P + \sigma^2\right) + Q = 0} \qquad (1)$$

is a pair $\{\iota, \sigma(\zeta)\}$ solving (1) where $\sigma(\zeta)$ is 2π-periodic.

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Theorem: A solution exists and it is unique.

ML, Sengupta, and Plunk (2019). Probably an earlier reference?

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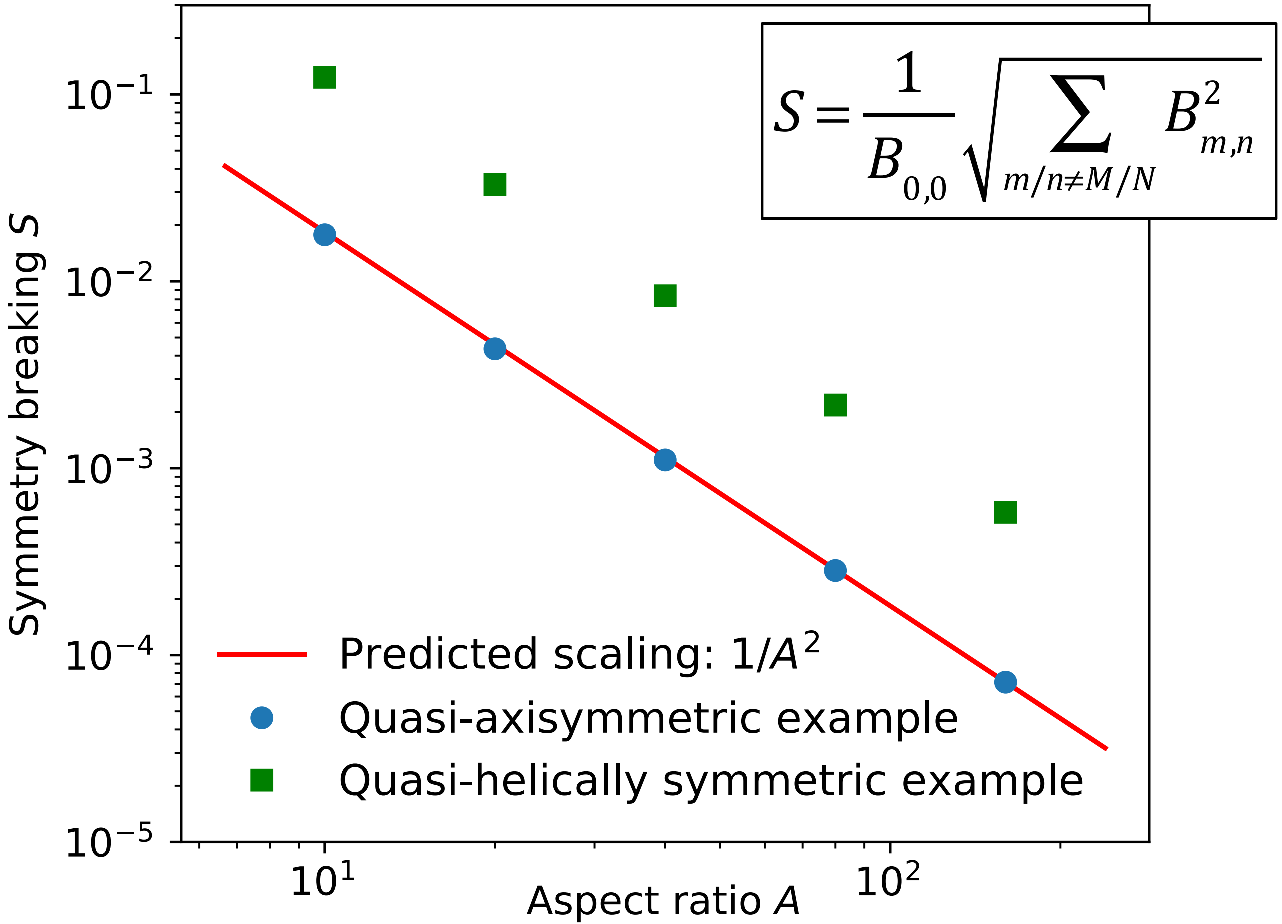
<u>Theorem:</u> A solution exists and it is unique.

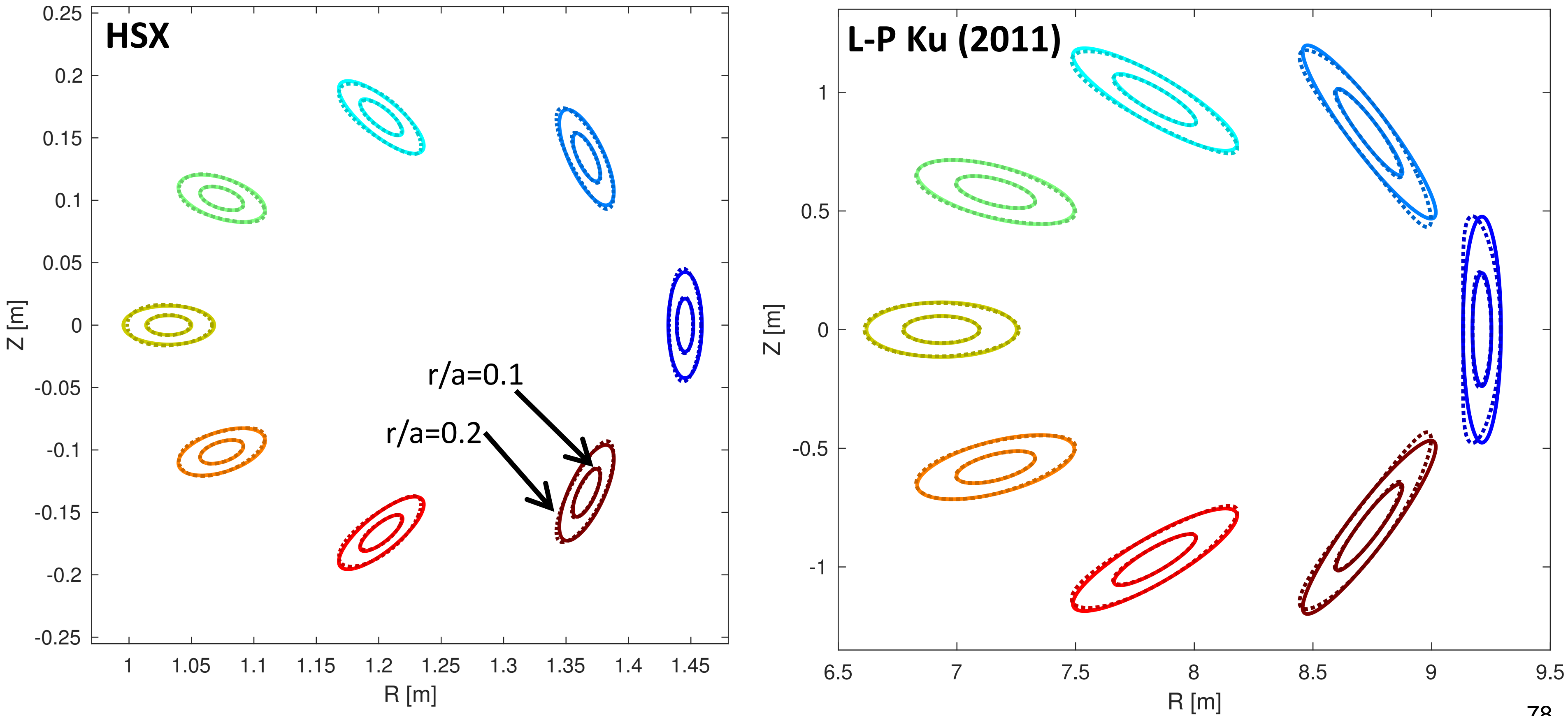
$\Rightarrow$ Numerical solution is very robust.

The symmetry-breaking Fourier amplitudes scale as predicted.



Quasi-helically symmetric configurations

Dotted: VMEC equilibrium
Solid: Garren-Boozer construction



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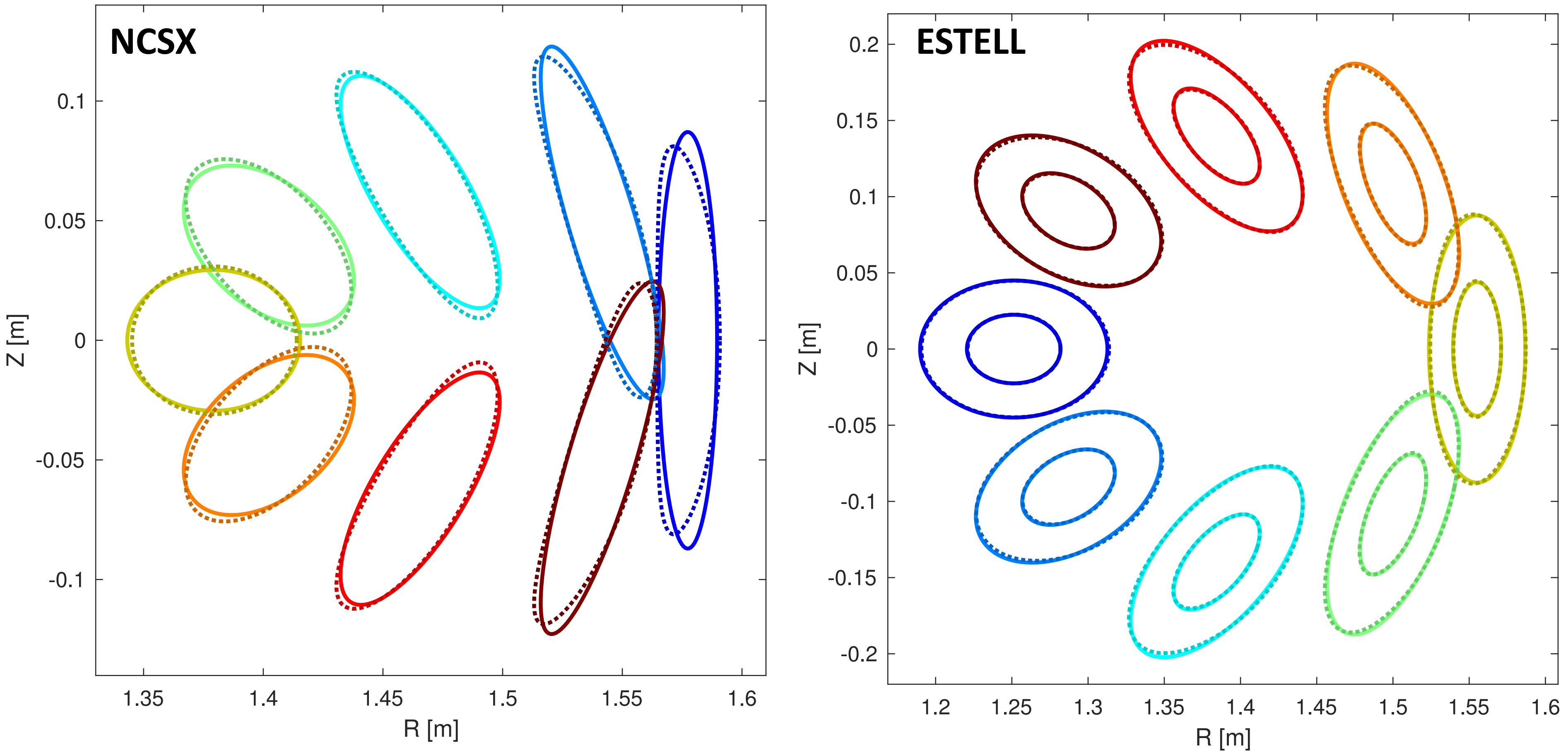

Omnigenity is a weaker confinement condition than quasisymmetry.

Definition of omnigenity: The radial drift has a time average of 0 for all particles.

$$\oint \left(\mathbf{v}_d \cdot \nabla r \right) dt = 0 \quad \forall \text{ magnetic moments \& energies.}$$

- J Cary & S Shasharina, *Physics of Plasmas* **4**, 3323 (1997).
- J Cary & S Shasharina, *Physical Review Letters* **78**, 674 (1997).
- P Helander & J Nührenberg, *Plasma Physics and Controlled Fusion* 51, 055004 (2009).
- M Landreman & P J Catto, *Physics of Plasmas* **19**, 056103 (2012).

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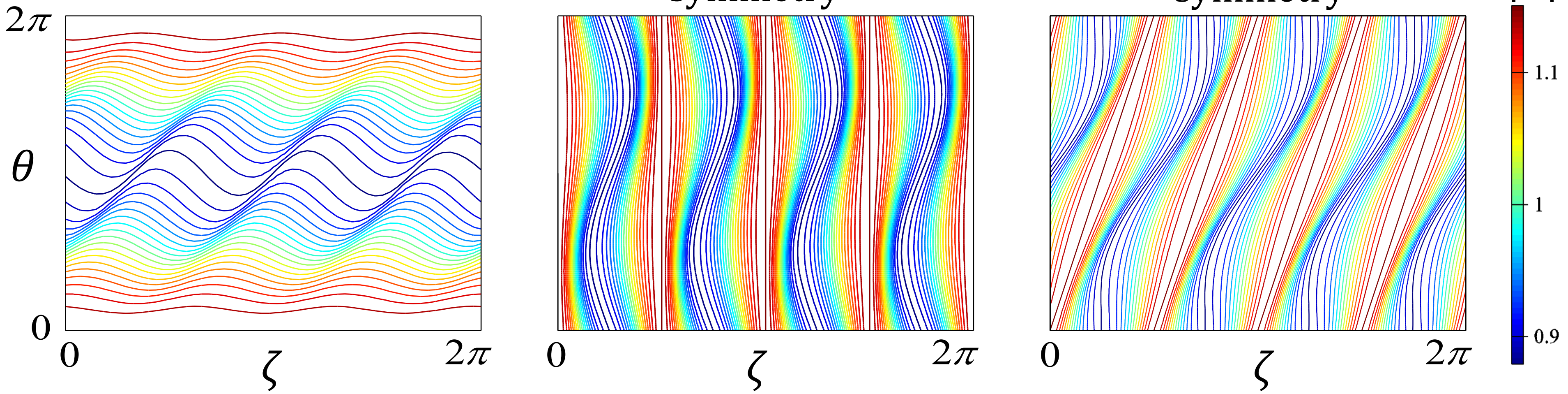
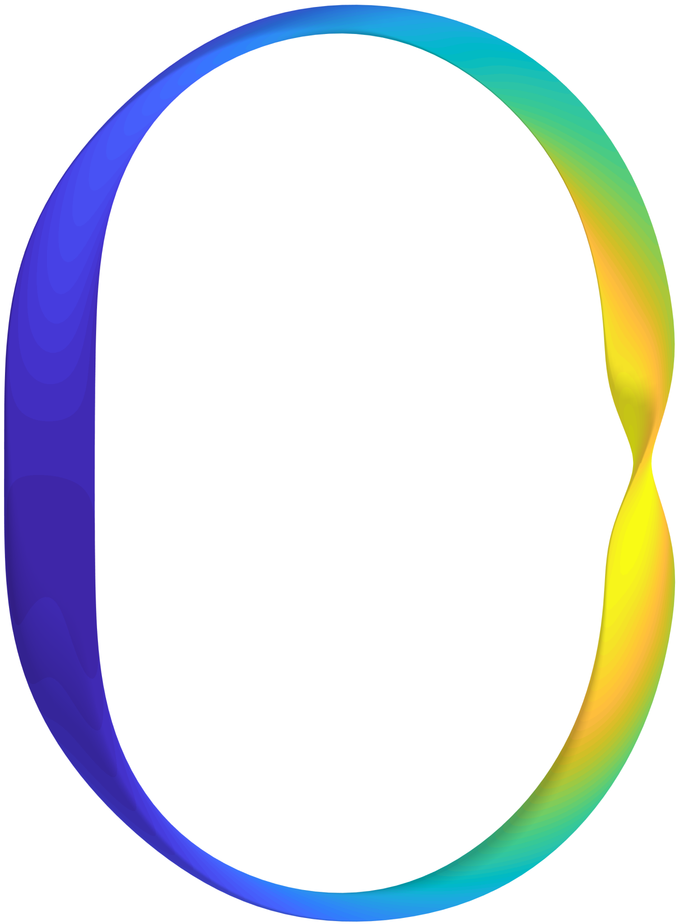
Generalized quasi-axisymmetry

Generalized quasi-poloidal symmetry

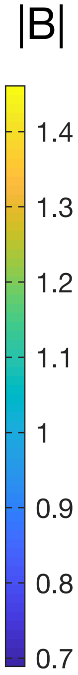
Generalized quasi-helical symmetry



The near-axis analysis can be generalized to construct omnigenous configurations



G G Plunk, ML, and P Helander,
In preparation



$$\boxed{\nabla_{\perp} B = B\kappa\mathbf{n}}$$

Quasi-poloidal symmetry is not possible near the axis, but omnigenity is.

Outline

- Construction for $O(r)$ quasisymmetry
  - Theory, & the number of solutions
  - Numerical results
  - Comparison to "real experiments"
  - The landscape of solutions
- Extensions
  - Omnigenity
  - $O(r^2)$ quasisymmetry

Extending the construction to higher order is tricky

- We can only "half-specify" the axis shape:
  - A curve like the axis is given by 2 functions, e.g. {curvature, torsion} or $\{R(\boldsymbol{\phi}), Z(\boldsymbol{\phi})\}$.
  - At $O(r)$, (# unknowns)-(# equations)=2 so we can specify (almost) any axis. But at $O(r^2)$, (# unknowns)-(# equations)=1 so we cannot.

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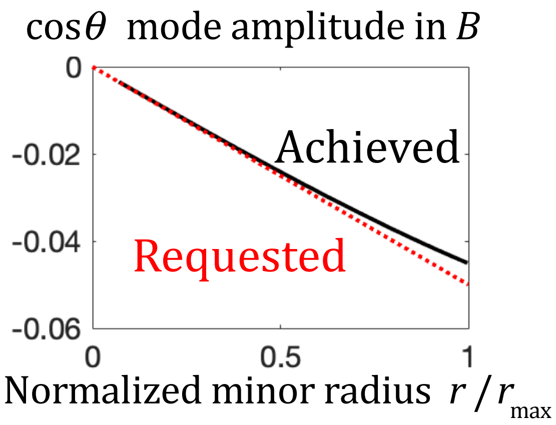
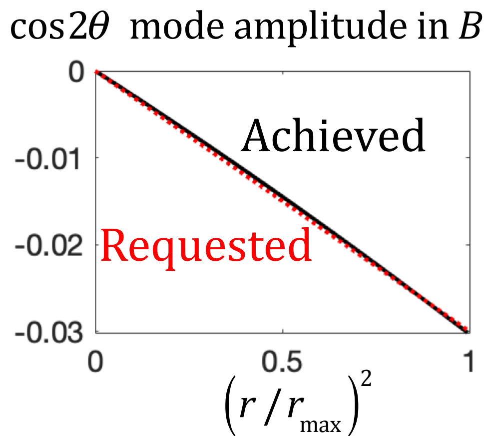
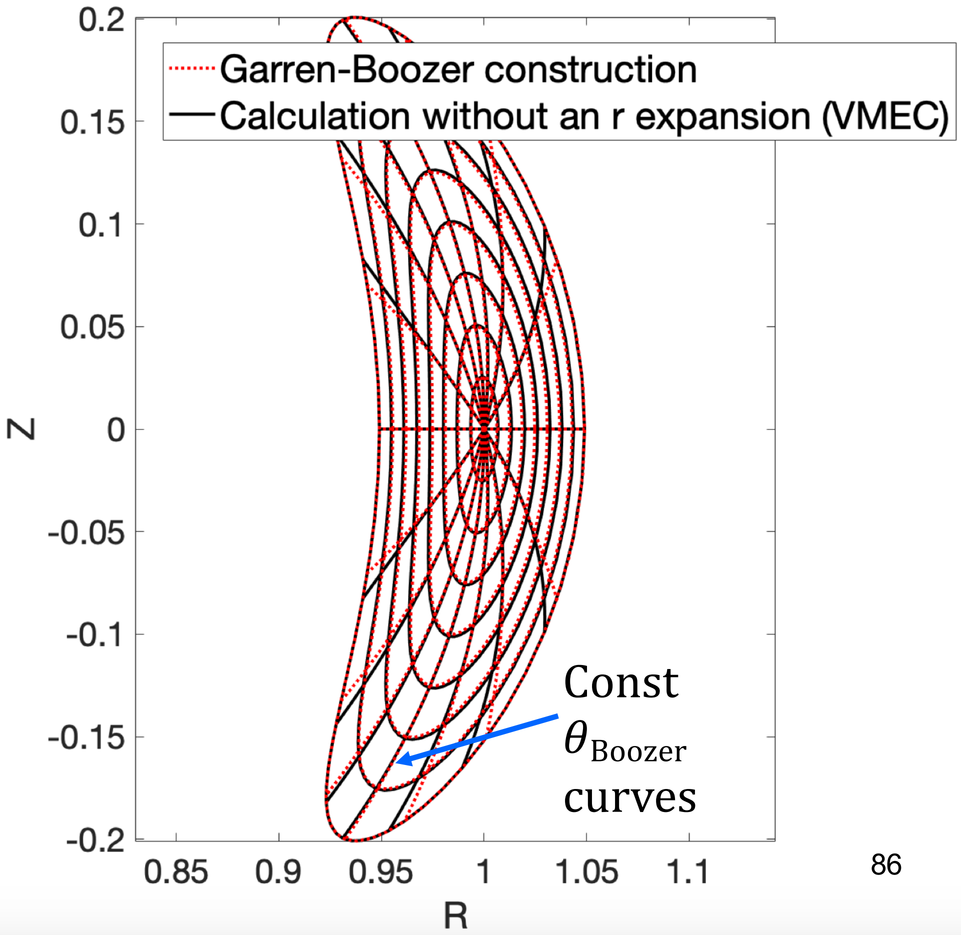
  - At $O(r)$, (# unknowns)-(# equations)=2 so we can specify (almost) any axis. But at $O(r^2)$, (# unknowns)-(# equations)=1 so we cannot.
- No existence & uniqueness theorem for solutions (yet).
- Magnetic shear (variation of rotational transform) does not appear until $O(r^3)$.

We are working to extend the construction to $O(r^2)$, enabling greater shaping

Axisymmetric example:







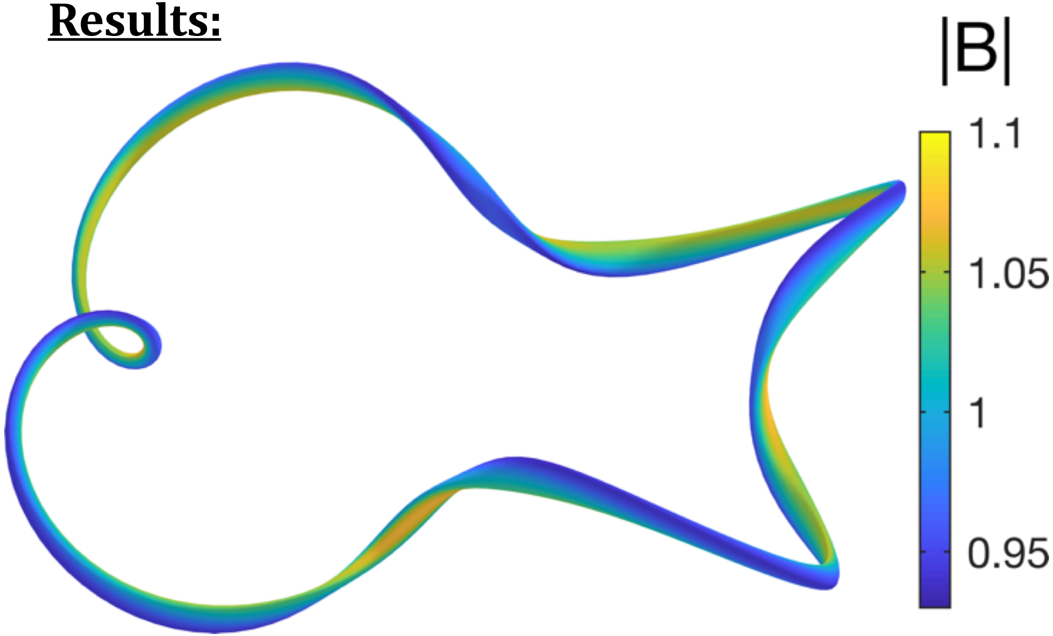
We now have a recipe for generating quasisymmetric VMEC input files: Set *r* to a small finite value *a*.

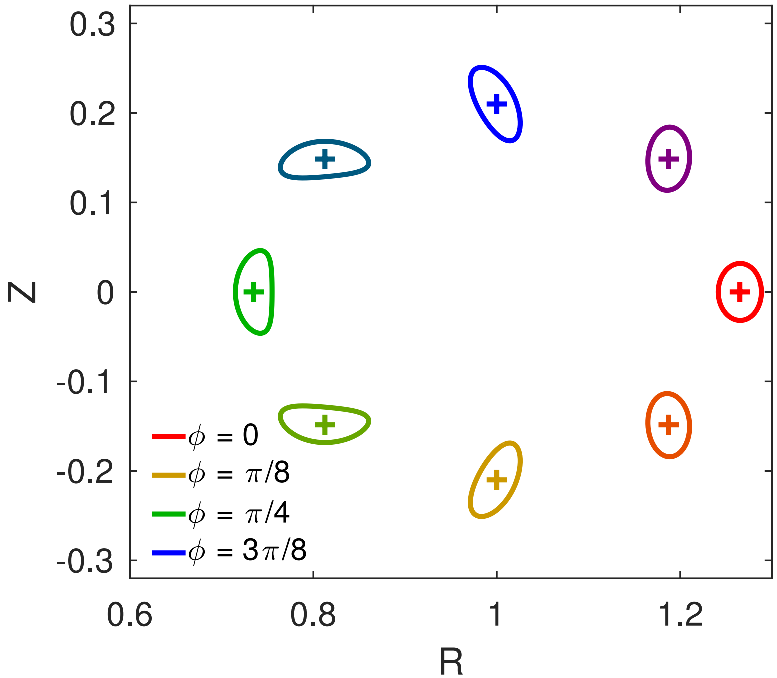
Inputs:

$$\text{axis shape } R_0(\phi) = 1 + 0.265\cos(4\phi), \quad Z_0(\phi) = -0.21\sin(4\phi),$$

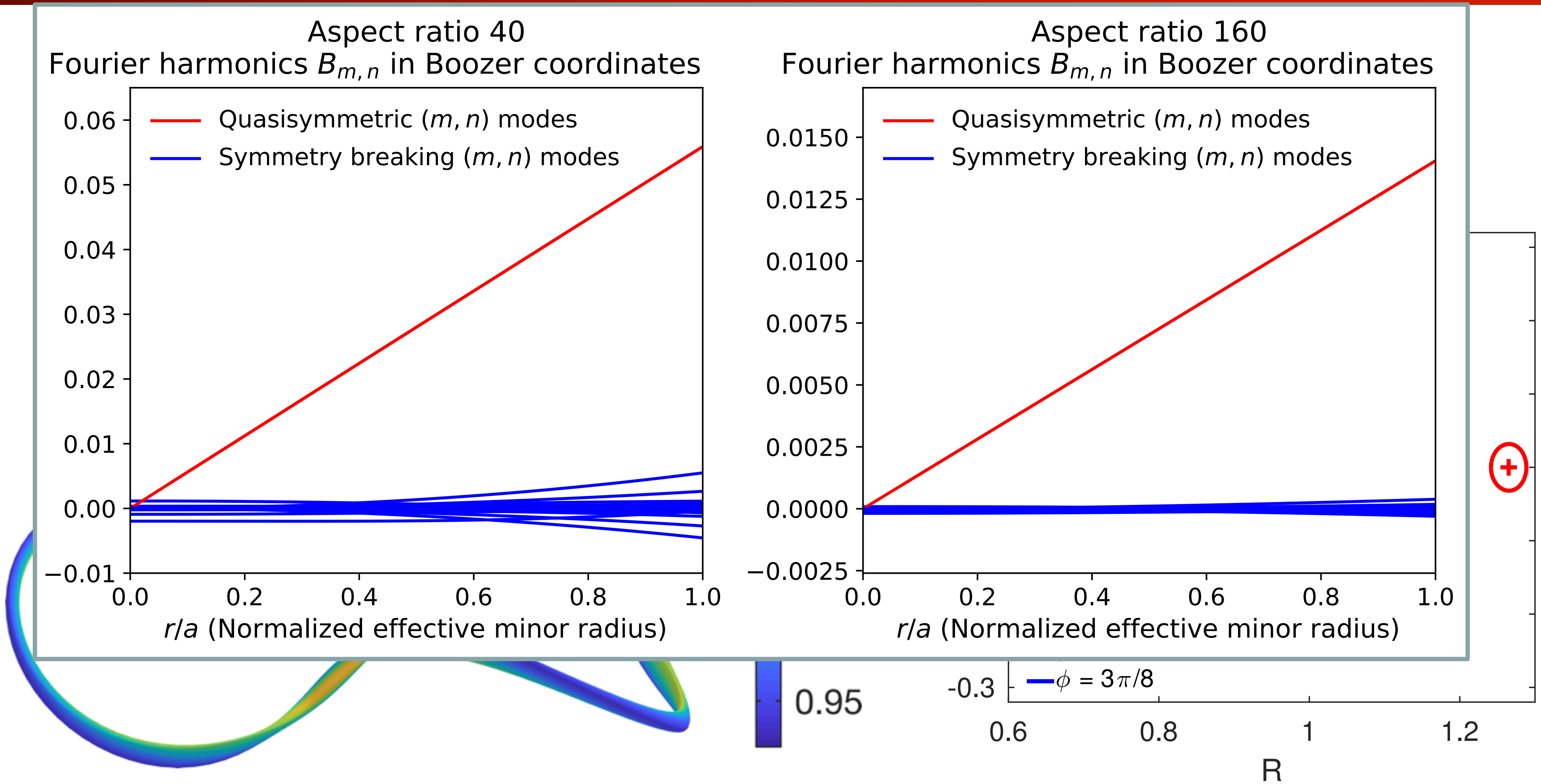
$$I_2 = 0, \quad \bar{\eta} = -2.25, \quad \sigma(0) = 0, \quad R/a = 40.$$

Results:





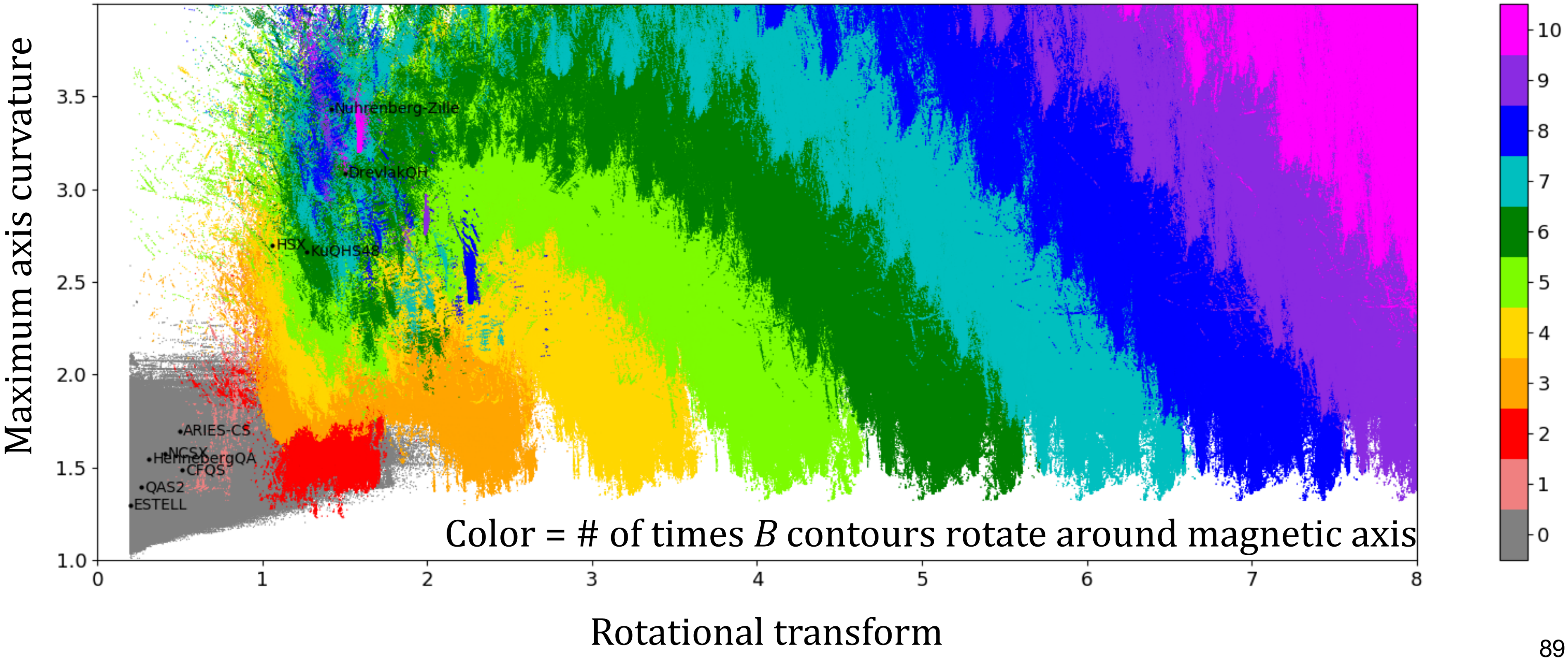
The construction can be verified by comparing to VMEC + BOOZ_XFORM.



The fast construction enables brute-force surveys of "all" quasisymmetric fields

Axis shape: $R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(j n_{fp} \phi), \quad Z_0(\phi) = 1 + \sum_{j=1}^{3} Z_j \sin(j n_{fp} \phi)$

$2.4\text{x}10^8$ configurations



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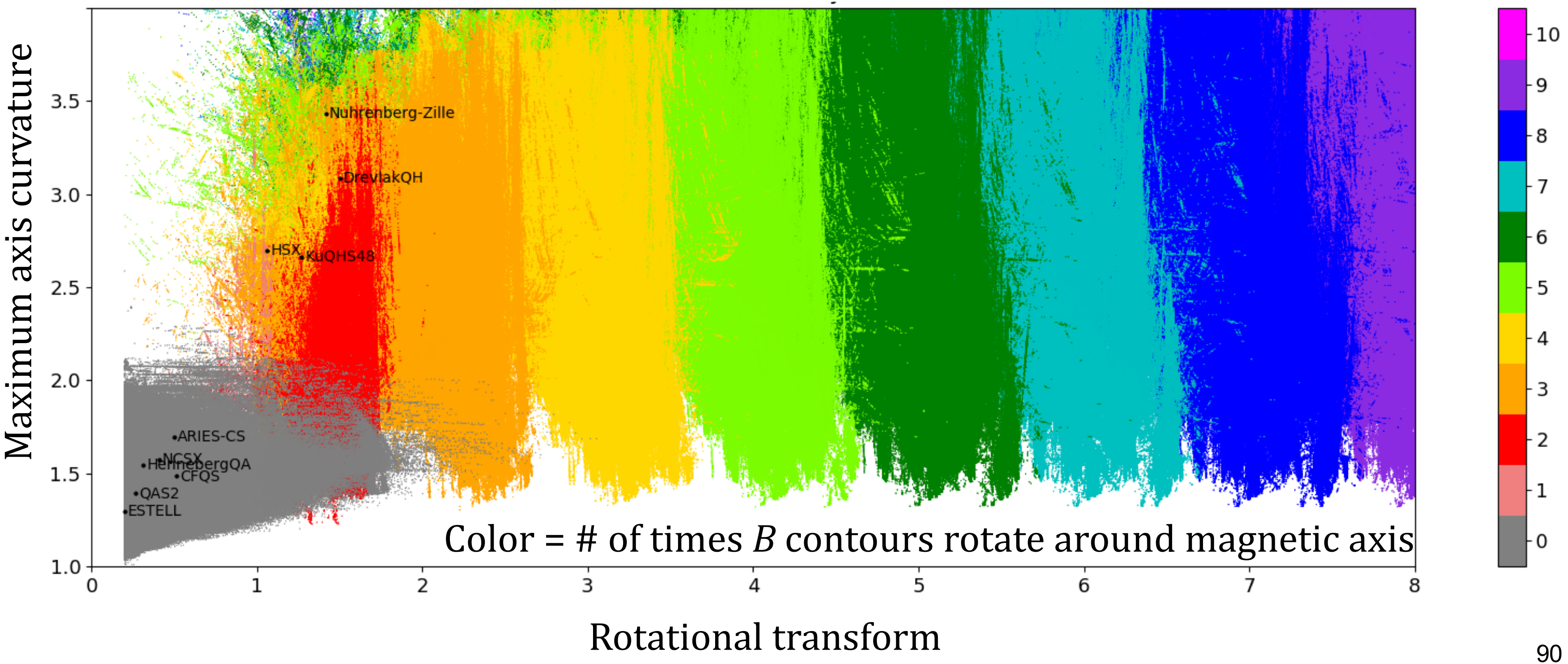
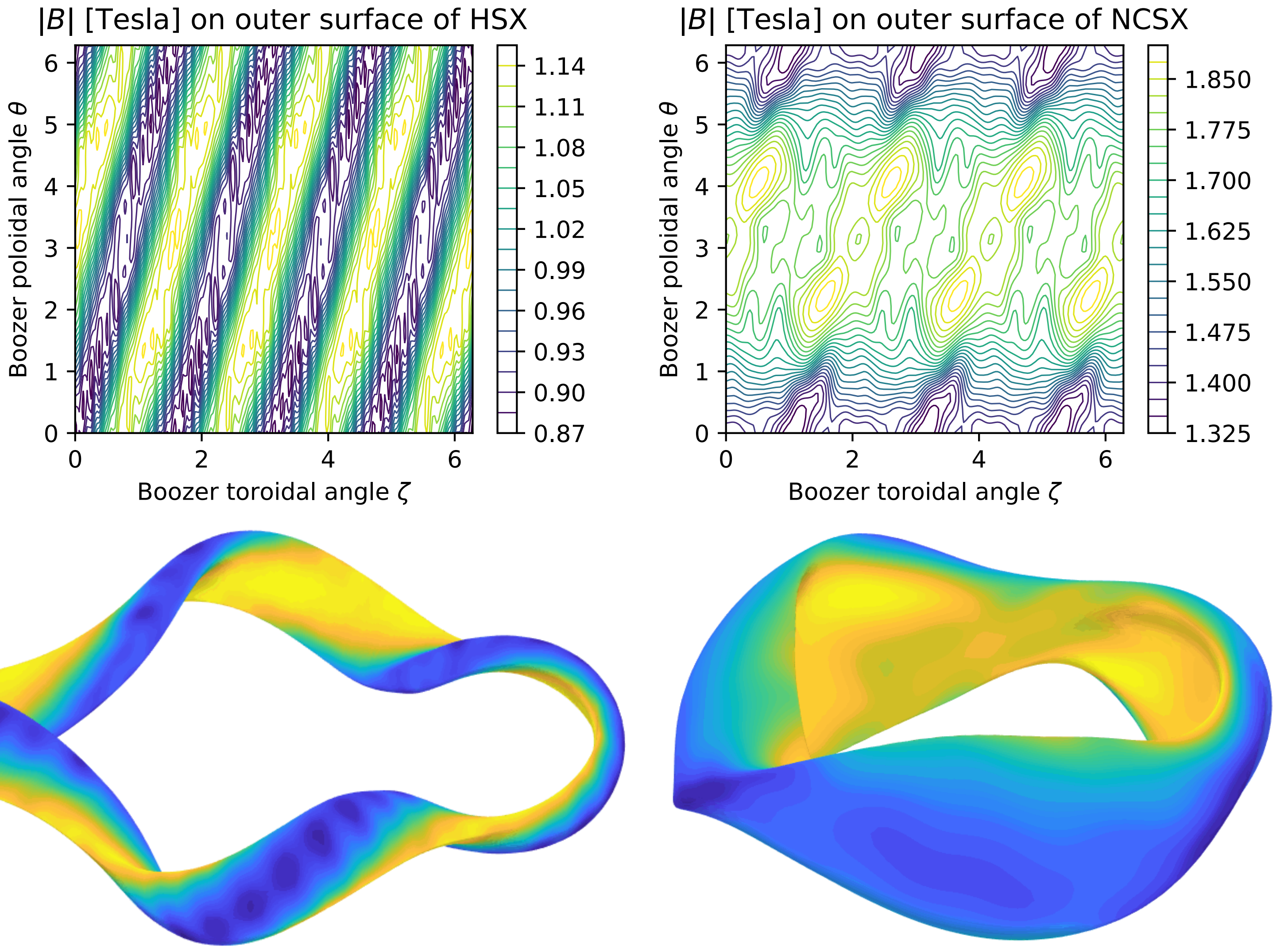


Quasisymmetric experiments to date actually have significant departures from symmetry.



Example of the $O(r^2)$ construction

Inputs: axis shape $R_0(\phi) = 1 + 0.173\cos(2\phi) + 0.0168\cos(4\phi) + 0.00101\cos(6\phi)$,

$$Z_0(\phi) = \quad 0.158\sin(2\phi) + 0.0165\sin(4\phi) + 0.000985\sin(6\phi),$$

$$I_2 = 0,\ \ \sigma(0) = 0,\ \ \bar{\eta} = 0.632,\ \ p_2 = 0,\ \ B_{2c} = -0.158,\ \ B_{2s} = 0,\ \ R/a = 10$$

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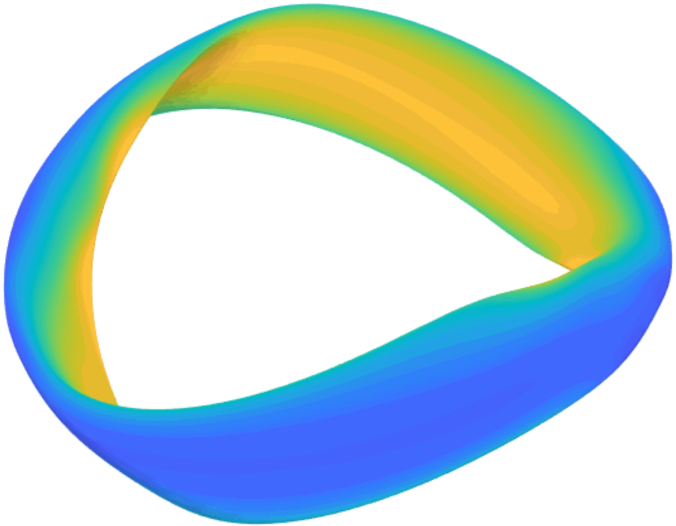
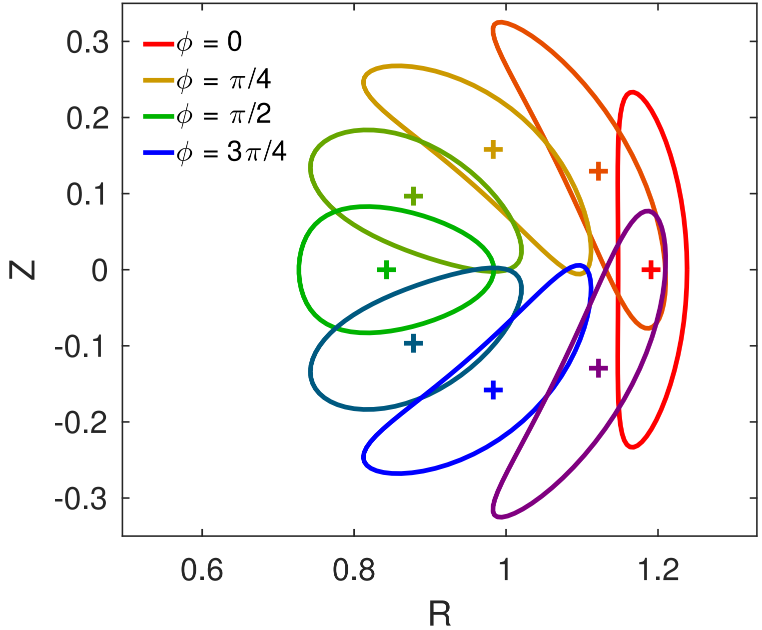
Results: $\iota = 0.424$





The $O(r^2)$ construction allows triangularity and more accurate quasisymmetry.

$$\mathbf{x}(r,\vartheta,\zeta)=\mathbf{x}_0(\zeta)+X(r,\vartheta,\zeta)\mathbf{n}(\zeta)+Y(r,\vartheta,\zeta)\mathbf{b}(\zeta)+Z(r,\vartheta,\zeta)\mathbf{t}(\zeta)$$

$$X(r,\vartheta,\zeta)=r\left[X_{1c}\cos\vartheta+X_{1s}\sin\vartheta\right]+r^2\left[X_{20}+X_{2c}\cos2\vartheta+X_{2s}\sin2\vartheta\right]+O(r^3)$$

Same for Y & Z.

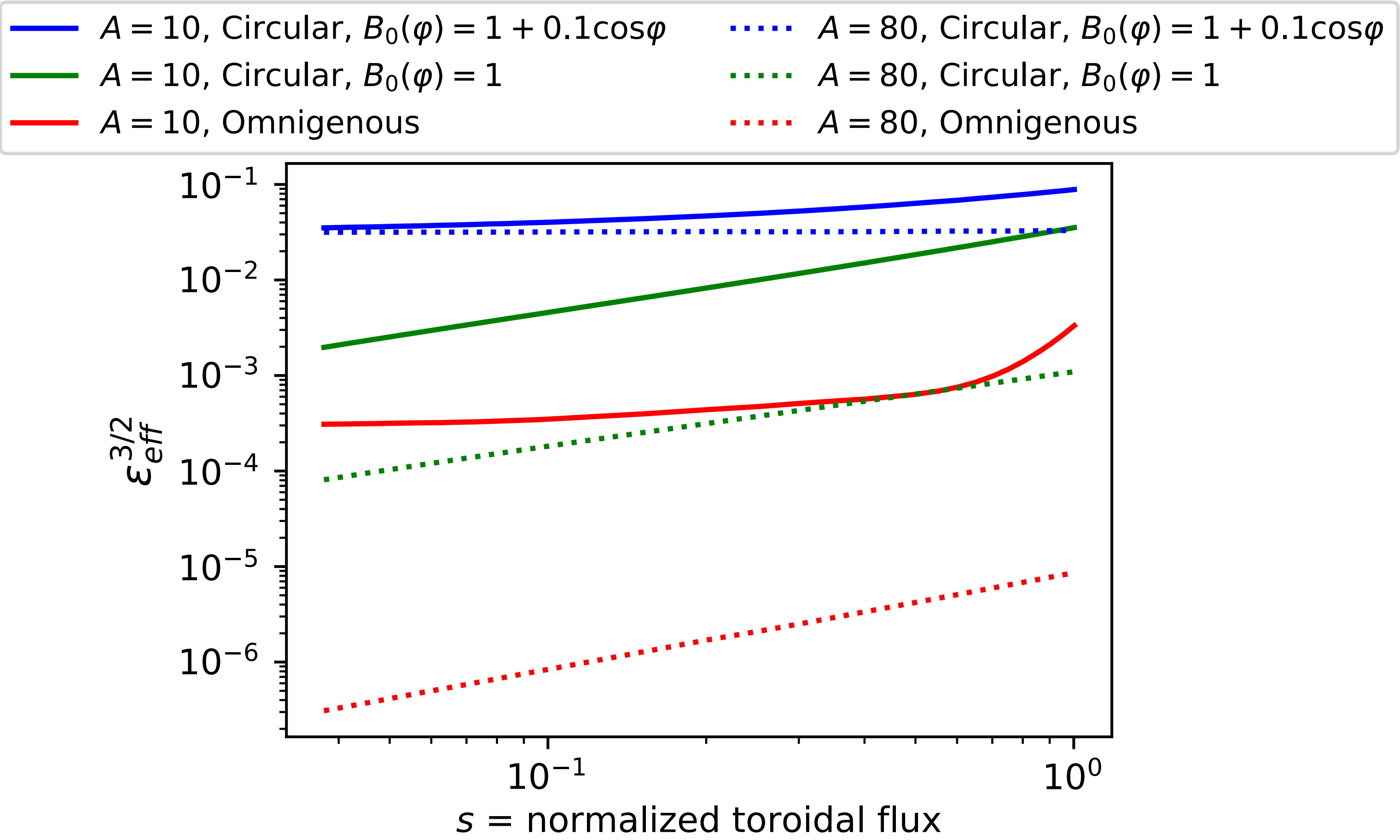
- 3 new input parameters: p_2, B_{2c}, B_{2s}.

$$p(r)=p_0+r^2p_2+O(r^4)$$

$$B(r,\vartheta,\varphi)=B_0+rB_0\bar{\eta}\cos\vartheta+r^2\left[B_{20}+B_{2c}\cos2\vartheta+B_{2s}\sin2\vartheta\right]+O(r^3)$$

- Net loss of 1 degree of freedom. My approach: $B_{20}(\zeta)$ is an output. Need to adjust inputs so $B_{20}(\zeta)\approx$ constant.
- Shafranov shift appears at this order. Matches textbook tokamak result (e.g. *Wesson, Hazeltine & Meiss*):

$$\left(R-R_0-\Delta\right)^2+z^2=r^2,\qquad \Delta=r^2\left(\frac{1}{8R_0}-\frac{\mu_0p_2R_0}{2\iota^2B_0^2}\right)$$



$A = 10$, Circular, $B_0(\varphi) = 1 + 0.1\cos\varphi$
$A = 10$, Circular, $B_0(\varphi) = 1$
$A = 10$, Omnigenous
$A = 80$, Circular, $B_0(\varphi) = 1 + 0.1\cos\varphi$
$A = 80$, Circular, $B_0(\varphi) = 1$
$A = 80$, Omnigenous
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$\epsilon_{eff}^{3/2}$
10^{-1}
10^{0}
s = normalized toroidal flux

Garren & Boozer (1991): Write position vector **x** using axis's Frenet frame, expand in small *r*

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}): \quad \frac{d\mathbf{x}_0}{d\ell}=\mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n}, \quad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b}, \quad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

$\mathbf{x}_0=$ magnetic axis, $\kappa=$ curvature, $\tau=$ torsion, $\mathbf{t}=$ tangent, $\mathbf{n}=$ normal, $\mathbf{b}=$ binormal

Results for quasisymmetry through $O(r)$:

$$\mathbf{x}(r,\theta,\zeta)=\mathbf{x}_0(\zeta)+r\frac{\bar{\eta}}{\kappa(\zeta)}\cos\vartheta\,\mathbf{n}(\zeta)+r\left[\frac{\kappa(\zeta)}{\bar{\eta}}\sin\vartheta+\frac{\sigma(\zeta)\kappa(\zeta)}{\bar{\eta}}\cos\vartheta\right]\mathbf{b}(\zeta)+O(r^2)$$

Toroidal angle $\zeta \propto$ axis arclength ℓ, $\quad \bar{\eta}=$ constant: $B=B_0\left[1+r\bar{\eta}\cos\vartheta+O(r^2)\right]$

$$\boxed{\frac{d\sigma}{d\zeta}+\iota\left[\frac{\bar{\eta}^4}{\kappa^4}+1+\sigma^2\right]-2\frac{\bar{\eta}^2}{\kappa^2}\left[I_2-\tau\right]=0}$$

$\vartheta=\theta-N\zeta$,

$\iota=$ rotational transform on axis,

$I_2=$ current density on axis

The size of the space of fields that are quasisymmetric to O(r) can be precisely understood.

<u>Inputs:</u>

- Shape of the magnetic axis. (Determines QA vs QH.)
- 3 real numbers:
  - I_2: Current density on the axis. (Usually 0).
  - $\sigma(0)$: Rotation of the elliptical flux surfaces at toroidal angle=0.
  - $\bar{\eta}$, which controls elongation and field strength: $B = B_0\left[1 + r\bar{\eta}\cos\vartheta + O\left(r^2\right)\right]$
- (Pressure doesn't matter to this order.)

Theorem: Given this data, a quasisymmetric solution exists, & it is unique.

$$\boxed{\frac{d\sigma}{d\zeta} + \iota\left[\frac{\bar{\eta}^4}{\kappa^4} + 1 + \sigma^2\right] - 2\frac{\bar{\eta}^2}{\kappa^2}\left[I_2 - \tau\right] = 0}$$

Conclusions

- The equations for quasisymmetric magnetic fields can be solved directly and rapidly if you expand about the magnetic axis.
- The resulting construction can be useful for generating new initial conditions for optimization.
- We precisely understand the size of the space of magnetic fields that are quasisymmetric near the axis (to $O(r)$).
- There is hope of definitively identifying all regions of parameter space with practical quasisymmetric fields (near the axis).
- We can discover qualitatively new magnetic configurations for fusion.

J Plasma Phys 84, 905840616 (2018) J Plasma Phys 85, 905850103 (2019) PPCF 61, 075001 (2019)

arXiv:1909.08919 arXiv:1908.10253 github.com/landreman/quasisymmetry

Parameter space (independent variables)

- Coil shapes: arbitrary 3D curves
- Coil currents
- Input parameters of the Garren-Boozer near-axis quasisymmetry equations:
  - Shape of magnetic axis (independent from the axis actually produced by coils!)
  - $\bar{\eta}$ $\qquad B = B_0\left[1 + r\bar{\eta}\cos\left(\theta - N\zeta\right) + O\left(r^2\right)\right]$

Objective function

$$f=\left(\frac{L_c-L_{c0}}{L_{c0}}\right)^2+\left(\frac{L_a-L_{a0}}{L_{a0}}\right)^2+\left(\frac{\iota-\iota_0}{\iota_0}\right)^2$$

$$+\oint_{\text{Garren-Boozer axis}} d\ell\left|\mathbf{B}_{coils}-\mathbf{B}_{Garren-Boozer}\right|^2+\oint_{\text{Garren-Boozer axis}} d\ell\left|\nabla\mathbf{B}_{coils}-\nabla\mathbf{B}_{Garren-Boozer}\right|^2$$

Differentiate Biot-Savart law

L_c = Total length of coils

L_{c0} = Target length of coils

L_a = Length of Garren-Boozer magnetic axis

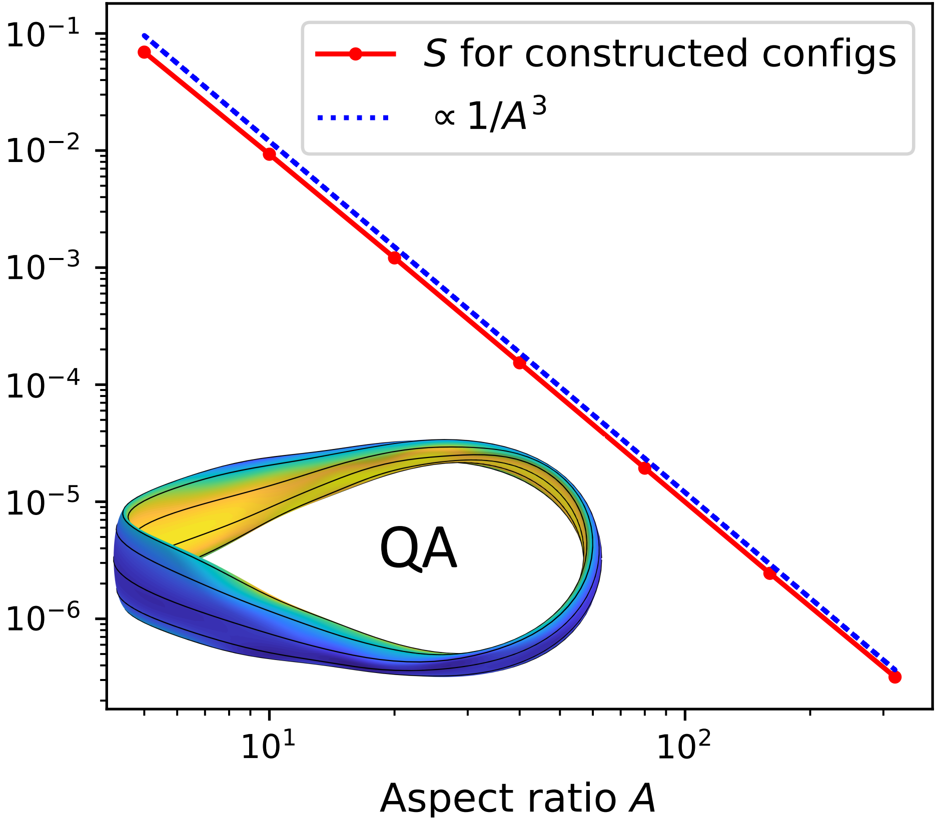
L_{a0} = Target length of magnetic axis

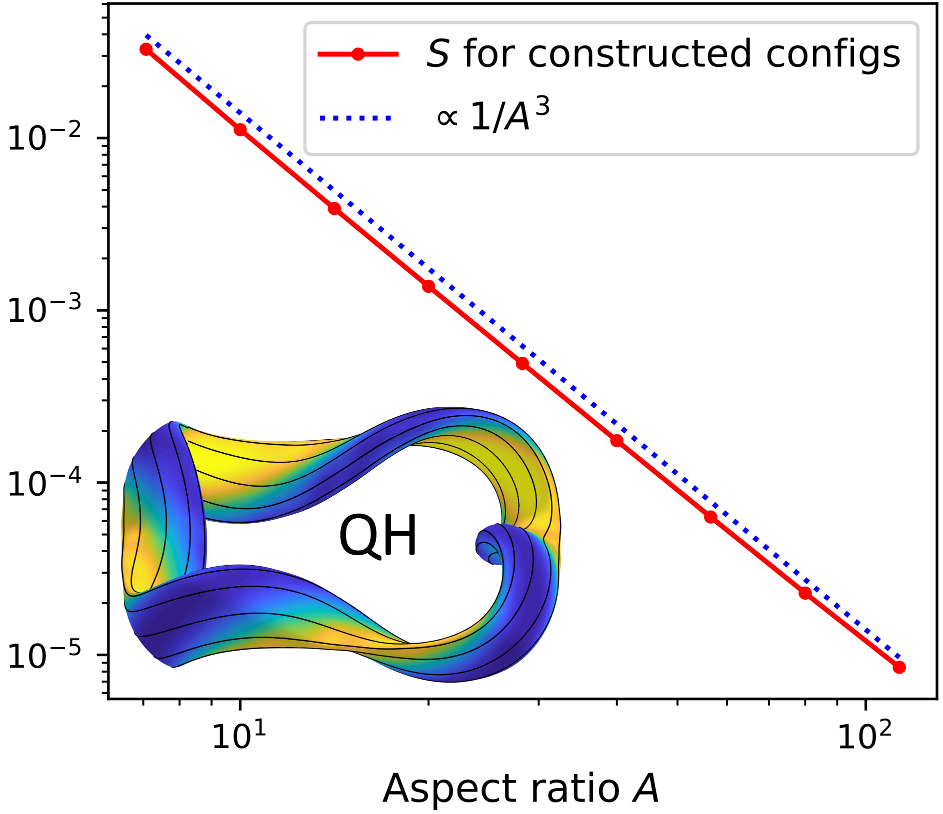
ι = Rotational transform from Garren-Boozer

ι_0 = Target rotational transport

We can now numerically demonstrate Garren & Boozer's scaling: $B_{nonsymm} \sim 1/A^3$

$$S = \frac{1}{B_0}\sqrt{\sum_{m,n\neq Nm} B^2_{m,n}} = \text{Symmetry-breaking}$$





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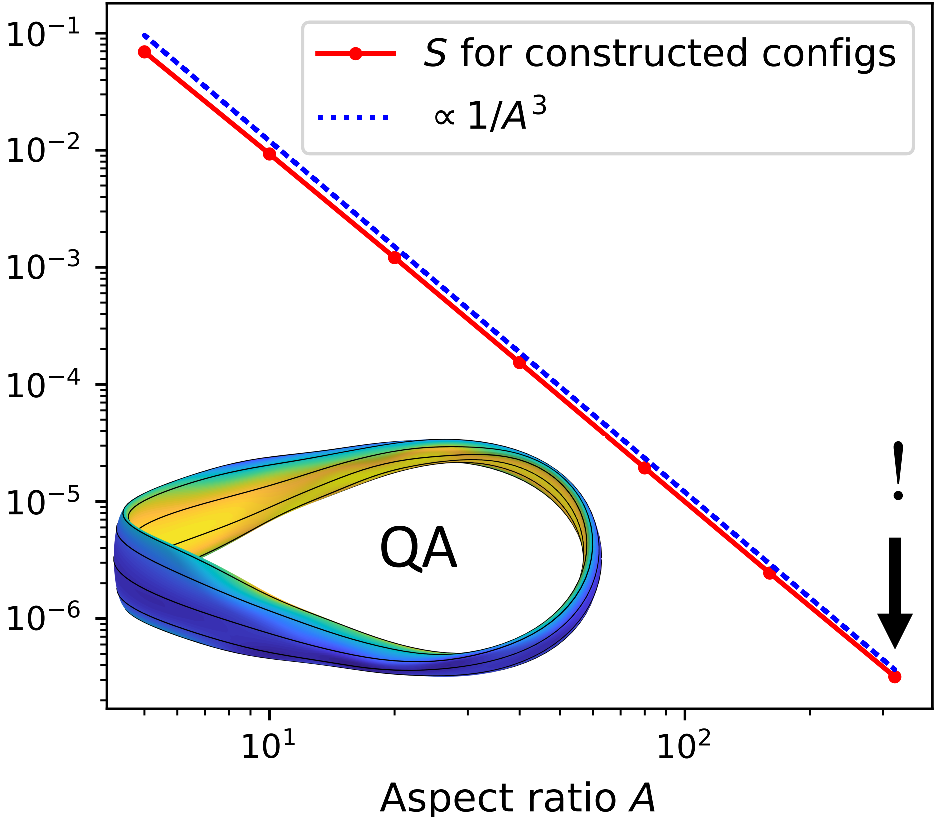
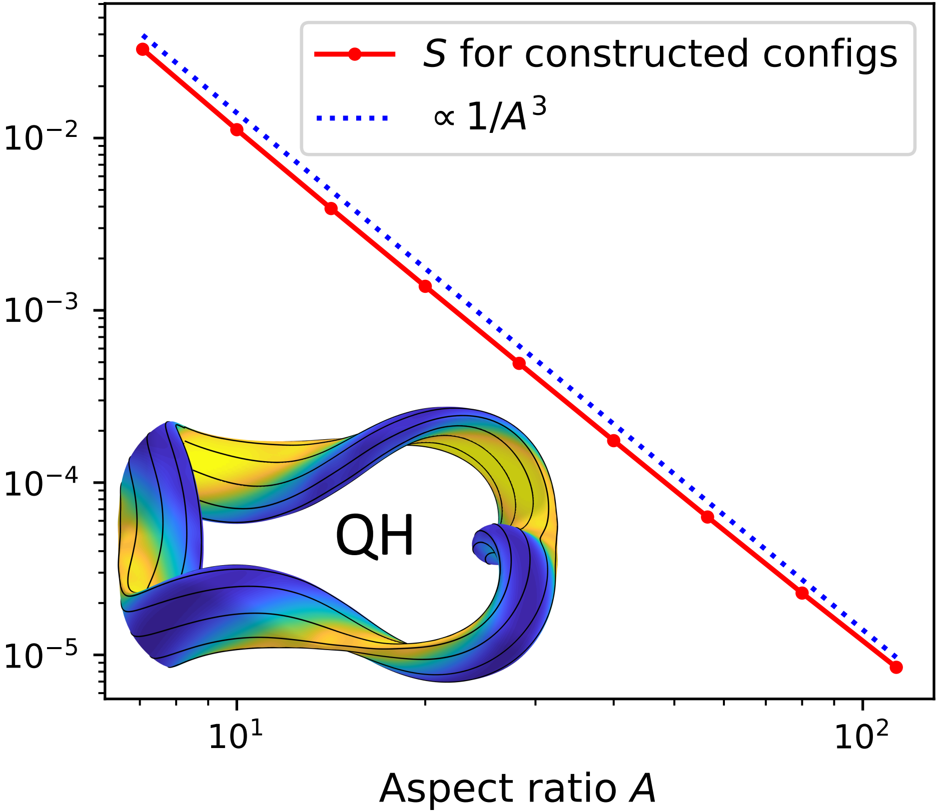
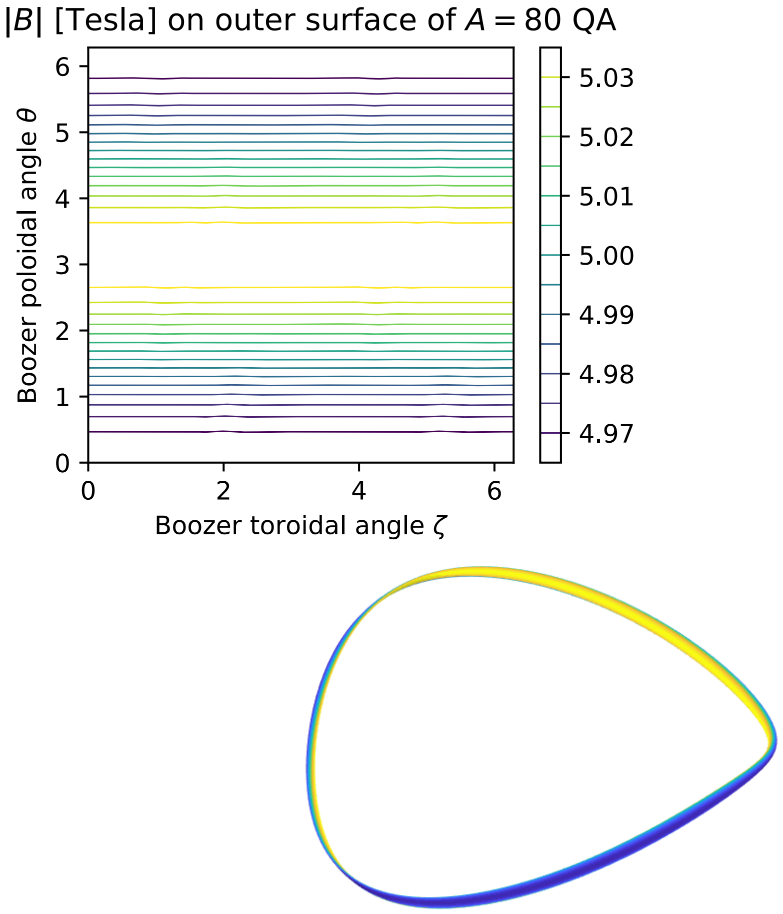
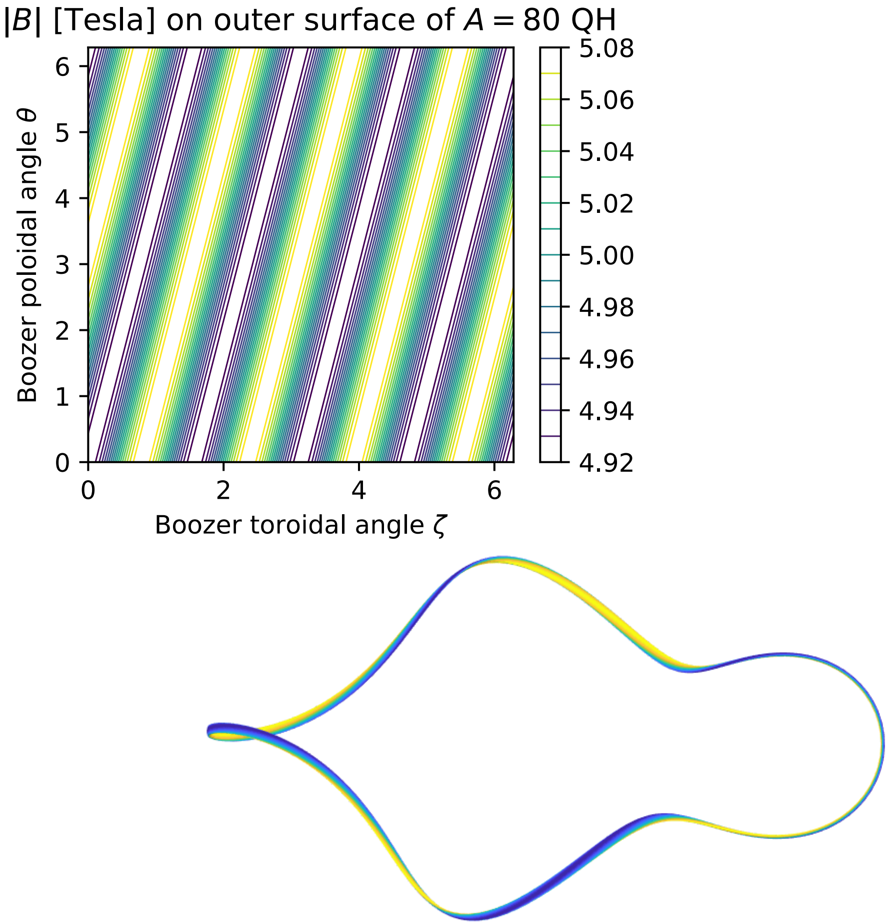




Quasisymmetry can be achieved to any desired precision, e.g. $B_{\text{nonsymm}} \leq B_{\text{Earth}}$





Garren & Boozer (1991): Write position vector **x** using axis's Frenet frame, expand in small *r*

$$\text{Frenet frame } \left(\mathbf{t},\mathbf{n},\mathbf{b}\right): \quad \frac{d\mathbf{x}_0}{d\ell}=\mathbf{t}, \qquad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n}, \qquad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b}, \qquad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

$$\mathbf{x}_0=\text{magnetic axis}, \qquad \kappa=\text{curvature}, \qquad \tau=\text{torsion}, \qquad \mathbf{t}=\text{tangent}, \qquad \mathbf{n}=\text{normal}, \qquad \mathbf{b}=\text{binormal}$$

$$\mathbf{x}\left(r,\theta,\zeta\right)=\mathbf{x}_0\left(\zeta\right)+X\left(r,\theta,\zeta\right)\mathbf{n}\left(\zeta\right)+Y\left(r,\theta,\zeta\right)\mathbf{b}\left(\zeta\right)+Z\left(r,\theta,\zeta\right)\mathbf{t}\left(\zeta\right), \qquad r=\sqrt{2\psi/B_0}$$

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$\mathbf{x}_0=$ magnetic axis, $\kappa=$ curvature, $\tau=$ torsion, $\mathbf{t}=$ tangent, $\mathbf{n}=$ normal, $\mathbf{b}=$ binormal

$$\mathbf{x}(r,\theta,\zeta)=\mathbf{x}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta), \qquad r=\sqrt{2\psi/B_0}$$

$$X(r,\theta,\zeta)=rX_1(\theta,\zeta)+r^2X_2(\theta,\zeta)+\ldots$$

Same for Y & Z.

Garren & Boozer (1991): Write position vector **x** using axis's Frenet frame, expand in small *r*

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$$\nabla r=\frac{\dfrac{\partial\mathbf{x}}{\partial\theta}\times\dfrac{\partial\mathbf{x}}{\partial\zeta}}{\dfrac{\partial\mathbf{x}}{\partial r}\cdot\dfrac{\partial\mathbf{x}}{\partial\theta}\times\dfrac{\partial\mathbf{x}}{\partial\zeta}},\quad \text{\& cyclic permutations}$$

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Same for Y & Z.

$$\nabla r=\frac{\dfrac{\partial\mathbf{x}}{\partial\theta}\times\dfrac{\partial\mathbf{x}}{\partial\zeta}}{\dfrac{\partial\mathbf{x}}{\partial r}\cdot\dfrac{\partial\mathbf{x}}{\partial\theta}\times\dfrac{\partial\mathbf{x}}{\partial\zeta}}, \quad \text{\& cyclic permutations}$$

$$\mathbf{B}=\frac{d\psi}{dr}\left[\nabla r\times\nabla\theta+\iota\left(r\right)\nabla\zeta\times\nabla r\right]$$

$$=\beta\left(r,\theta,\zeta\right)\frac{d\psi}{dr}\nabla r+I\left(r\right)\nabla\theta+G\left(r\right)\nabla\zeta$$

Garren & Boozer (1991): Write position vector **x** using axis's Frenet frame, expand in small *r*

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}): \quad \frac{d\mathbf{x}_0}{d\ell}=\mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n}, \quad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b}, \quad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

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$$\mathbf{x}(r,\theta,\zeta)=\mathbf{x}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta), \qquad r=\sqrt{2\psi/B_0}$$

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$$=\beta(r,\theta,\zeta)\frac{d\psi}{dr}\nabla r+I(r)\nabla\theta+G(r)\nabla\zeta$$

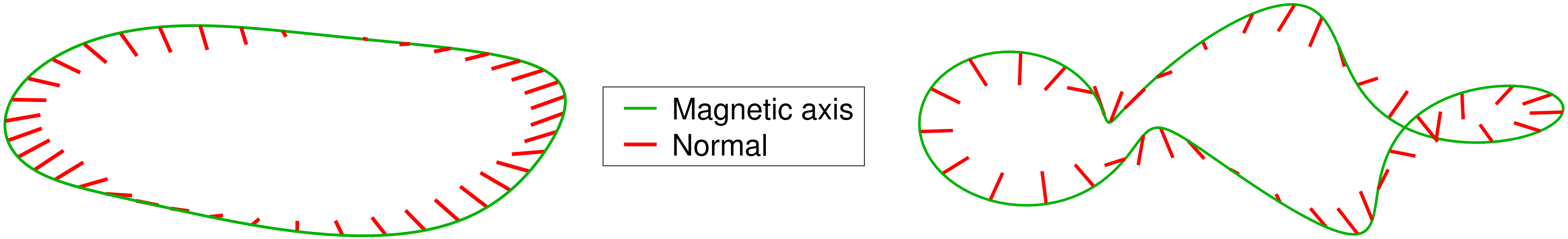
$$(\nabla\times\mathbf{B})\times\mathbf{B}=\mu_0\frac{dp}{dr}\nabla r, \qquad B(r,\theta,\zeta)=B_0+rB_{1c}\cos(\theta-N\zeta)+O(r^2).$$

Expand in $r\ll\kappa^{-1}$.

Garren & Boozer's equations yield a practical algorithm

<u>Inputs:</u>

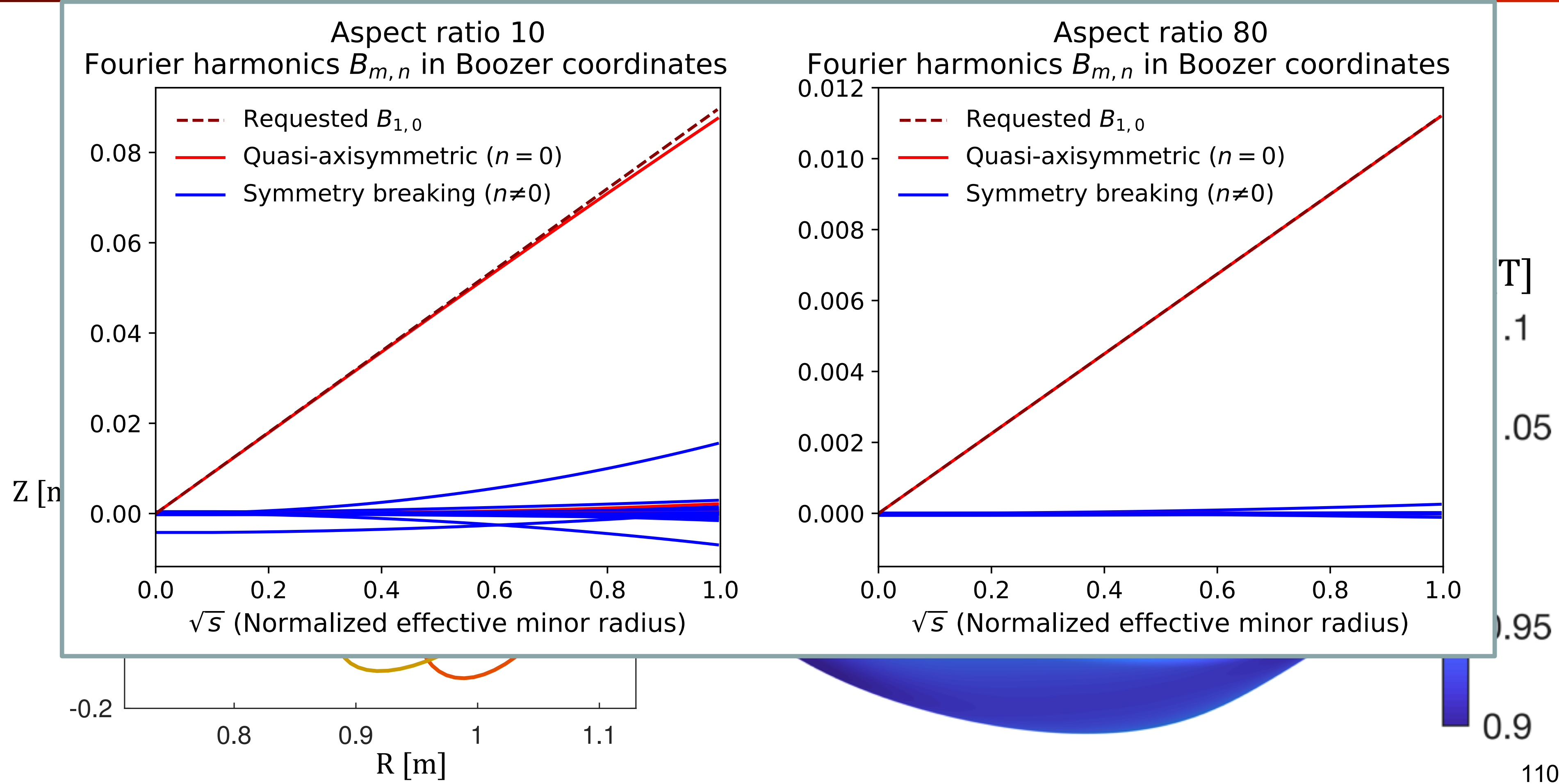
- Shape of the magnetic axis, with $\kappa \neq 0$. (Determines QA vs QH.)
- 3 real numbers:
  - I_2: Current density on the axis. (Usually 0).
  - $\sigma(0)$: Rotation of the elliptical flux surfaces at toroidal angle=0.
  - $\bar{\eta}$, which controls elongation and field strength: $B = B_0\left[1 + r\bar{\eta}\cos\left(\theta - N\zeta\right) + O\left(r^2\right)\right]$
- (Pressure doesn't matter to this order.)



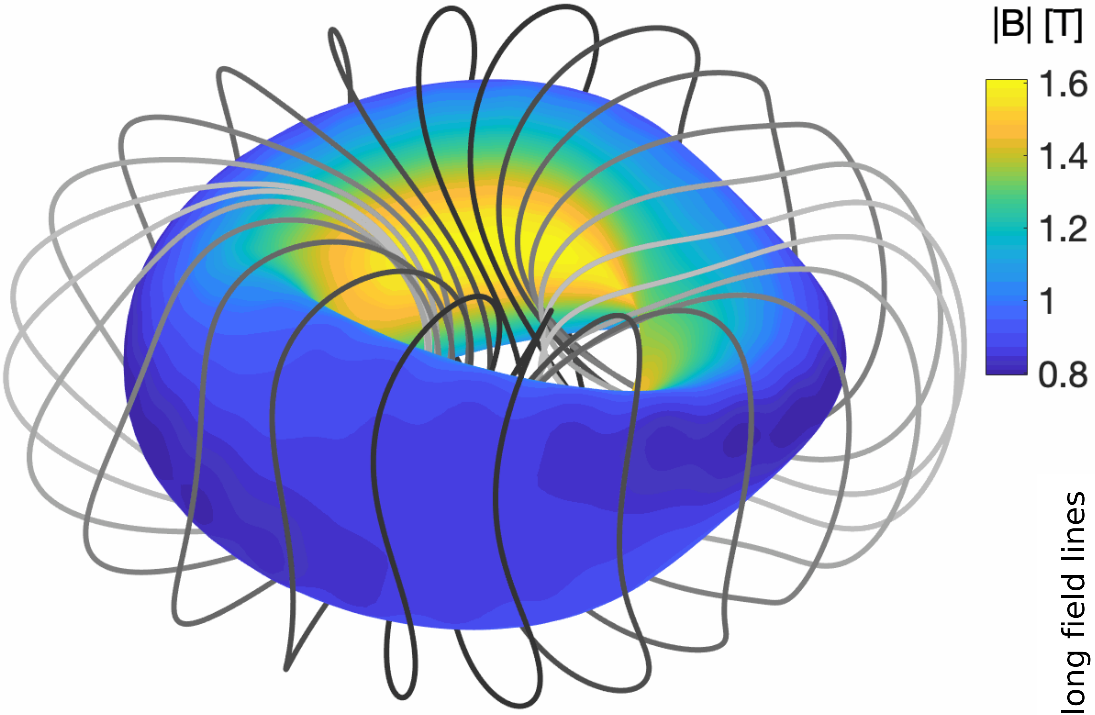
$\Rightarrow$ $N = 0$: Quasi-axisymmetry

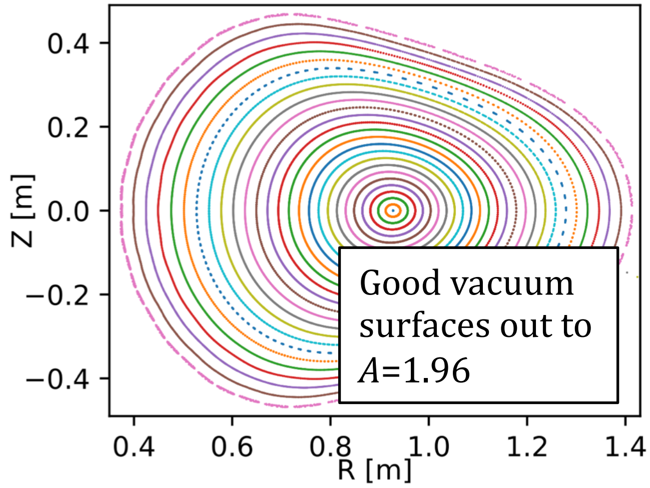
$\Rightarrow$ $N \neq 0$: Quasi-helical symmetry

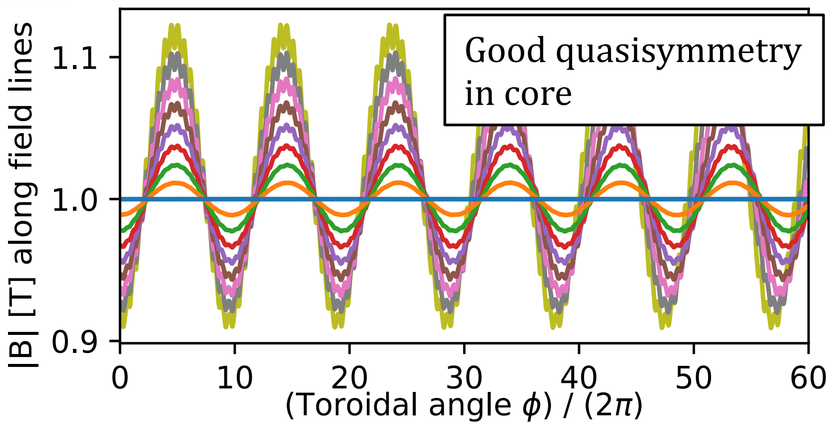
The construction can be verified by running an MHD equilibrium code (VMEC) which does not make the expansion.



The axis expansion enables a combined (1-stage) coil + quasisymmetry optimization using analytic derivatives







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