A new & faster method to generate transport-optimized stellarators

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github.com/landreman/quasisymmetry
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$$\min_X f(X)$$

Parameter space: $X = \text{toroidal boundary shapes}$

Objective: $f = \sum_{m,n \neq Nm} B_{m,n}^2 (r_0)$ where $B(r,\theta,\zeta) = \sum_{m,n} B_{m,n} (r) \exp(im\theta - in\zeta)$
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where $$B(r,\theta,\zeta) = \sum_{m,n} B_{m,n}(r) \exp(im\theta - in\zeta)$$

- Computationally expensive.
- What is the size & character of the solution space?
- Result depends on initial condition, so cannot be sure you’ve found all solutions.
Expansion about the magnetic axis can be a powerful practical tool for generating quasisymmetric & omnigenous stellarators

- Accurate at least in the core of any configuration.
- Hasn’t been considered much since numerical optimization began in ~1980s.
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- Complements the traditional optimization approach:
  - Many orders of magnitude faster.
  - You can parameterize all possible solutions.
  - Can generate new initial conditions that can be refined by optimization.
We have translated analytic work by Garren & Boozer (1991) into practical algorithms.

Fundamental equations:

\[ x(r, \theta, \zeta) = x_0(\zeta) + \text{Taylor series in } r = \sqrt{2\psi / B_0}, \]

\[ (\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p, \]

\[ \mathbf{B} = \nabla \psi \times \nabla \theta + i \nabla \zeta \times \nabla \psi = \beta \nabla \psi + I(\psi) \nabla \theta + G(\psi) \nabla \zeta, \quad B(r, \theta, \zeta) = B_0 \left[ 1 + r \bar{\eta} \cos(\theta - N \zeta) + O(r^2) \right] \]

Mercier's inverse expansion of \( \psi(x) \) gives equivalent results.

Rogerio Jorge & Wrick Sengupta.
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\]

**Algorithm Inputs:**

- **Shape of the magnetic axis, with \( \kappa \neq 0 \). (Determines QA vs QH.)**
- **3 numbers:**
  - \( I_2 \): Current density on the axis. (Usually 0).
  - Rotation of the elliptical flux surfaces at \( \zeta=0 \). (Usually 0).
  - \( \bar{\eta} \), which controls elongation and field strength.

**Theorem:** Given this data, a unique \( O(r) \) quasisymmetric solution exists.

\[\Rightarrow \text{The space of configurations that are quasisymmetric to } O(r) \text{ is precisely understood.}\]
We have translated analytic work by Garren & Boozer (1991) into practical algorithms.

**Fundamental equations:**

\[
x(r, \theta, \zeta) = x_0(\zeta) + \text{Taylor series in } r = \sqrt{2\psi / B_0}, \quad \left(\nabla \times \mathbf{B}\right) \times \mathbf{B} = \mu_0 \nabla \rho,
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\mathbf{B} = \nabla \psi \times \nabla \theta + i \nabla \zeta \times \nabla \psi = \beta \nabla \psi + I(\psi) \nabla \theta + G(\psi) \nabla \zeta, \quad B(r, \theta, \zeta) = B_0 \left[ 1 + r\overline{\eta} \cos(\theta - N\zeta) + O(r^2) \right]
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  - Rotation of the elliptical flux surfaces at \( \zeta = 0 \). (Usually 0).
  - \( \overline{\eta} \), which controls elongation and field strength.

**Outputs:**

- Shape of the surfaces around the axis. (Elongation & rotation of ellipses.)
- Rotational transform on axis.

---

Example $O(r)$ construction: quasi-axisymmetry

**Inputs:**
axis shape $R_0(\phi) = 1 + 0.045 \cos(3\phi)$ \[ \text{m} \], 
$I_2 = 0$, $\bar{\eta} = -0.9$.

$Z_0(\phi) = -0.045 \sin(3\phi)$ \[ \text{m} \], $\sigma(0) = 0$.

Plug in $r = 0.1$ m.

**Results:**

Inputs:
- $0.8$ $0.9$ $1$ $1.1$ $\text{R} \text{[m]}$
- $0$ $0.1$ $0.2$ $\text{Z} \text{[m]}$

Results:
- $\eta = -0.9$
- $I_2 = 0$
- $\bar{\eta} = -0.9$
- $\sigma(0) = 0$

Plug in $r = 0.1$ m.
Extending the construction to $O(r^2)$, you get triangularity and better quasisymmetry.

$$\lambda = 0.42$$

$$\lambda = 1.14$$
The construction can be verified by running an MHD equilibrium code (VMEC) which does not make the expansion.

Fourier harmonics $B_{m,n}$ in Boozer coordinates [T]

- Requested $B_{1,0}$
- Quasi-axisymmetric ($n = 0$)
- Symmetry-breaking ($n \neq 0$)

Fourier harmonics $B_{m,n}$ in Boozer coordinates [T]

- Requested $B_{1,4}$
- Quasisymmetric ($n = 4m$)
- Symmetry-breaking ($n \neq 4m$)
The near-axis analysis can be generalized to construct omnigenous configurations.

\[ G G \text{ Plunk, ML, and } P \text{ Helander, arXiv:1909.08919, Accepted in J Plasma Phys} \]

Omnigenity: \[ \oint \left( \mathbf{v}_d \cdot \nabla \psi \right) dt = 0 \]

\( \forall \) magnetic moments & energies.
The fast construction enables brute-force surveys of “all” quasisymmetric fields.

Axis shape: $R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(j n_{fp} \phi)$, $Z_0(\phi) = \sum_{j=1}^{3} Z_j \sin(j n_{fp} \phi)$

$2.4 \times 10^8$ configurations

Color $= N = \#$ of times $B$ contours rotate around magnetic axis
Brute-force searching is already yielding some new configurations.

Quasi-helical symmetry with

1 field period

2 field periods
Brute-force searching is already yielding some new configurations.

Quasi-helical symmetry with

1 field period

2 field periods
The axis expansion enables a combined (1-stage) coil + quasisymmetry optimization using analytic derivatives.

\[
\min_x f(X)
\]

\[
X = \{ \text{Coil shapes, axis shape, } \bar{\eta} \}
\]

\[
f = \int \left| B_{\text{Biot-Savart}} - B_{\text{Near-axis quasisymmetry}} \right|^2
\]

\[
+ \left( \frac{l_{\text{axis}} - l_{\text{target}}}{l_{\text{target}}} \right)^2
\]

\[
+ \left( \text{other terms} \right)
\]

With Andrew Giuliani, Georg Stadler, Antoine Cerfon (NYU)
Conclusions

• The near-axis expansion enables transport-optimized stellarator configurations to be generated orders of magnitude faster than before.

• We now precisely understand the space of quasisymmetric fields to $O(r)$.

• There is hope of definitively identifying all regions of parameter space with practical quasisymmetric & omnigenous fields (near the axis).

• Much more can be done, e.g. gyrokinetic & MHD analysis near axis.
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- Much more can be done, e.g. gyrokinetic & MHD analysis near axis.
- We might discover qualitatively new magnetic configurations for fusion?
Extra slides
Advantages of stellarators: steady-state, no disruptions, no power recirculated for current drive, no Greenwald limit, don’t rely on plasma for confinement.
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• But, alpha losses & neoclassical transport would be too large unless you carefully choose the geometry.

$$\oint (v_d \cdot \nabla r) dt = 0 \text{ in axisymmetry, } \neq 0 \text{ in a general stellarator.}$$
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• A solution: quasisymmetry

\[ B = B(r, \theta - N\zeta) \Rightarrow \oint (v_d \cdot \nabla r) dt = 0. \]

Guiding-center Lagrangian in Boozer coordinates depends on \((\theta, \zeta)\) only through \(B=|B|\).
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\[ B = B(r, \theta - N\zeta) \quad \Rightarrow \quad \int (v_d \cdot \nabla r) \, dt = 0. \]

(Or, weaker conditions like omnigenity)

Boozer angles

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- Steady-state
- No disruptions
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• How do you find MHD equilibria with these properties?
• Theory for $O(r)$ quasisymmetry
• Comparison to optimized configurations
• The landscape of solutions
• $O(r)$ omnigenity
• $O(r^2)$ quasisymmetry
• Higher order: Calculate $B_3$ so symmetry-breaking can be minimized.

• Examine MHD & gyrokinetic stability using the expansion.

• Can anything be proved about the number or character of $O(r^2)$ solutions?

• Is there an analogous construction to give quasisymmetry at an off-axis surface?

• Check coil feasibility for newly discovered configurations.
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• There is hope of definitively identifying all regions of parameter space with practical quasisymmetric & omnigenous fields (near the axis).

• Much more can be done, e.g. gyrokinetic & MHD analysis near axis.
The near-axis analysis can be generalized to construct omnigenous configurations.


Omnigenity: \[ \oint_{\text{bounce}} (\mathbf{v}_d \cdot \nabla \psi) \, dt = 0 \quad \forall \text{ magnetic moments & energies}. \]

Implications for \( B(r,\theta,\zeta) \):
- All \( B \) contours close toroidally, helically, or poloidally.
- Distance along \( B \) between bounce points is the same for every field line on a flux surface.

[Cary & Shasharina (1997)]
The near-axis analysis can be generalized to construct omnigenous configurations


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- All \(B\) contours close toroidally, helically, or poloidally.
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- Relates \(B(\psi, \theta, \zeta)\) to \(x_0, X, Y, Z\) near axis for any equilibrium, not just quasisymmetry.
The near-axis analysis can be generalized to construct omnigenous configurations

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\oint (v_d \cdot \nabla \psi) \, dt = 0 \quad \forall \text{ magnetic moments & energies.}
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Garren \& Boozer (1991):

Relates \(B(\psi, \theta, \zeta)\) to \(x_0, X, Y, Z\) near axis for \textit{any} equilibrium, not just quasisymmetry.

\[ (\text{Cary \& Shasharina } 1997) + (\text{Garren \& Boozer } 1991) = (\text{Plunk et al } 2019) \]
The near-axis analysis can be generalized to construct omnigenous configurations.

$1/\nu$ transport computed by VMEC + BOOZ_XFORM + NEO codes scales as expected
The $O(r^2)$ construction allows triangularity, Shafranov shift, and more accurate quasisymmetry.

\[
x(r, \vartheta, \zeta) = x_0(\zeta) + X(r, \vartheta, \zeta) n(\zeta) + Y(r, \vartheta, \zeta) b(\zeta) + Z(r, \vartheta, \zeta) t(\zeta)
\]

\[
X(r, \vartheta, \zeta) = r \left[ X_{1c} \cos \vartheta + X_{1s} \sin \vartheta \right] + r^2 \left[ X_{20} + X_{2c} \cos 2\vartheta + X_{2s} \sin 2\vartheta \right] + O(r^3)
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Same for $Y$ & $Z$. 

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Same for $Y$ & $Z$.

- 3 new input parameters: $p_2, B_{2c}, B_{2s}$.

\[ p(r) = p_0 + r^2 p_2 + O(r^4) \]

\[ B(r, \vartheta, \varphi) = B_0 + rB_0 \bar{\eta} \cos \vartheta + r^2 \left[ B_{20} + B_{2c} \cos 2\vartheta + B_{2s} \sin 2\vartheta \right] + O(r^3) \]
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$x(r, \vartheta, \zeta) = x_0(\zeta) + X(r, \vartheta, \zeta)n(\zeta) + Y(r, \vartheta, \zeta)b(\zeta) + Z(r, \vartheta, \zeta)t(\zeta)$

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- 3 new input parameters: $p_2, B_{2c}, B_{2s}$.
- $p(r) = p_0 + r^2 p_2 + O(r^4)$
- $B(r, \vartheta, \phi) = B_0 + rB_0 \bar{\eta} \cos \vartheta + r^2 \left[ B_{20} + B_{2c} \cos 2\vartheta + B_{2s} \sin 2\vartheta \right] + O(r^3)$

- Net loss of 1 degree of freedom. My approach: $B_{20}(\zeta)$ is an output. Need to adjust inputs so $B_{20}(\zeta) \approx$ constant.
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• Net loss of 1 degree of freedom. My approach: $B_{20}(\zeta)$ is an output. Need to adjust inputs so $B_{20}(\zeta) \approx$ constant.

• Minimize $X_2$ & $Y_2$, to maximize the $r$ at which surfaces begin to self-intersect.
Expansion about the magnetic axis can be a powerful practical tool for generating quasisymmetric & omnigenous stellarators.

Complements the traditional optimization approach:

- Many orders of magnitude faster.
- You can parameterize all possible solutions.
- Can generate initial conditions that can be refined by optimization.
The construction can be fit to quasisymmetric stellarators designed by optimization

- Adopt the same axis shape.
- Fit $\bar{\eta}$ to minimize difference in the shapes of a near-axis surface.
The direct construction gives an accurate match to the on-axis rotational transform in quasisymmetric stellarators designed by optimization.

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Dotted: VMEC equilibrium
Solid: Garren-Boozer construction

r/a=0.1
r/a=0.2
We now have a recipe for generating quasisymmetric VMEC input files: Set $r$ to a small finite value $a$.

**Inputs:**
- Axis shape $R_0(\phi) = 1 + 0.32 \cos(4\phi)$,
- $Z_0(\phi) = 0.35 \sin(4\phi)$,
- $I_2 = 0$,
- $\bar{\eta} = 1.5$,
- $\sigma(0) = 0$,
- $R/a = 18$.

**Results:**
We now have a recipe for generating quasisymmetric VMEC input files:

Set \( r \) to a small finite value \( \alpha \).

**Inputs:**

axis shape \( R_0(\phi) = 1 + 0.32 \cos(4\phi) \),
\( Z_0(\phi) = 0.35 \sin(4\phi) \),

\( \sigma(0) = 0 \), \( I_2 = 0 \), \( \bar{\eta} = 1.5 \), \( R/a = 18 \).

**Results:**

\( \eta = 1.5 \), \( R/a = 18.0 \).
The construction can be verified by comparing to VMEC + BOOZ_XFORM.

The plots show the aspect ratio 18 and 80 Fourier harmonics $B_{m,n}$ in Boozer coordinates. The graphs display the normalized effective minor radius $r/a$ across different values of $\phi$, with red and blue lines indicating quasisymmetric and symmetry breaking modes, respectively.

Inputs:
- $I_2 = 0$
- $\sigma_0(\phi) = 0$

Results:
- $\eta = 1.5$
- $R/a = 18$. 

The plots also include annotations for $\phi = 3\pi/8$.
Alternative method to generate a finite-thickness boundary: find coils to make a skinny surface, then see what you get outside.

$$\frac{R}{a} = 5$$

$$\phi = 0$$

$$\phi = \pi / 6$$

$$\phi = \pi / 3$$
Alternative method to generate a finite-thickness boundary: find coils to make a skinny surface, then see what you get outside.

\[ \frac{R}{a} = 5 \]

\[ \phi = 0 \]

\[ \phi = \pi / 6 \]

\[ \phi = \pi / 3 \]
J × B = \nabla p \implies \nabla_\perp B = B\kappa n

So B contours rotate about axis with the same topology as n.
Quasi-axisymmetry vs quasi-helical symmetry is determined purely by the axis shape

\[ \mathbf{J} \times \mathbf{B} = \nabla \rho \quad \Rightarrow \quad \nabla \perp \mathbf{B} = B \kappa \mathbf{n} \]

So \( B \) contours rotate about axis with the same topology as \( \mathbf{n} \).

\[ \mathbf{n} \text{ does not rotate about the axis as you follow the axis around.} \]

\[ \Rightarrow \quad \text{Quasi-axisymmetry} \]

\[ B = B(r, \theta) \]

\[ \mathbf{n} \text{ rotates about the axis 4 times as you follow the axis around.} \]

\[ \Rightarrow \quad \text{Quasi-helical symmetry} \]

\[ B = B(r, \theta - 4\zeta) \]
The fast construction enables brute-force surveys of "all" quasisymmetric fields.

Axis shape:  
\[ R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(jn_{fp} \phi), \quad Z_0(\phi) = 1 + \sum_{j=1}^{3} Z_j \sin(jn_{fp} \phi) \]

2.4x10^8 configurations

Color = # of times B contours rotate around magnetic axis
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Axis shape: \( R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(jn_{fp}\phi), \quad Z_0(\phi) = 1 + \sum_{j=1}^{3} Z_j \sin(jn_{fp}\phi) \n\)

4x10^6 configurations

Quasi-axisymmetric solutions only

Color = \( n_{fp} \) (discrete rotational symmetry)
The construction enables fast scans over parameter space.

E.g. Scan over \( (R_{0c}, Z_{0s}, \bar{\eta}) \) where magnetic axis shape is

\[
R_0(\phi) = 1 + R_{0c} \cos(4\phi)
\]
\[
Z_0(\phi) = Z_{0s} \sin(4\phi)
\]

274,560 solutions generated in <30s on a laptop.

Quasi-axisymmetry

Quasi-helical symmetry
All stellarators built to date have ‘stellarator symmetry’, which is unrelated to quasisymmetry.
You can make a quasi-axisymmetric stellarator without stellarator symmetry.

**Inputs:**  
axis shape \( R_0(\phi) = 1 + 0.042\cos(3\phi) \),  
\( Z_0(\phi) = -0.042\sin(3\phi) - 0.025\cos(3\phi) \),  
\( I_2 = 0 \),  
\( \bar{\eta} = -1.1 \),  
\( \sigma(0) = -0.6 \),  
\( \eta = -1.1 \).  

**Results:**  
\( R/a = 10 \)
You can make a quasi-axisymmetric stellarator without stellarator symmetry.

Aspect ratio 10

Fourier harmonics $B_{m,n}$ in Boozer coordinates

- Quasi-axisymmetric $(m, n)$ modes
- Symmetry breaking $(m, n)$ modes

Solid = $\cos(m\theta - n\zeta)$ modes
Dashed = $\sin(m\theta - n\zeta)$ modes

Inputs:

- $0.7$ $0.8$ $0.9$ $1$ $1.1$ $1.2$

Results:

- $(R/a = 10)$

Aspect ratio 80

Fourier harmonics $B_{m,n}$ in Boozer coordinates

- Quasi-axisymmetric $(m, n)$ modes
- Symmetry breaking $(m, n)$ modes

Solid = $\cos(m\theta - n\zeta)$ modes
Dashed = $\sin(m\theta - n\zeta)$ modes

Inputs:

- $0.7$ $0.8$ $0.9$ $1$ $1.1$ $1.2$

Results:

- $(R/a = 80)$
We will expand in the skinniness of the inner flux surfaces.

Define effective radius $r$ by $\Psi = \pi r^2 B_{axis}$.

"Aspect ratio" $\frac{R}{r} \gg 1$
Theory: Expand position vector using Frenet frame, equate 2 representations of $B$. 

Frenet frame $(t, n, b)$: 
\[ \frac{\partial r_0}{\partial \ell} = t, \quad \frac{dt}{d\ell} = \kappa n, \quad \frac{dn}{d\ell} = -\kappa t + \tau b, \quad \frac{db}{d\ell} = -\tau n \]

$r_0 =$ magnetic axis, $\kappa =$ curvature, $\tau =$ torsion 

$t =$ tangent, $n =$ normal, $b =$ binormal
Theory: Write position vector using Frenet frame

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\]

\(\mathbf{r}_0\) = magnetic axis, \(\kappa\) = curvature, \(\tau\) = torsion, \(t\) = tangent, \(n\) = normal, \(b\) = binormal

\[
\mathbf{r}(r, \theta, \zeta) = \mathbf{r}_0(\zeta) + X(r, \theta, \zeta)\mathbf{n}(\zeta) + Y(r, \theta, \zeta)\mathbf{b}(\zeta) + Z(r, \theta, \zeta)\mathbf{t}(\zeta)
\]
Theory: Write position vector using Frenet frame, expand in small $r = (\text{flux})^{1/2}$

Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$: \[
\frac{d\mathbf{r}_0}{d\ell} = \mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell} = \kappa \mathbf{n}, \quad \frac{d\mathbf{n}}{d\ell} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \frac{d\mathbf{b}}{d\ell} = -\tau \mathbf{n}
\]

$r_0 =$ magnetic axis, $\kappa =$ curvature, $\tau =$ torsion, $\mathbf{t} =$ tangent, $\mathbf{n} =$ normal, $\mathbf{b} =$ binormal

\[
\mathbf{r}(r, \theta, \zeta) = \mathbf{r}_0(\zeta) + X(r, \theta, \zeta)\mathbf{n}(\zeta) + Y(r, \theta, \zeta)\mathbf{b}(\zeta) + Z(r, \theta, \zeta)\mathbf{t}(\zeta)
\]

\[
= \mathbf{r}_0(\zeta) + rX_{1c}(\zeta)\cos \theta \mathbf{n}(\zeta) + r\left[ Y_{1s}(\zeta)\sin \theta + Y_{1c}(\zeta)\cos \theta \right] \mathbf{b}(\zeta) + O(r^2)
\]

Using magnetohydrodynamic equilibrium $(\mathbf{J} \times \mathbf{B} = \nabla p)$
Theory: Write position vector using Frenet frame, expand in small $r = (\text{flux})^{1/2}$

Frenet frame $(t, n, b)$:
\[
\frac{dr_0}{d\ell} = t, \quad \frac{dt}{d\ell} = \kappa n, \quad \frac{dn}{d\ell} = -\kappa t + \tau b, \quad \frac{db}{d\ell} = -\tau n
\]

$r_0 =$ magnetic axis, $\kappa =$ curvature, $\tau =$ torsion, $t =$ tangent, $n =$ normal, $b =$ binormal

\[
r(r, \theta, \zeta) = r_0(\zeta) + X(r, \theta, \zeta) n(\zeta) + Y(r, \theta, \zeta) b(\zeta) + Z(r, \theta, \zeta) t(\zeta)
\]

\[
= r_0(\zeta) + rX_{1c}(\zeta) \cos \theta n(\zeta) + r \left[ Y_{1s}(\zeta) \sin \theta + Y_{1c}(\zeta) \cos \theta \right] b(\zeta) + O(r^2)
\]

\[
X_{1c}(\zeta) = \frac{\eta}{\kappa(\zeta)}, \quad Y_{1s}(\zeta) = \frac{\kappa(\zeta)}{\eta}, \quad Y_{1c}(\zeta) = \frac{\sigma(\zeta) \kappa(\zeta)}{\eta}
\]

Toroidal angle $\zeta \propto \text{arclength}$, $\eta =$ constant: $B = B_0 \left[ 1 + r\eta \cos(\theta - N\phi) + O(r^2) \right]$}

\[
\frac{d\sigma}{d\zeta} + \eta \left[ \frac{\eta^4}{\kappa^4} + 1 + O^2 \right] - 2 \frac{\eta^2}{\kappa^2} \left[ I_2 - \tau \right] = 0
\]

$I_2 =$ current density
Theory: Expand position vector using Frenet frame, equate 2 representations of $B$.

Frenet frame $(t,n,b)$:

\[ \frac{\partial r_0}{\partial \ell} = t, \quad \frac{dt}{d\ell} = \kappa n, \quad \frac{dn}{d\ell} = -\kappa t + \tau b, \quad \frac{db}{d\ell} = -\tau n \]

$r_0$ = magnetic axis, $\kappa$ = curvature, $\tau$ = torsion
\[ t = \text{tangent}, \quad n = \text{normal}, \quad b = \text{binormal} \]

\[ r(r,\theta,\zeta) = r_0(\zeta) + X(r,\theta,\zeta)n(\zeta) + Y(r,\theta,\zeta)b(\zeta) + Z(r,\theta,\zeta)t(\zeta) \]
Theory: Expand position vector using Frenet frame, equate 2 representations of B.

Frenet frame \( (t, n, b) \):
\[
\frac{\partial r_0}{\partial \ell} = t, \quad \frac{dt}{d\ell} = \kappa n, \quad \frac{dn}{d\ell} = -\kappa t + \tau b, \quad \frac{db}{d\ell} = -\tau n
\]

\( r_0 = \) magnetic axis, \( \kappa = \) curvature, \( \tau = \) torsion
\( t = \) tangent, \( n = \) normal, \( b = \) binormal

\[
\vec{r}(r, \theta, \zeta) = r_0(\zeta) + X(r, \theta, \zeta)n(\zeta) + Y(r, \theta, \zeta)b(\zeta) + Z(r, \theta, \zeta)t(\zeta)
\]

\[
X(r, \theta, \zeta) = r \left[ X_{1s}(\zeta)\sin\theta + X_{1c}(\zeta)\cos\theta \right] + O(r^2). \quad \text{Same for} \ Y, Z.
\]
Theory: Expand position vector using Frenet frame, equate 2 representations of B.

Frenet frame \( (t, n, b) \):

\[
\frac{dr_0}{d\ell} = t, \quad \frac{dt}{d\ell} = \kappa n, \quad \frac{dn}{d\ell} = -\kappa t + \tau b, \quad \frac{db}{d\ell} = -\tau n
\]

- \( r_0 \) = magnetic axis, \( \kappa \) = curvature, \( \tau \) = torsion
- \( t \) = tangent, \( n \) = normal, \( b \) = binormal

\[
r(r, \theta, \zeta) = r_0(\zeta) + X(r, \theta, \zeta)n(\zeta) + Y(r, \theta, \zeta)b(\zeta) + Z(r, \theta, \zeta)t(\zeta)
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X(r, \theta, \zeta) = r\left[X_{1s}(\zeta)\sin \theta + X_{1c}(\zeta)\cos \theta \right] + O(r^2). \quad \text{Same for } Y, Z.
\]

\[
B = B_r \nabla r + B_\theta \nabla \theta + B_\zeta \nabla \zeta, \quad B = \nabla \psi \times \nabla \theta + i \nabla \zeta \times \nabla \psi
\]
Theory: Expand position vector using Frenet frame, equate 2 representations of $B$.

Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$: \[
\begin{align*}
\frac{\partial \mathbf{r}_0}{\partial \ell} &= \mathbf{t}, & \frac{d \mathbf{t}}{d \ell} &= \kappa \mathbf{n}, & \frac{d \mathbf{n}}{d \ell} &= -\kappa \mathbf{t} + \tau \mathbf{b}, & \frac{d \mathbf{b}}{d \ell} &= -\tau \mathbf{n}
\end{align*}
\]

$r_0$ = magnetic axis, $\kappa$ = curvature, $\tau$ = torsion
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\mathbf{r}(r, \theta, \zeta) = \mathbf{r}_0(\zeta) + X(r, \theta, \zeta) \mathbf{n}(\zeta) + Y(r, \theta, \zeta) \mathbf{b}(\zeta) + Z(r, \theta, \zeta) \mathbf{t}(\zeta)
\]

$X(r, \theta, \zeta) = r \left[ X_{1s}(\zeta) \sin \theta + X_{1c}(\zeta) \cos \theta \right] + O(r^2)$. Same for $Y, Z$.

$\mathbf{B} = B_r \nabla r + B_\theta \nabla \theta + B_\zeta \nabla \zeta$, \quad $\mathbf{B} = \nabla \psi \times \nabla \theta + i \nabla \zeta \times \nabla \psi$

Dual relations: $\nabla r = \left[ \frac{\partial \mathbf{r}}{\partial r} \cdot \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \zeta} \right]^{-1} \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \zeta}$, cyclic permutations.
The rotational transform computed by VMEC converges to the value computed by the Garren-Boozer approach.
The ODE is solved with spectral accuracy using pseudospectral discretization + Newton iteration.

Uniform grid in $\phi$ with $N$ points: $\phi_1 = 0$, $\phi_2 = 2\pi / (Nn_{fp})$, ..., $\phi_N = 2\pi(N - 1) / (Nn_{fp})$.

Vector of $N$ unknowns: \( \left( t, \sigma(\phi_2), \sigma(\phi_3), ..., \sigma(\phi_N) \right)^T \)

$N$ equations: impose ODE at $\phi_1$, ..., $\phi_N$.

\[ \frac{d\sigma}{d\phi} \rightarrow D\sigma \quad \text{where } D \text{ is a pseudospectral differentiation matrix.} \]
Of 10 configurations examined, the fit is less good for 2
The configurations with relatively poor fits can be explained by their larger symmetry-breaking.
The conventional approach to finding quasisymmetric fields works but has shortcomings.

Want magnetic field strength $B$ to have quasisymmetry: 

$$B = B(r, \theta - N\zeta)$$
The conventional approach to finding quasisymmetric fields works but has shortcomings

Want magnetic field strength $B$ to have quasisymmetry:  

$$B = B(r, \theta - N\zeta)$$

$$\min_x f(X)$$

Parameter space: $X = \text{toroidal boundary shapes}$

Objective:  

$$f = \sum_{m,n \neq Nm} B_{m,n}^2(r_0)$$  

where  

$$B(r,\theta,\zeta) = \sum_{m,n} B_{m,n}(r)\exp(i\theta - in\zeta)$$
The conventional approach to finding quasisymmetric fields works but has shortcomings

Want magnetic field strength $B$ to have quasisymmetry: $B = B(r, \theta - N\zeta)$

$$\min_x f(X)$$

Parameter space: $X = \text{toroidal boundary shapes}$

Objective: $f = \sum_{m,n \neq Nm} B_{m,n}^2 (r_0)$ where $B(r,\theta,\zeta) = \sum_{m,n} B_{m,n}(r) \exp(i m\theta - i n\zeta)$

- Computationally expensive.
- What is the size & character of the solution space?
- Result depends on initial condition, so cannot be sure you’ve found all solutions.
Alternative: expand equations near the magnetic axis

A key ingredient of the theory is the Frenet frame of the magnetic axis

\[
\text{Frenet frame } (t, n, b) : \quad \frac{dx_0}{d\ell} = t, \quad \frac{dt}{d\ell} = \kappa n, \quad \frac{dn}{d\ell} = -\kappa t + \tau b, \quad \frac{db}{d\ell} = -\tau n
\]

\(x_0=\text{magnetic axis}, \quad \kappa=\text{curvature}, \quad \tau=\text{torsion}, \quad t=\text{tangent}, \quad n=\text{normal}, \quad b=\text{binormal}

A key ingredient of the theory is the Frenet frame of the magnetic axis

\[
\text{Frenet frame } (t, n, b): \quad \frac{dx_0}{d\ell} = t, \quad \frac{dt}{d\ell} = \kappa n, \quad \frac{dn}{d\ell} = -\kappa t + \tau b, \quad \frac{db}{d\ell} = -\tau n
\]

\(x_0\) = magnetic axis, \(\kappa\) = curvature, \(\tau\) = torsion, \(t\) = tangent, \(n\) = normal, \(b\) = binormal

Garren & Boozer (1991): Write position vector $x$ using axis’s Frenet frame, expand in small $r$

Frenet frame $(t, n, b)$:

$$\frac{dx_0}{d\ell} = t, \quad \frac{dt}{d\ell} = \kappa n, \quad \frac{dn}{d\ell} = -\kappa t + \tau b, \quad \frac{db}{d\ell} = -\tau n$$

$x_0$ = magnetic axis, $\kappa$ = curvature, $\tau$ = torsion, $t$ = tangent, $n$ = normal, $b$ = binormal

$$x(r, \theta, \zeta) = x_0(\zeta) + X(r, \theta, \zeta)n(\zeta) + Y(r, \theta, \zeta)b(\zeta) + Z(r, \theta, \zeta)t(\zeta), \quad r \propto \sqrt{\psi}$$
The size of the space of fields that are quasisymmetric to $O(r)$ can be precisely understood.

Given $P(\zeta) > 0$, $Q(\zeta)$, and $\sigma(0)$, with $P(\zeta)$ and $Q(\zeta)$

$2\pi$-periodic, bounded, and integrable, a solution to

$$\frac{d\sigma}{d\zeta} + i(P + \sigma^2) + Q = 0$$

(1)

is a pair $\{i, \sigma(\zeta)\}$ solving (1) where $\sigma(\zeta)$ is $2\pi$-periodic.
The size of the space of fields that are quasisymmetric to $O(r)$ can be precisely understood.

Given $P(\zeta)>0$, $Q(\zeta)$, and $\sigma(0)$, with $P(\zeta)$ and $Q(\zeta)$

$2\pi$-periodic, bounded, and integrable, a solution to

$$\frac{d\sigma}{d\zeta}+i(P+\sigma^2)+Q=0 \quad (1)$$

is a pair $\{i, \sigma(\zeta)\}$ solving (1) where $\sigma(\zeta)$ is $2\pi$-periodic.

**Theorem:** A solution exists and it is unique.

*ML, Sengupta, and Plunk (2019).* Probably an earlier reference?
The size of the space of fields that are quasisymmetric to $O(r)$ can be precisely understood.

Given $P(\zeta)>0$, $Q(\zeta)$, and $\sigma(0)$, with $P(\zeta)$ and $Q(\zeta)$ $2\pi$-periodic, bounded, and integrable, a solution to

$$\frac{d\sigma}{d\zeta} + i(P + \sigma^2) + Q = 0 \quad (1)$$

is a pair $\{i, \sigma(\zeta)\}$ solving (1) where $\sigma(\zeta)$ is $2\pi$-periodic.

**Theorem:** A solution exists and it is unique.

$\Rightarrow$ Numerical solution is very robust.
The symmetry-breaking Fourier amplitudes scale as predicted.

\[ S = \frac{1}{B_{0,0}} \sqrt{\sum_{m/n \neq M/N} B_{m,n}^2} \]

Predicted scaling: \( 1/A^2 \)

- Quasi-axisymmetric example
- Quasi-helically symmetric example
Quasi-helically symmetric configurations

Dotted: VMEC equilibrium  
Solid: Garren-Boozer construction

HSX

L-P Ku (2011)

Dotted: VMEC equilibrium  
Solid: Garren-Boozer construction
Quasi-axisymmetric configurations

Dotted: VMEC equilibrium
Solid: Garren-Boozer construction

NCSX

ESTELL
Omnigenity is a weaker confinement condition than quasisymmetry.

Definition of omnigenity: The radial drift has a time average of 0 for all particles.

$$\oint (v_d \cdot \nabla r) dt = 0 \quad \forall \text{ magnetic moments & energies.}$$

Omnigenity is a weaker confinement condition than quasisymmetry.

Definition of omnigenity: The radial drift has a time average of 0 for all particles.

\[ \oint (v_d \cdot \nabla r) dt = 0 \quad \forall \text{ magnetic moments & energies.} \]
The near-axis analysis can be generalized to construct omnigenous configurations

$G G$ Plunk, ML, and P Helander, *In preparation*

Quasi-poloidal symmetry is not possible near the axis, but omnigenity is.

$\nabla B = B k n$
• Construction for $O(r)$ quasisymmetry
  – Theory, & the number of solutions
  – Numerical results
  – Comparison to “real experiments”
  – The landscape of solutions

• Extensions
  – Omnigenity
  – $O(r^2)$ quasisymmetry
We can only “half-specify” the axis shape:

- A curve like the axis is given by 2 functions, e.g. \{curvature, torsion\} or \{R(\phi), Z(\phi)\}.

- At $O(r)$, (# unknowns)-(# equations)=2 so we can specify (almost) any axis. But at $O(r^2)$, (# unknowns)-(# equations)=1 so we cannot.
Extending the construction to higher order is tricky

- We can only “half-specify” the axis shape:
  - A curve like the axis is given by 2 functions, e.g. \( \{ \text{curvature, torsion} \} \) or \( \{ R(\phi), Z(\phi) \} \).
  - At \( O(r) \), \((\# \text{ unknowns})-(\# \text{ equations})=2\) so we can specify (almost) any axis. But at \( O(r^2) \), \((\# \text{ unknowns})-(\# \text{ equations})=1\) so we cannot.

- No existence & uniqueness theorem for solutions (yet).

- Magnetic shear (variation of rotational transform) does not appear until \( O(r^3) \).
We are working to extend the construction to $O(r^2)$, enabling greater shaping.

Axisymmetric example:

- $\cos \theta$ mode amplitude in $B$
- $\cos 2\theta$ mode amplitude in $B$

Graphs showing:
- Achieved
- Requested

Normalized minor radius $r/r_{max}$

- Garren-Boozer construction
- Calculation without an $r$ expansion (VMEC)

Contour lines

$\theta_{Boozer}$ curves
We now have a recipe for generating quasisymmetric VMEC input files:
Set $r$ to a small finite value $a$.

**Inputs:**
axis shape $R_0(\phi) = 1 + 0.265\cos(4\phi)$,
$Z_0(\phi) = -0.21\sin(4\phi),
I_z = 0,
\bar{\eta} = -2.25,
\sigma(0) = 0,
R / a = 40.$

**Results:**

![Image of magnetic field lines and phase portraits]
The construction can be verified by comparing to VMEC + BOOZ_XFORM.

\[
\begin{align*}
R_0 = & 1 + 0.265 \cos 4 \phi, \\
Z_0 = & -0.21 \sin 4 \phi,
\end{align*}
\]

Inputs:
\[
\begin{align*}
0.6 & \quad 0.8 & \quad 1 & \quad 1.2 \\
-0.3 & -0.2 & -0.1 & 0 & 0.1 & 0.2 & 0.3
\end{align*}
\]

Results:
\[
R/a = 40
\]
The fast construction enables brute-force surveys of “all” quasisymmetric fields.

Axis shape:
\[
R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(j n_{fp} \phi), \quad Z_0(\phi) = 1 + \sum_{j=1}^{3} Z_j \sin(j n_{fp} \phi)
\]

2.4x10^8 configurations

Maximum axis curvature

Rotational transform

Color = # of times B contours rotate around magnetic axis
The fast construction enables brute-force surveys of “all” quasisymmetric fields

Axis shape: \( R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(j n_{fp} \phi), \quad Z_0(\phi) = 1 + \sum_{j=1}^{3} Z_j \sin(j n_{fp} \phi) \)

2.4x10^8 configurations

Color = # of times B contours rotate around magnetic axis
Quasisymmetric experiments to date actually have significant departures from symmetry.
Example of the $O(r^2)$ construction

**Inputs:**

axis shape $R_0(\phi) = 1 + 0.173 \cos(2\phi) + 0.0168 \cos(4\phi) + 0.00101 \cos(6\phi)$,

$Z_0(\phi) = 0.158 \sin(2\phi) + 0.0165 \sin(4\phi) + 0.000985 \sin(6\phi)$,

$I_2 = 0$, $\sigma(0) = 0$, $\bar{\eta} = 0.632$, $p_2 = 0$, $B_{2c} = -0.158$, $B_{2s} = 0$, $R/a = 10$
Example of the \(O(r^2)\) construction

**Inputs:**

\[
R_0(\phi) = 1 + 0.173\cos(2\phi) + 0.0168\cos(4\phi) + 0.00101\cos(6\phi),
\]
\[
Z_0(\phi) = 0.158\sin(2\phi) + 0.0165\sin(4\phi) + 0.000985\sin(6\phi),
\]
\[I_2 = 0, \quad \sigma(0) = 0, \quad \bar{\eta} = 0.632, \quad p_2 = 0, \quad B_{2c} = -0.158, \quad B_{2s} = 0, \quad R/a = 10
\]

**Results:** \(\iota = 0.424\)
The $O(r^2)$ construction allows triangularity and more accurate quasisymmetry.

\[
\mathbf{x}(r, \vartheta, \zeta) = \mathbf{x}_0(\zeta) + X(r, \vartheta, \zeta)\mathbf{n}(\zeta) + Y(r, \vartheta, \zeta)\mathbf{b}(\zeta) + Z(r, \vartheta, \zeta)\mathbf{t}(\zeta)
\]

\[
X(r, \vartheta, \zeta) = r\left[ X_{1c} \cos \vartheta + X_{1s} \sin \vartheta \right] + r^2\left[ X_{20} + X_{2c} \cos 2\vartheta + X_{2s} \sin 2\vartheta \right] + O(r^3)
\]

- 3 new input parameters: $p_2, B_{2c}, B_{2s}$.

\[
p(r) = p_0 + r^2 p_2 + O(r^4)
\]

\[
B(r, \vartheta, \phi) = B_0 + rB_0 \bar{\eta} \cos \vartheta + r^2\left[ B_{20} + B_{2c} \cos 2\vartheta + B_{2s} \sin 2\vartheta \right] + O(r^3)
\]

- Net loss of 1 degree of freedom. My approach: $B_{20}(\zeta)$ is an output. Need to adjust inputs so $B_{20}(\zeta) \approx$ constant.

- Shafranov shift appears at this order. Matches textbook tokamak result (e.g. Wesson, Hazeltine & Meiss):

\[
\left( R - R_0 - \Delta \right)^2 + Z^2 = r^2, \quad \Delta = r^2\left( \frac{1}{8R_0} - \frac{\mu_0 p_2 R_0}{2l^2 B_0^2} \right)
\]

Frenet frame $(t,n,b)$: 
\[
\frac{dx_0}{d\ell} = t, \quad \frac{dt}{d\ell} = \kappa n, \quad \frac{dn}{d\ell} = -\kappa t + \tau b, \quad \frac{db}{d\ell} = -\tau n
\]

$x_0 =$ magnetic axis, \hspace{0.5cm} $\kappa =$ curvature, \hspace{0.5cm} $\tau =$ torsion, \hspace{0.5cm} $t =$ tangent, \hspace{0.5cm} $n =$ normal, \hspace{0.5cm} $b =$ binormal

Results for quasisymmetry through $O(r)$:

\[
x(r,\theta,\zeta) = x_0(\zeta) + r \left[ \frac{\kappa(\zeta)}{\kappa'(\zeta)} \cos \vartheta n(\zeta) + r \left[ \frac{\kappa(\zeta)}{\kappa'(\zeta)} \sin \vartheta + r \frac{\kappa(\zeta)}{\kappa'(\zeta)} \cos \vartheta \right] b(\zeta) + O(r^2) \right]
\]

Toroidal angle $\zeta \propto$ axis arclength $\ell$, \hspace{0.5cm} $\bar{\eta} =$ constant: 

\[
B = B_0 \left[ 1 + r\bar{\eta} \cos \vartheta + O(r^2) \right]
\]

\[
\frac{d\vartheta}{d\zeta} + \iota \left[ \frac{\bar{\eta}^4}{\kappa^4} + 1 + O^2 \right] - 2 \frac{\bar{\eta}^2}{\kappa^2} \left[ I_2 - \tau \right] = 0
\]

$\vartheta = \theta - N \zeta$, \hspace{0.5cm} $\iota =$ rotational transform on axis, \hspace{0.5cm} $I_2 =$ current density on axis
The size of the space of fields that are quasisymmetric to $O(r)$ can be precisely understood.

**Inputs:**
- Shape of the magnetic axis. (Determines QA vs QH.)
- 3 real numbers:
  - $I_2$: Current density on the axis. (Usually 0).
  - $\sigma(0)$: Rotation of the elliptical flux surfaces at toroidal angle=0.
  - $\bar{\eta}$, which controls elongation and field strength: $B = B_0 \left[ 1 + r \bar{\eta} \cos \vartheta + O(r^2) \right]$
- (Pressure doesn’t matter to this order.)

**Theorem:** Given this data, a quasisymmetric solution exists, & it is unique.

\[
\frac{d\sigma}{d\zeta} + t \left[ \frac{\bar{\eta}^4}{\kappa^4} + 1 + O^2 \right] - 2 \frac{\bar{\eta}^2}{\kappa^2} \left[ I_2 - \tau \right] = 0
\]
Conclusions

• The equations for quasisymmetric magnetic fields can be solved directly and rapidly if you expand about the magnetic axis.

• The resulting construction can be useful for generating new initial conditions for optimization.

• We precisely understand the size of the space of magnetic fields that are quasisymmetric near the axis (to $O(r)$).

• There is hope of definitively identifying all regions of parameter space with practical quasisymmetric fields (near the axis).

• We can discover qualitatively new magnetic configurations for fusion.
Parameter space (independent variables)

- Coil shapes: arbitrary 3D curves
- Coil currents
- Input parameters of the Garren-Boozer near-axis quasisymmetry equations:
  - Shape of magnetic axis (independent from the axis actually produced by coils!)
  - \( \bar{\eta} \)
  
  \[
  B = B_0 \left[ 1 + r\bar{\eta}\cos(\theta - N\zeta) + O\left(r^2\right) \right]
  \]
Objective function

\[
f = \left( \frac{L_c - L_{c0}}{L_{c0}} \right)^2 + \left( \frac{L_a - L_{a0}}{L_{a0}} \right)^2 + \left( \frac{\iota - \iota_0}{\iota_0} \right)^2
\]

\[
+ \oint_{\text{Garren-Boozer axis}} d\ell \left| \mathbf{B}_{\text{coils}} - \mathbf{B}_{\text{Garren-Boozer}} \right|^2 + \oint_{\text{Garren-Boozer axis}} d\ell \left| \nabla \mathbf{B}_{\text{coils}} - \nabla \mathbf{B}_{\text{Garren-Boozer}} \right|^2
\]

\begin{align*}
L_c & = \text{Total length of coils} \\
L_{c0} & = \text{Target length of coils} \\
L_a & = \text{Length of Garren-Boozer magnetic axis} \\
L_{a0} & = \text{Target length of magnetic axis} \\
\iota & = \text{Rotational transform from Garren-Boozer} \\
\iota_0 & = \text{Target rotational transport}
\end{align*}

Differentiate Biot-Savart law
We can now numerically demonstrate Garren & Boozer’s scaling: $B_{\text{nonsymm}} \sim 1/A^3$

$$S = \frac{1}{B_0} \sqrt{\sum_{m,n \neq Nm} B_{m,n}^2}$$

= Symmetry-breaking

$\sum_{m,n \neq Nm} S_{m,n}$

$S$ for constructed configs

$\propto 1/A^3$

10^{-1}

10^{-2}

10^{-3}

10^{-4}

10^{-5}

10^{-6}

Aspect ratio $A$

10^1

10^2

$QA$

$QH$

10^{-1}

10^{-2}

10^{-3}

10^{-4}

10^{-5}

Aspect ratio $A$

10^1

10^2
We can now numerically demonstrate Garren & Boozer’s scaling: $B_{\text{nonsymm}} \sim 1/A^3$

$$S = \frac{1}{B_0} \sqrt{\sum_{m,n \neq Nm} B_{m,n}^2} = \text{Symmetry-breaking}$$

$S = \frac{1}{B_0} \sqrt{\sum_{m,n \neq Nm} B_{m,n}^2} = \text{Symmetry-breaking}$

$Q_A$

$Q_H$

Aspect ratio $A$

Aspect ratio $A$
Quasisymmetry can be achieved to any desired precision, e.g. $B_{\text{nonsymm}} \leq B_{\text{Earth}}$
Garren & Boozer (1991): Write position vector $x$ using axis's Frenet frame, expand in small $r$

Frenet frame $(t, n, b)$:
\[
\begin{align*}
\frac{dx_0}{d\ell} &= t, \\
\frac{dt}{d\ell} &= \kappa n, \\
\frac{dn}{d\ell} &= -\kappa t + \tau b, \\
\frac{db}{d\ell} &= -\tau n
\end{align*}
\]

$x_0 = $ magnetic axis, \quad $\kappa = $ curvature, \quad $\tau = $ torsion, \quad $t = $ tangent, \quad $n = $ normal, \quad $b = $ binormal

\[
\begin{align*}
x(r, \theta, \zeta) &= x_0(\zeta) + X(r, \theta, \zeta)n(\zeta) + Y(r, \theta, \zeta)b(\zeta) + Z(r, \theta, \zeta)t(\zeta), \\
r &= \sqrt{2\psi / B_0}
\end{align*}
\]
Garren & Boozer (1991): Write position vector \( \mathbf{x} \) using axis's Frenet frame, expand in small \( r \)

Frenet frame \( (\mathbf{t}, \mathbf{n}, \mathbf{b}) \):

\[
\frac{d\mathbf{x}_0}{d\ell} = \mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell} = \kappa \mathbf{n}, \quad \frac{d\mathbf{n}}{d\ell} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \frac{d\mathbf{b}}{d\ell} = -\tau \mathbf{n}
\]

\( \mathbf{x}_0 = \) magnetic axis, \( \kappa = \) curvature, \( \tau = \) torsion, \( \mathbf{t} = \) tangent, \( \mathbf{n} = \) normal, \( \mathbf{b} = \) binormal

\[
\mathbf{x}(r, \theta, \zeta) = \mathbf{x}_0(\zeta) + X(r, \theta, \zeta)\mathbf{n}(\zeta) + Y(r, \theta, \zeta)\mathbf{b}(\zeta) + Z(r, \theta, \zeta)\mathbf{t}(\zeta), \quad r = \sqrt{2\psi / B_0}
\]

\[
X(r, \theta, \zeta) = rX_1(\theta, \zeta) + r^2X_2(\theta, \zeta) + \ldots
\]

Same for \( Y \& Z \).
Garren & Boozer (1991): Write position vector $\mathbf{x}$ using axis’s Frenet frame, expand in small $r$

Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$: 
\[
\frac{d\mathbf{x}_0}{d\ell} = \mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell} = \kappa \mathbf{n}, \quad \frac{d\mathbf{n}}{d\ell} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \frac{d\mathbf{b}}{d\ell} = -\tau \mathbf{n}
\]

$x_0$ = magnetic axis, $\kappa$ = curvature, $\tau$ = torsion, $\mathbf{t}$ = tangent, $\mathbf{n}$ = normal, $\mathbf{b}$ = binormal

\[
\mathbf{x}(r, \theta, \zeta) = x_0(\zeta) + X(r, \theta, \zeta)\mathbf{n}(\zeta) + Y(r, \theta, \zeta)\mathbf{b}(\zeta) + Z(r, \theta, \zeta)\mathbf{t}(\zeta), \quad r = \sqrt{2\psi / B_0}
\]

$X(r, \theta, \zeta) = rX_1(\theta, \zeta) + r^2 X_2(\theta, \zeta) + \ldots$ \quad Same for $Y$ & $Z$.

\[
\nabla r = \frac{\partial \mathbf{x} \times \partial \mathbf{x}}{\partial r \frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \zeta}}, \quad \text{& cyclic permutations}
\]
Frenet frame \((\mathbf{t}, \mathbf{n}, \mathbf{b})\): \[
\frac{d\mathbf{x}_0}{d\ell} = \mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell} = \kappa \mathbf{n}, \quad \frac{d\mathbf{n}}{d\ell} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \frac{d\mathbf{b}}{d\ell} = -\tau \mathbf{n}
\]

\(x_0\) = magnetic axis, \(\kappa\) = curvature, \(\tau\) = torsion, \(\mathbf{t}\) = tangent, \(\mathbf{n}\) = normal, \(\mathbf{b}\) = binormal

\[
\mathbf{x}(r, \theta, \zeta) = x_0(\zeta) + X(r, \theta, \zeta)\mathbf{n}(\zeta) + Y(r, \theta, \zeta)\mathbf{b}(\zeta) + Z(r, \theta, \zeta)\mathbf{t}(\zeta), \quad r = \sqrt{2\psi / B_0}
\]

\[
X(r, \theta, \zeta) = rX_1(\theta, \zeta) + r^2X_2(\theta, \zeta) + \ldots \quad \text{Same for } Y \& Z.
\]

\[
\nabla r = \frac{\partial \mathbf{x} \times \partial \mathbf{x}}{\partial \theta \partial \zeta} \quad \text{& cyclic permutations}
\]

B:
\[
B = \frac{d\psi}{dr} \left[ \nabla r \times \nabla \theta + I(r) \nabla \zeta \times \nabla r \right]
\]

\[
= \beta(r, \theta, \zeta) \frac{d\psi}{dr} \nabla r + I(r) \nabla \theta + G(r) \nabla \zeta
\]
Frenet frame \((\mathbf{t}, \mathbf{n}, \mathbf{b})\): \[
\frac{d\mathbf{x}_0}{d\ell} = \mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell} = \kappa \mathbf{n}, \quad \frac{dn}{d\ell} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \frac{db}{d\ell} = -\tau \mathbf{n}
\]
\(x_0\) = magnetic axis, \(\kappa\) = curvature, \(\tau\) = torsion, \(\mathbf{t}\) = tangent, \(\mathbf{n}\) = normal, \(\mathbf{b}\) = binormal

\[
x(r, \theta, \zeta) = x_0(\zeta) + X(r, \theta, \zeta)\mathbf{n}(\zeta) + Y(r, \theta, \zeta)\mathbf{b}(\zeta) + Z(r, \theta, \zeta)\mathbf{t}(\zeta), \quad r = \sqrt{2\psi / B_0}
\]

\(X(r, \theta, \zeta) = rX_1(\theta, \zeta) + r^2X_2(\theta, \zeta) + \ldots\) \quad \text{Same for } Y \& Z.

\[
\nabla r = \frac{\partial \mathbf{x} \times \partial \mathbf{x}}{\partial \theta \times \partial \zeta}, \quad \text{& cyclic permutations}
\]
\[
\nabla \times \mathbf{B} = \mu_0 \frac{dp}{dr} \nabla r, \quad \mathbf{B}(r, \theta, \zeta) = B_0 + rB_{1c} \cos(\theta - N\zeta) + O(r^2). \quad \text{Expand in } r \ll \kappa^{-1}.
\]
Garren & Boozer’s equations yield a practical algorithm

Inputs:

• Shape of the magnetic axis, with $\kappa \neq 0$. (Determines QA vs QH.)

• 3 real numbers:
  – $I_2$: Current density on the axis. (Usually 0).
  – $\sigma(0)$: Rotation of the elliptical flux surfaces at toroidal angle=0.
  – $\eta$, which controls elongation and field strength: $B = B_0 \left[ 1 + r\eta \cos(\theta - N\zeta) + O(r^2) \right]$

• (Pressure doesn’t matter to this order.)

$\Rightarrow \quad N = 0$: Quasi-axisymmetry

$\Rightarrow \quad N \neq 0$: Quasi-helical symmetry
The construction can be verified by running an MHD equilibrium code (VMEC) which does not make the expansion.

\[ \phi(0) = 1 + 0.045 \cos 3\phi \]

\[ Z(\phi) = -0.045 \sin 3\phi \]

\[ \eta = -0.9 \]

\[ \sigma(\phi) = 0 \]

The inputs are:

- \( R/a = 10 \)

The results are:

- \( B_{1,0} \)
- Quasi-axisymmetric \( n = 0 \)
- Symmetry breaking \( n \neq 0 \)
The axis expansion enables a combined (1-stage) coil + quasisymmetry optimization using analytic derivatives.

Good vacuum surfaces out to $A=1.96$

Good quasisymmetry in core

With Andrew Giuliani, Georg Stadler, Antoine Cerfon (NYU)