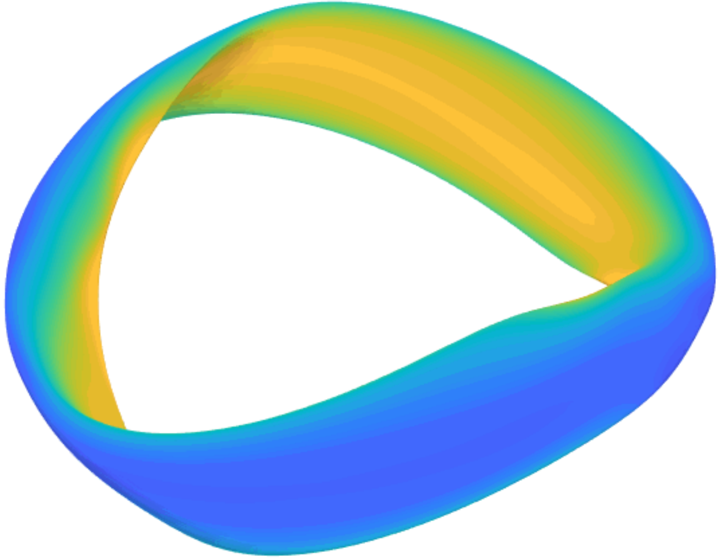
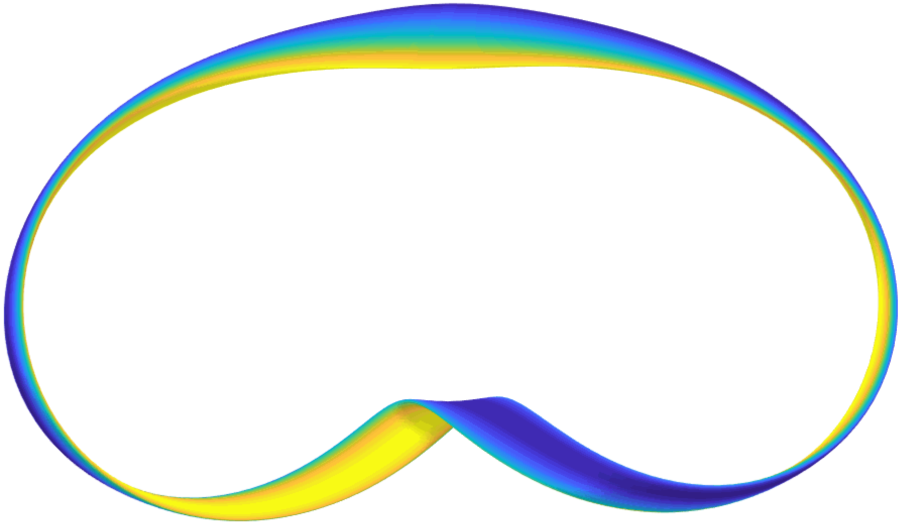
Optimized stellarators without optimization:

Direct construction of stellarator shapes with good confinement





Matt Landreman[1]

Wrick Sengupta[2], Gabe Plunk[3]

1. University of Maryland 2. NYU 3. Max Planck Institute for Plasma Physics

J Plasma Phys (2019) PPCF (2019) https://github.com/landreman/quasisymmetry

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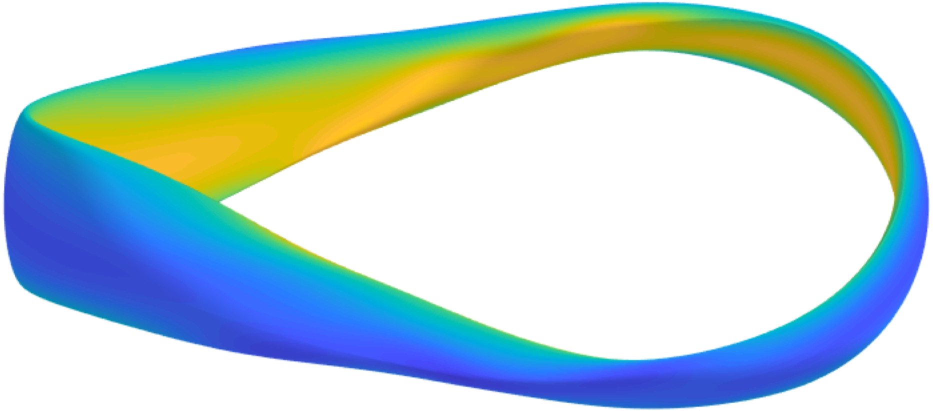

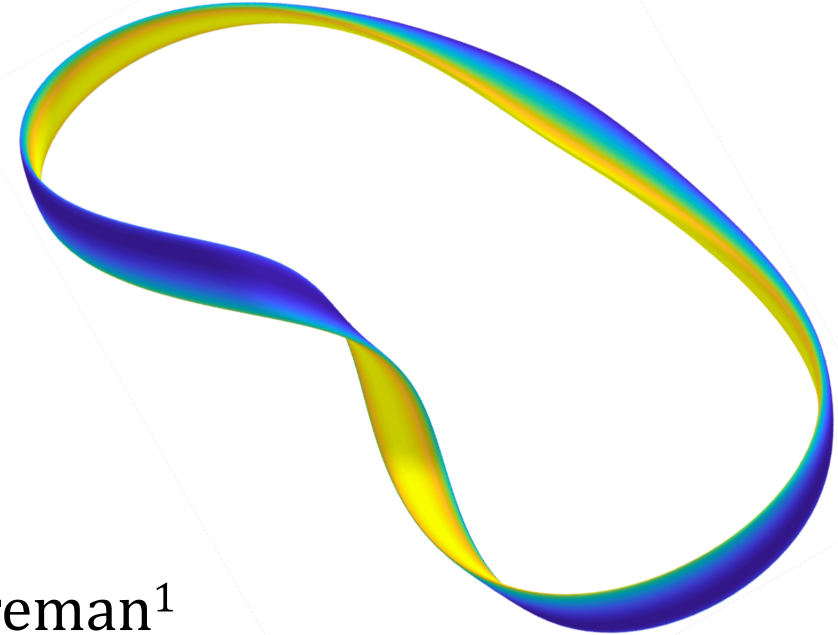

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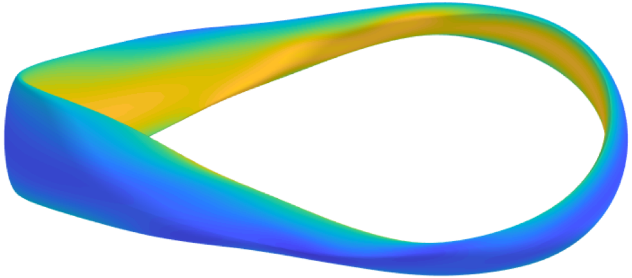
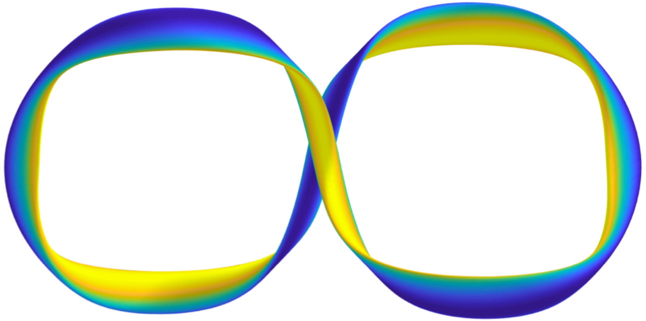
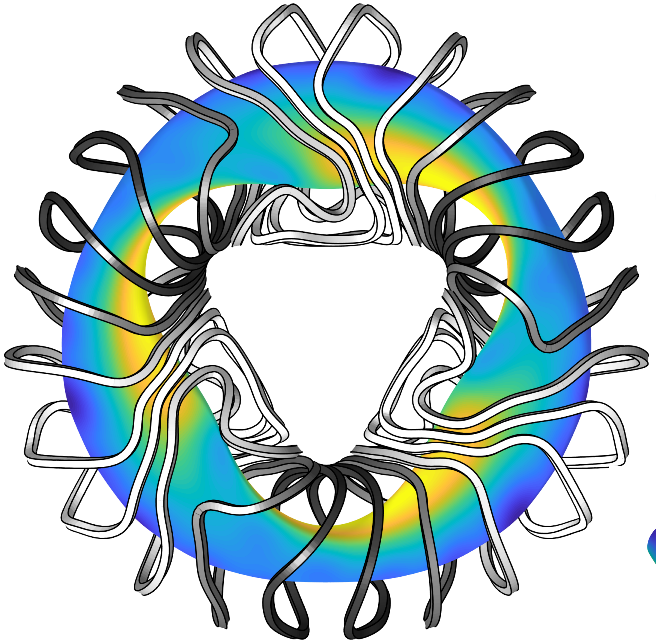
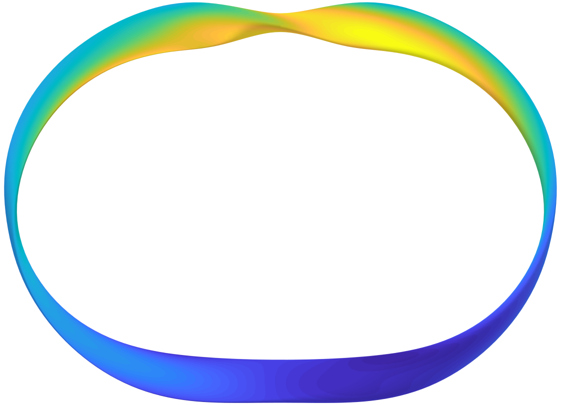
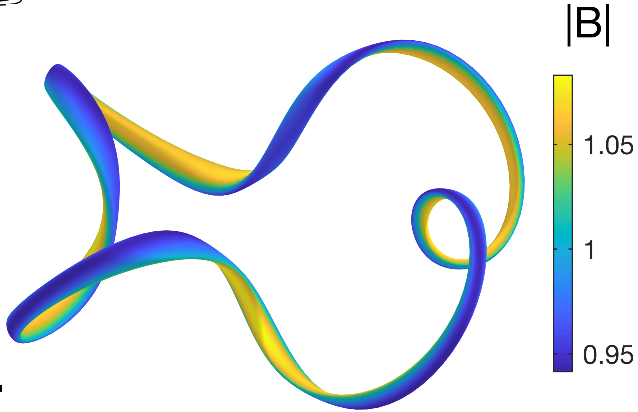
Expansion about the magnetic axis can be a powerful practical tool for generating quasisymmetric & omnigenous stellarators











Complements the stellopt approach:

- Orders of magnitude faster.
- You can parameterize *all* possible solutions.
- Can generate initial conditions that can be refined by stellopt.

Outline

- Theory
- $O(r)$ quasisymmetry
  - Algorithm
  - Comparison to optimized configurations
  - The landscape of solutions
- $O(r^2)$ quasisymmetry

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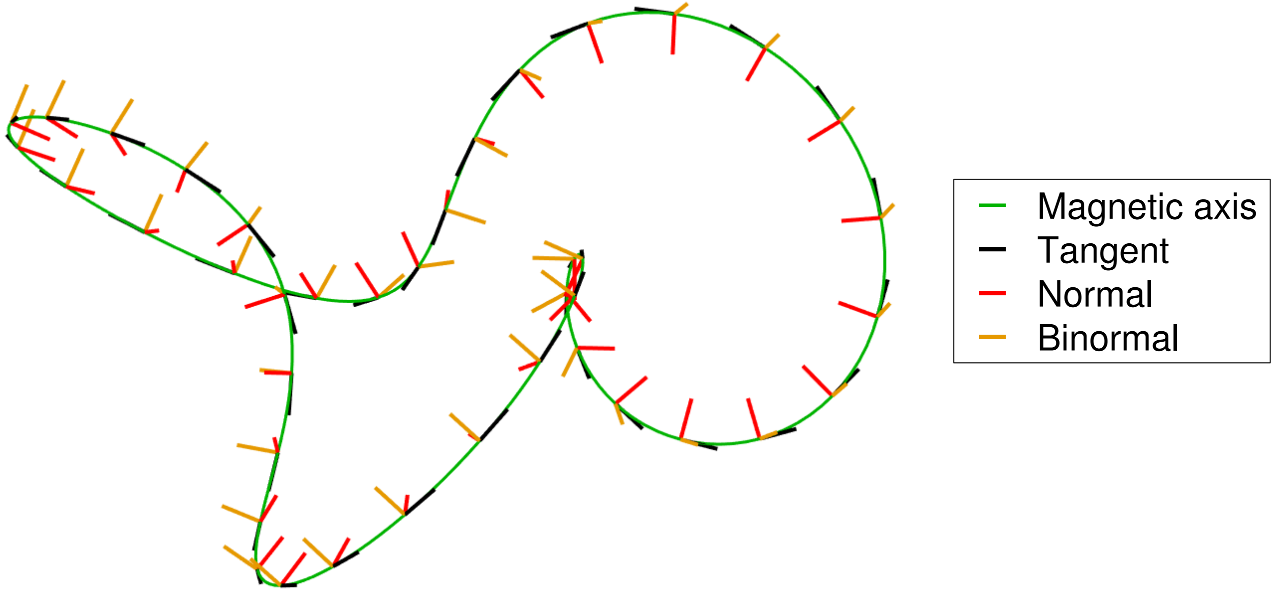
A key ingredient of the theory is the Frenet frame of the magnetic axis

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}): \quad \frac{d\mathbf{x}_0}{d\ell}=\mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n}, \quad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b}, \quad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

$\mathbf{x}_0 =$ magnetic axis, $\kappa =$ curvature, $\tau =$ torsion, $\mathbf{t} =$ tangent, $\mathbf{n} =$ normal, $\mathbf{b} =$ binormal

Mercier (1964), Solovev & Shafranov (1970), Lortz & Nührenberg (1976), Garren & Boozer (1991)



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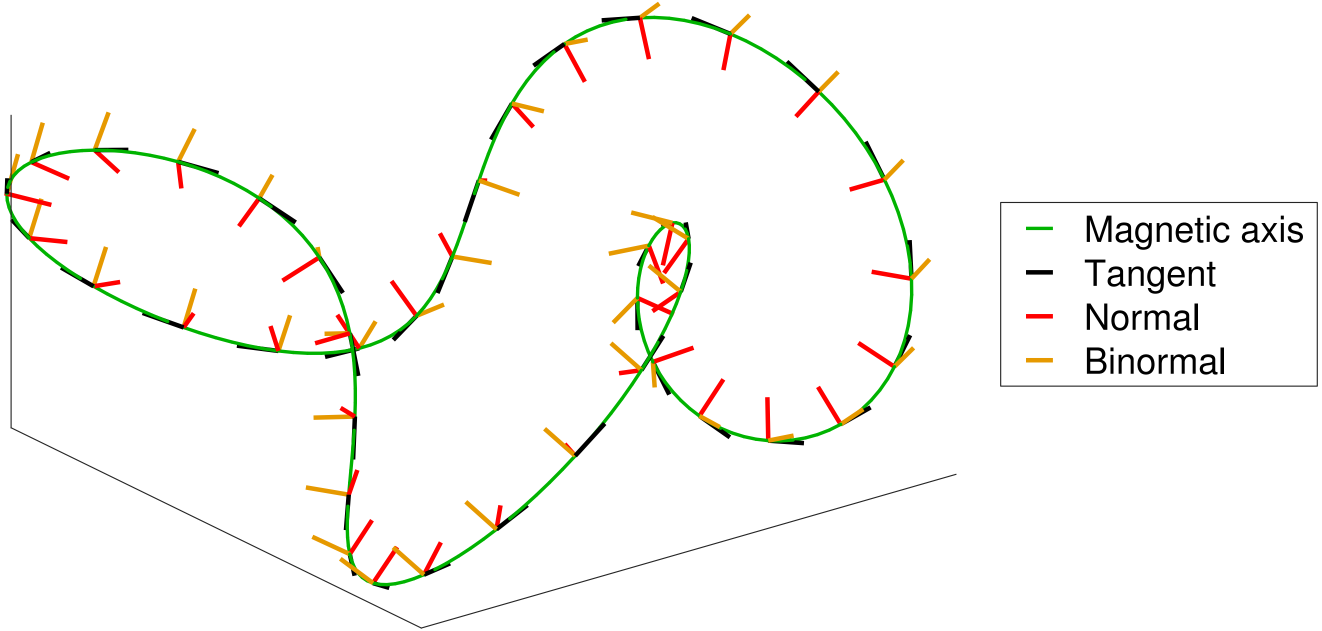

Garren & Boozer (1991): Write position vector x using axis's Frenet frame, expand in small *r*

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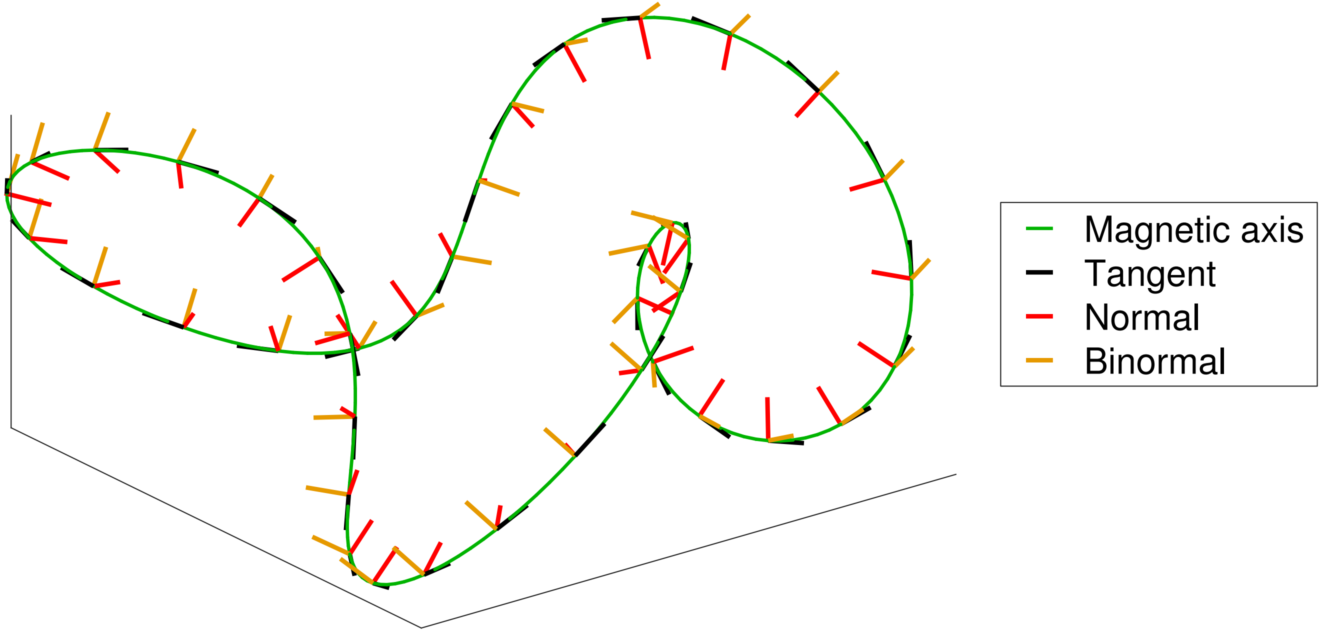

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Same for Y & Z.

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$$\mathbf{B}=\frac{d\psi}{dr}\left[\nabla r\times\nabla\theta+\iota(r)\nabla\zeta\times\nabla r\right]$$
$$=\beta(r,\theta,\zeta)\frac{d\psi}{dr}\nabla r+I(r)\nabla\theta+G(r)\nabla\zeta$$

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$$\mathbf{J}\times\mathbf{B}=\frac{dp}{dr}\nabla r, \qquad B(r,\theta,\zeta)=B_0+rB_{1c}\cos(\theta-N\zeta)+O(r^2)$$

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<u>Results for quasisymmetry through $O(r)$:</u>

$$\mathbf{x}\left(r,\theta,\zeta\right)=\mathbf{x}_0\left(\zeta\right)+r\frac{\bar{\eta}}{\kappa\left(\zeta\right)}\cos\vartheta\,\mathbf{n}\left(\zeta\right)+r\left[\frac{\kappa\left(\zeta\right)}{\bar{\eta}}\sin\vartheta+\frac{\sigma\left(\zeta\right)\kappa\left(\zeta\right)}{\bar{\eta}}\cos\vartheta\right]\mathbf{b}\left(\zeta\right)+O\left(r^2\right)$$

Toroidal angle $\zeta\propto$ axis arclength ℓ, $\quad\bar{\eta}=$ constant: $B=B_0\left[1+r\bar{\eta}\cos\vartheta+O\left(r^2\right)\right]$

$$\frac{d\sigma}{d\zeta}+\iota\left[\frac{\bar{\eta}^4}{\kappa^4}+1+\sigma^2\right]-2\frac{\bar{\eta}^2}{\kappa^2}\left[I_2-\tau\right]=0$$

$\vartheta=\theta-N\zeta$,

$\iota=$ rotational transform on axis,

$I_2=$ current density on axis

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The size of the space of fields that are quasisymmetric to O(r) can be precisely understood.

Inputs:

- Shape of the magnetic axis.
- 3 real numbers:
  - I_2: Current density on the axis. (Usually 0).
  - $\sigma(0)$: Rotation of the elliptical flux surfaces at toroidal angle=0.
  - $\bar{\eta}$, which controls elongation and field strength: $B = B_0\left[1 + r\bar{\eta}\cos\vartheta + O\left(r^2\right)\right]$
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Outputs:

- Shape of the surfaces around the axis. (Elongation & rotation of ellipses.)
- Rotational transform on axis.

Quasi-axisymmetry vs quasi-helical symmetry is determined purely by the axis shape

$$\mathbf{J}\times\mathbf{B} = \cancel{\nabla p} \qquad \Rightarrow \qquad \boxed{\nabla_{\perp} B = B\kappa\mathbf{n}}$$

So B contours rotate about axis with the same topology as $\mathbf{n}$.

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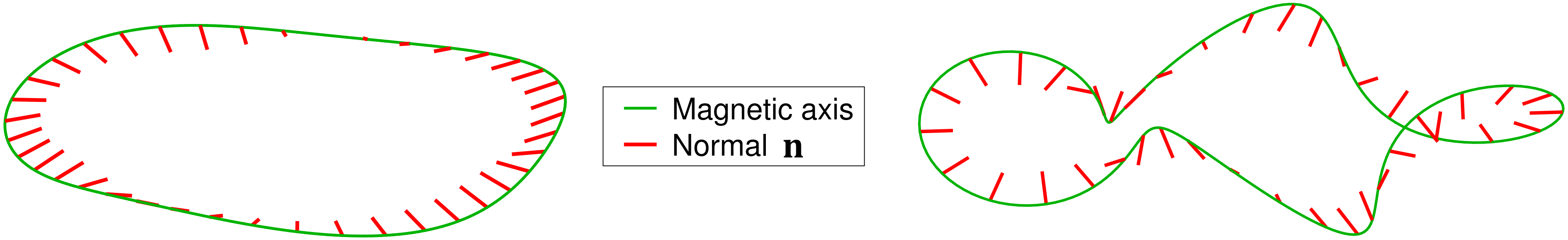


$\mathbf{n}$ does not rotate about the axis as you follow the axis around.

$\Rightarrow$ Quasi-axisymmetry

$$B=B(r,\theta)$$

$\mathbf{n}$ rotates about the axis 4 times as you follow the axis around.

$\Rightarrow$ Quasi-helical symmetry

$$B=B(r,\theta-4\zeta)$$

We now have a recipe for generating quasisymmetric VMEC input files: Set *r* to a small finite value *a*.

Inputs:

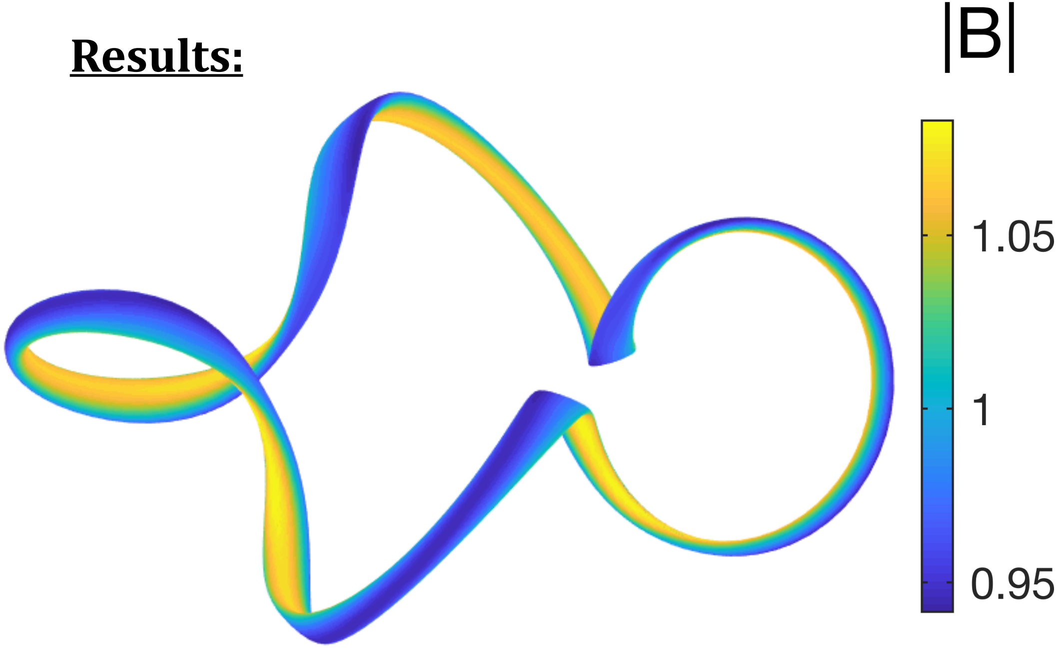
axis shape $R_0(\phi) = 1 + 0.32\cos(4\phi)$,

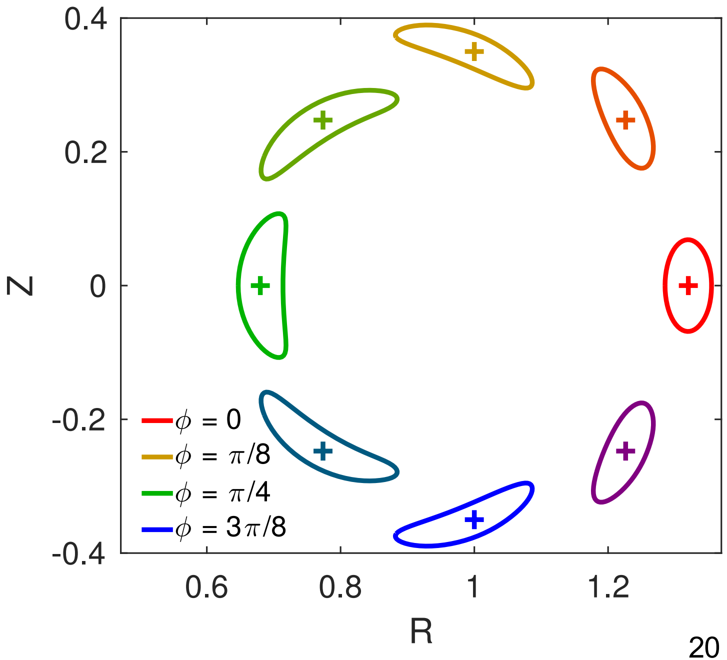
$Z_0(\phi) = 0.35\sin(4\phi)$,

$I_2 = 0$, $\bar{\eta} = 1.5$,

$\sigma(0) = 0$, $R/a = 18$.

Results:





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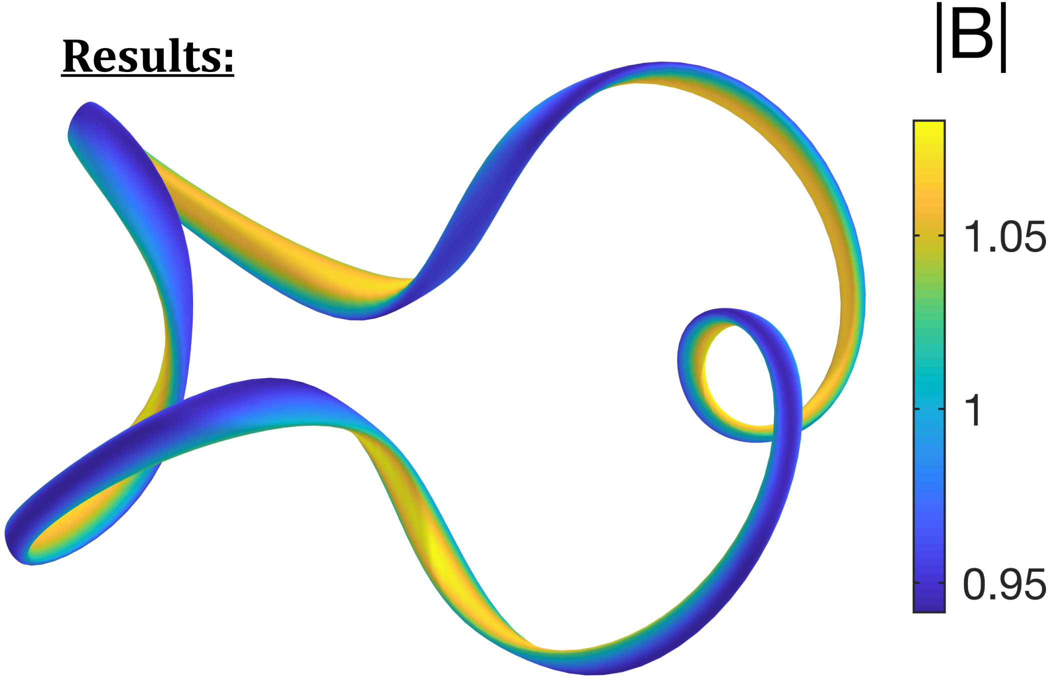
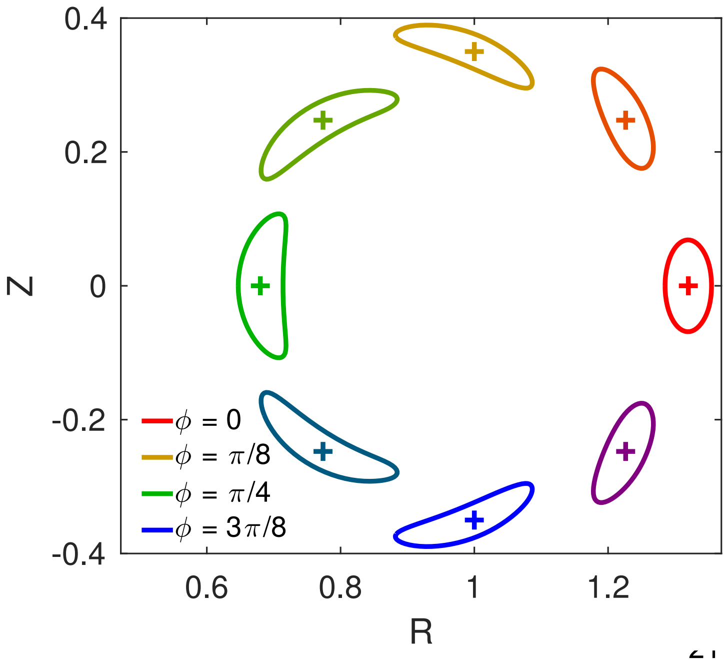
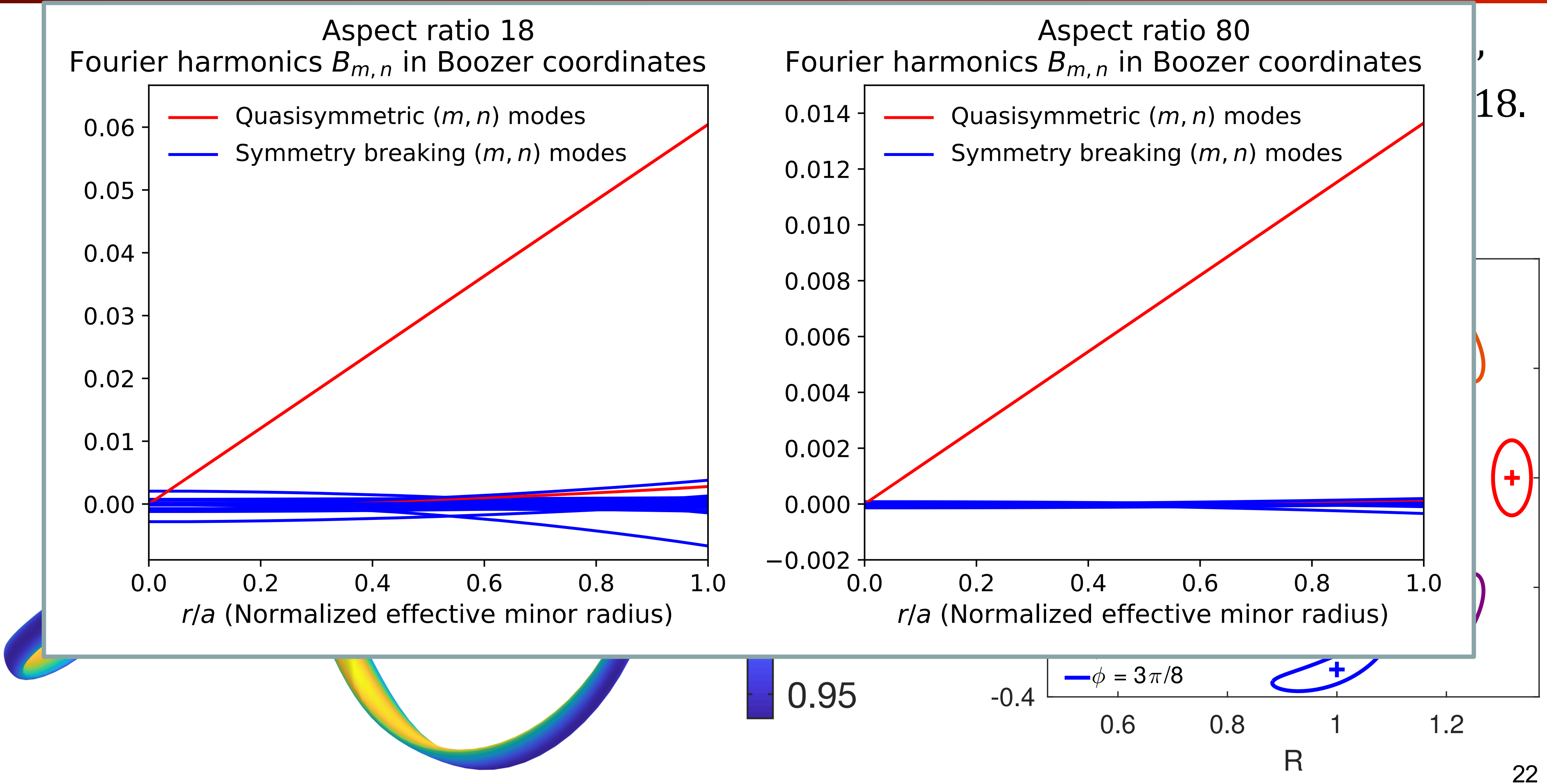


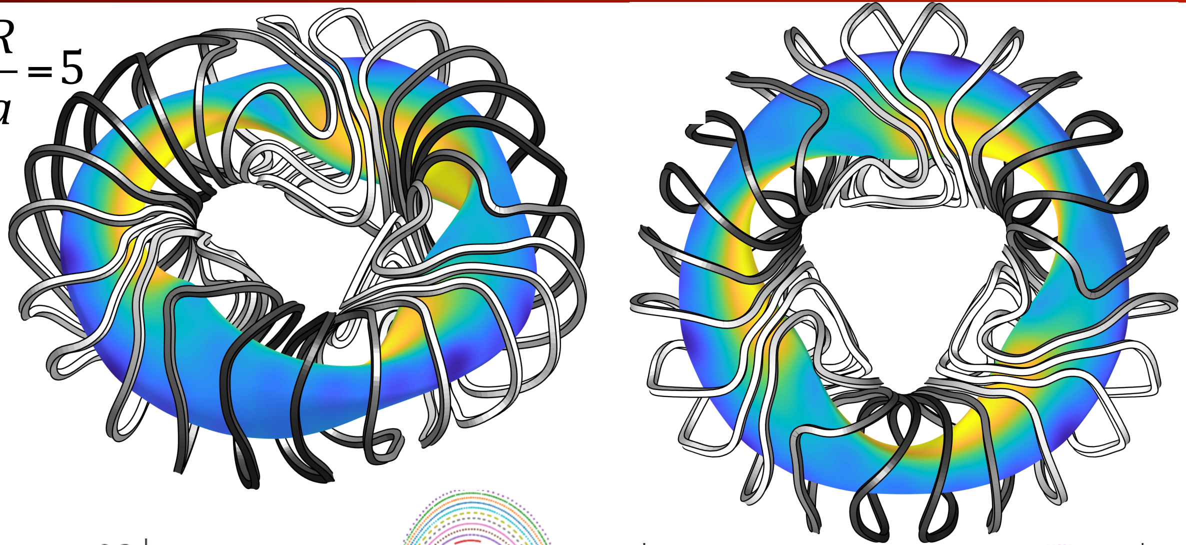


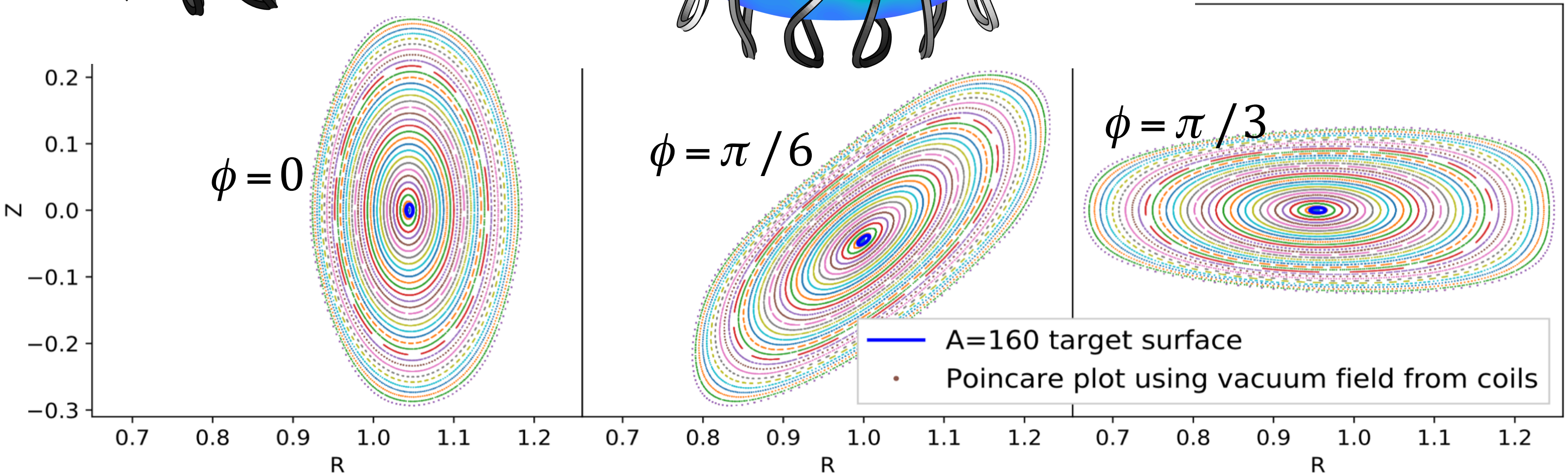
The construction can be verified by comparing to VMEC + BOOZ_XFORM.



Alternative method to generate a finite-thickness boundary: find coils to make a skinny surface, then see what you get outside.

$\frac{R}{a} = 5$





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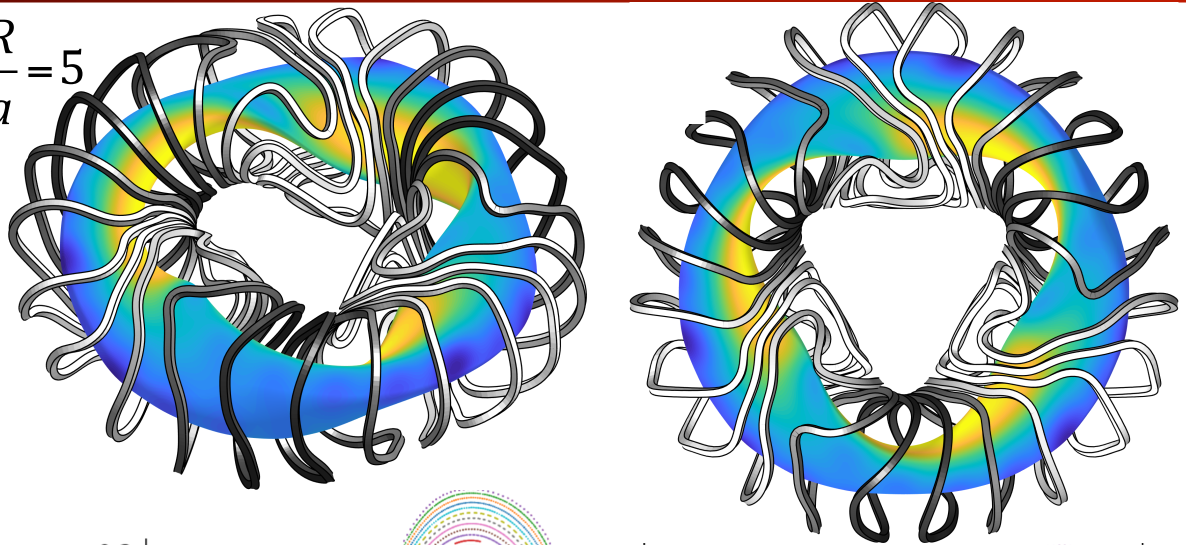

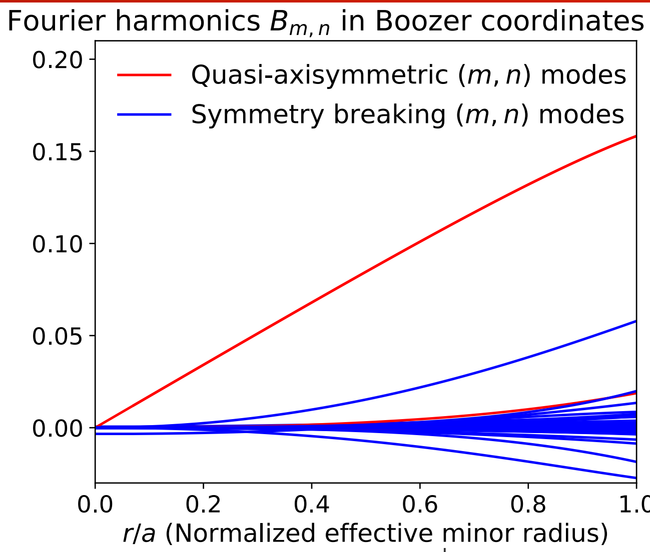

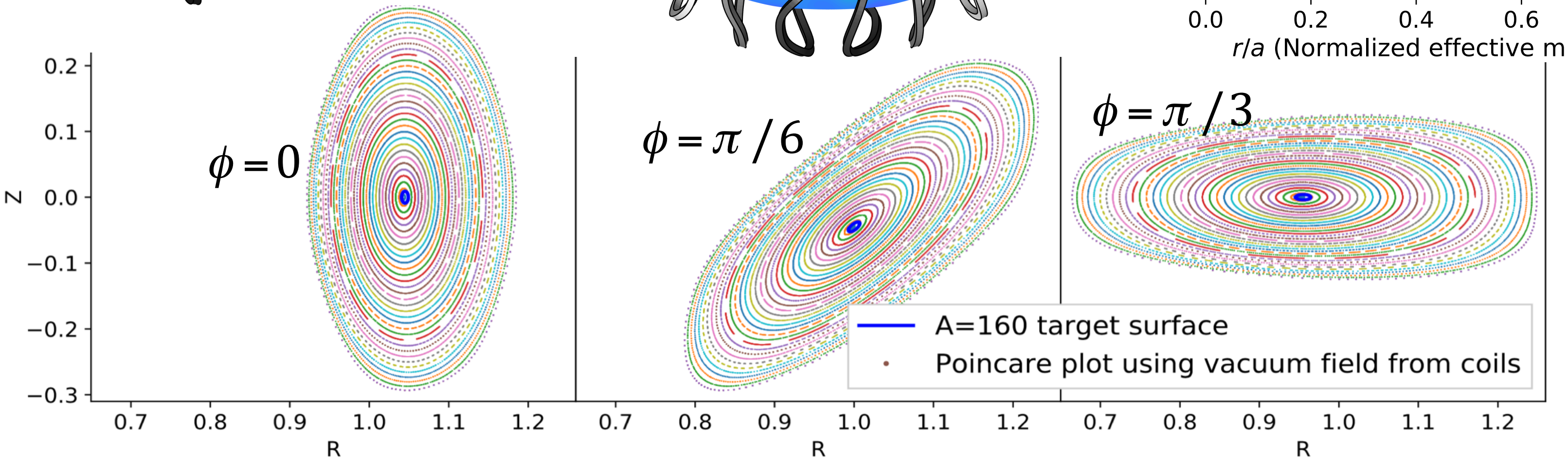

The construction can be fit to quasisymmetric stellarators designed by optimization

- Adopt the same axis shape.
- Fit $\bar{\eta}$ to minimize difference in the shapes of a near-axis surface.

The direct construction gives an accurate match to the on-axis rotational transform in quasisymmetric stellarators designed by optimization

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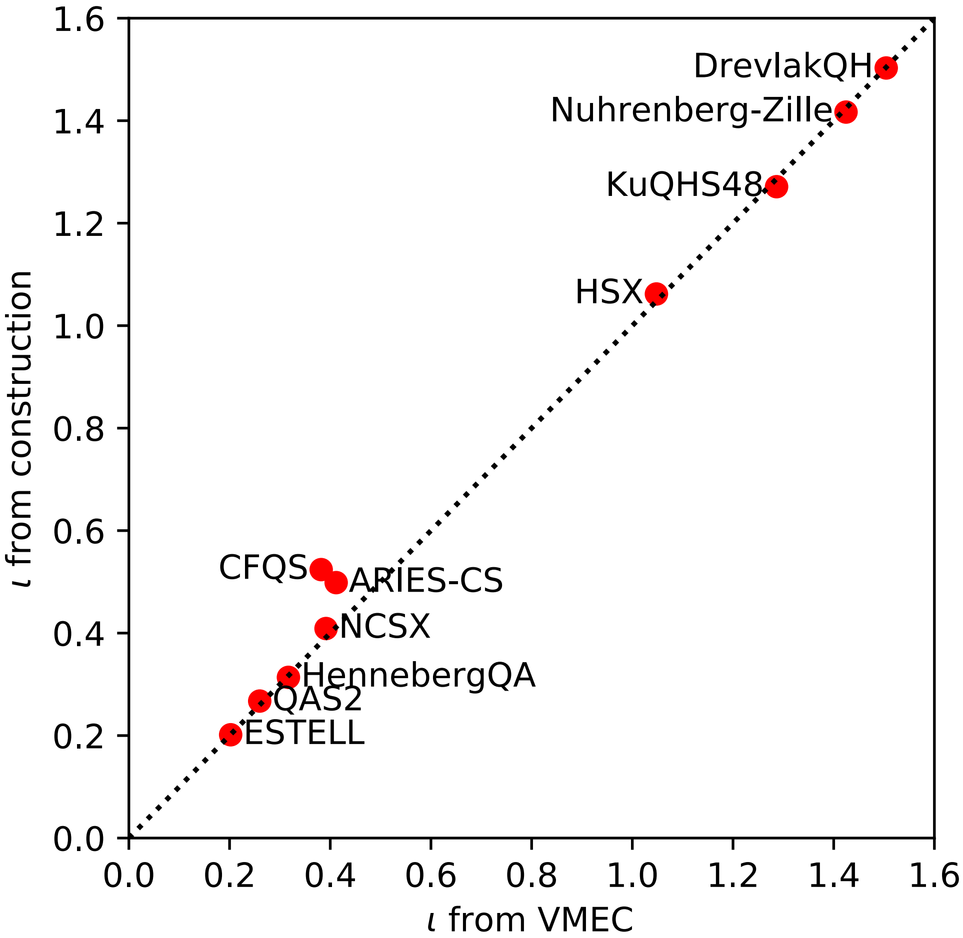
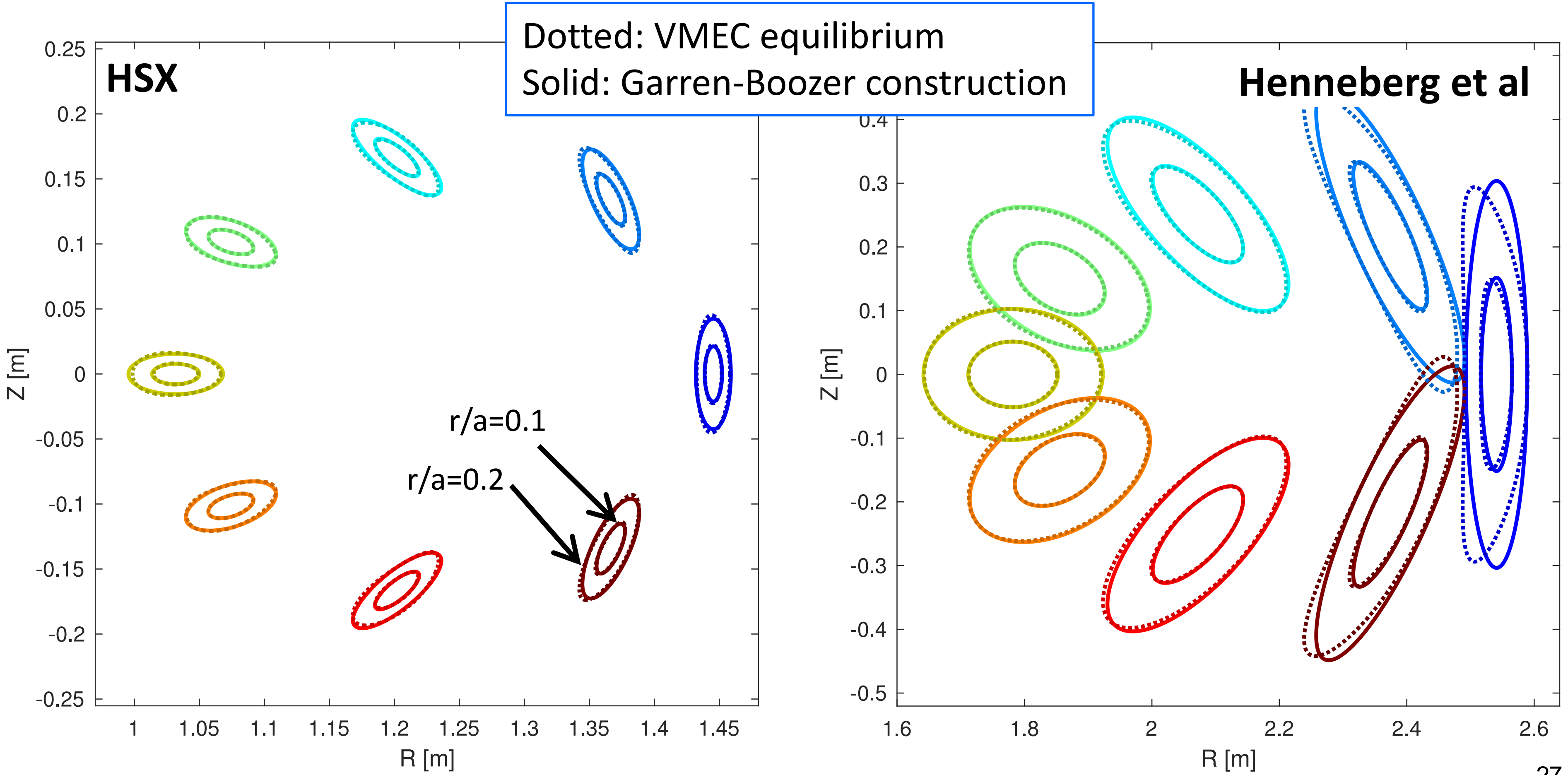


The direct construction gives an accurate match to the near-axis surface shapes of quasisymmetric stellarators designed by optimization

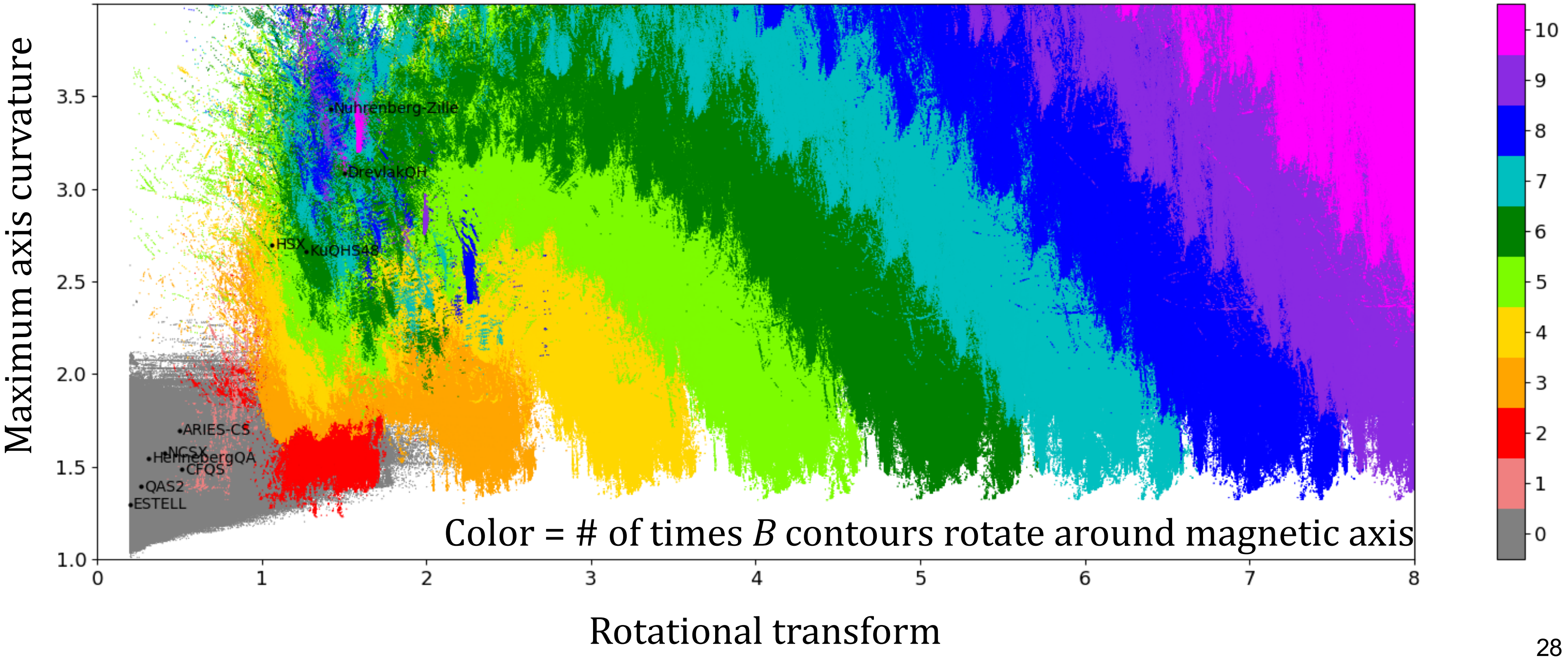
Dotted: VMEC equilibrium
Solid: Garren-Boozer construction



The fast construction enables brute-force surveys of "all" quasisymmetric fields

Axis shape: $R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(j n_{fp} \phi), \quad Z_0(\phi) = 1 + \sum_{j=1}^{3} Z_j \sin(j n_{fp} \phi)$

2.4×10^8 configurations



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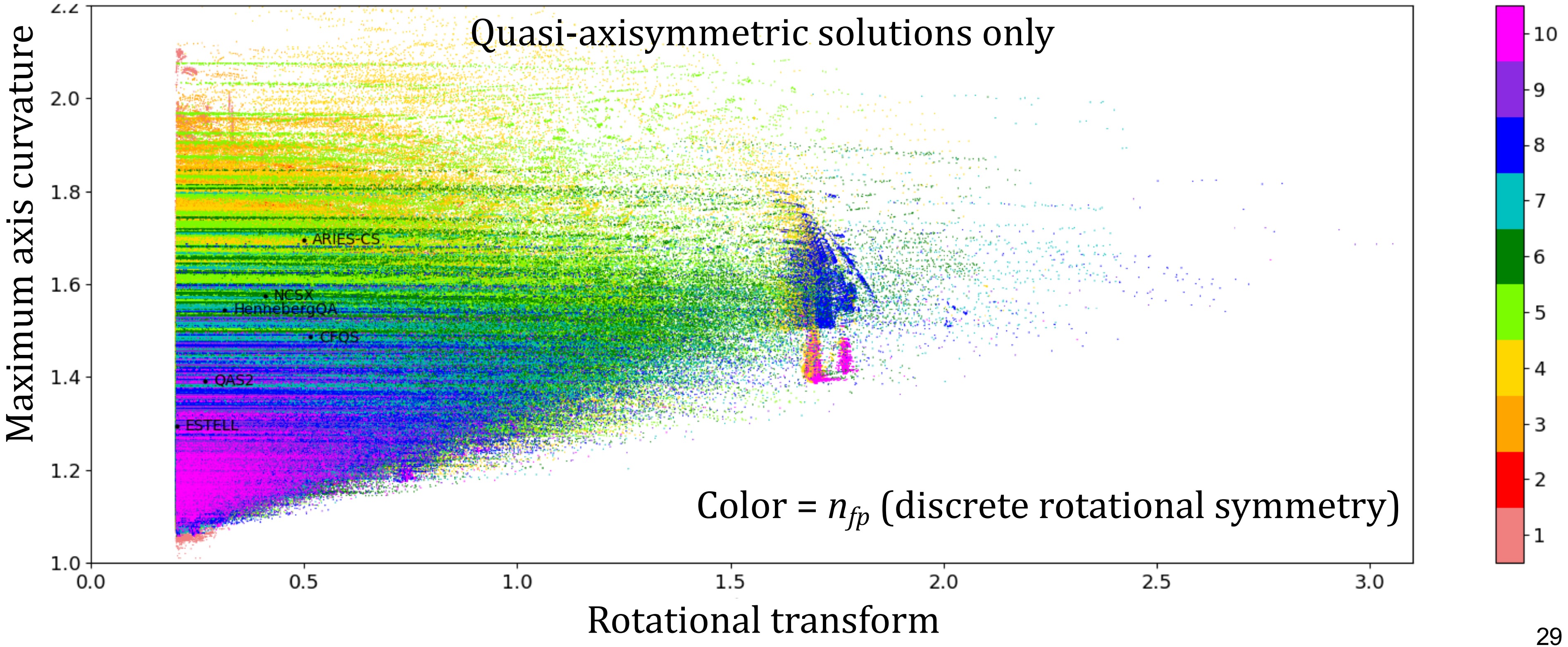
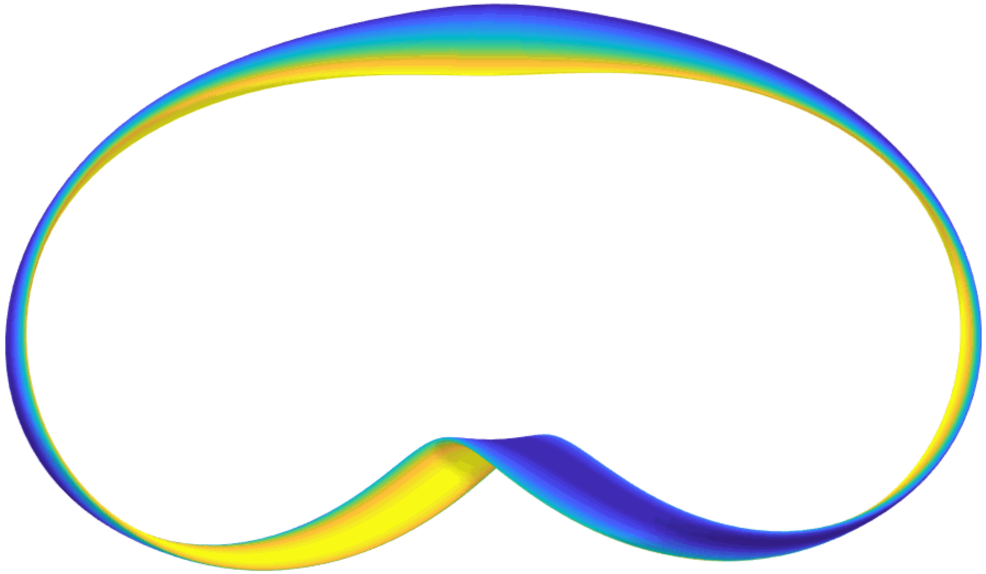
4×10^6 configurations



Brute-force searching is already yielding some new configurations

Quasi-helical symmetry with

1 field period



2 field periods

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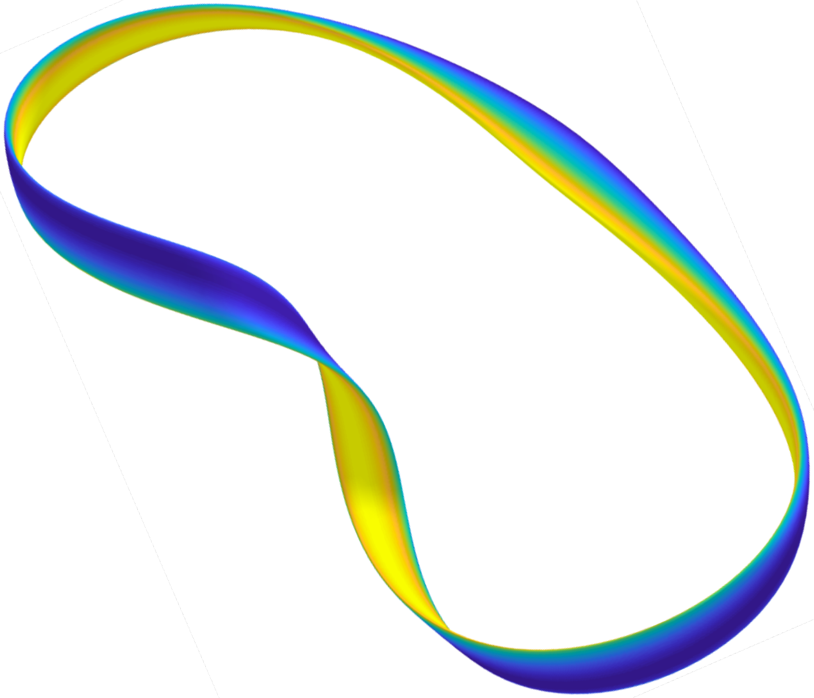

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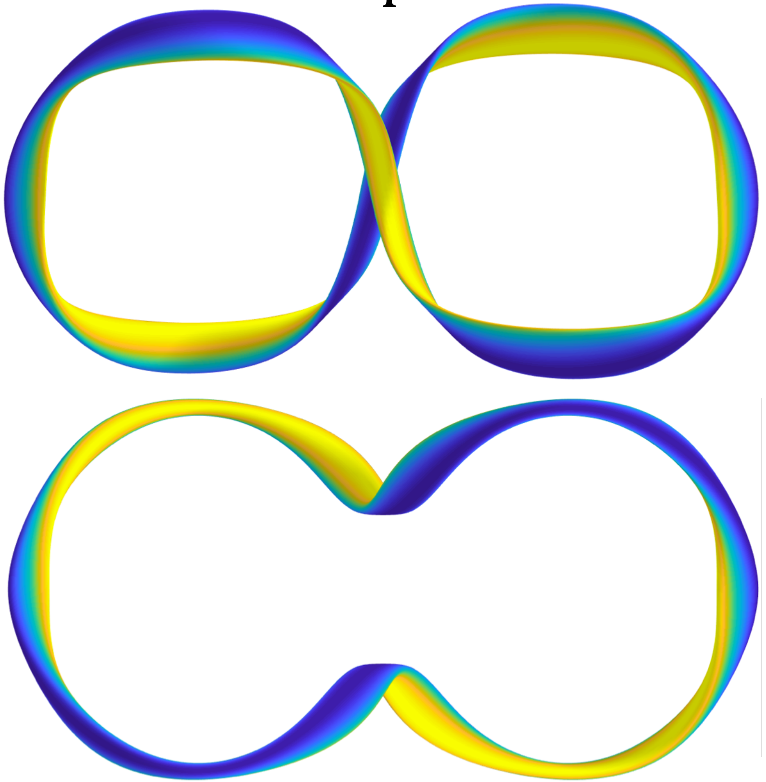

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- $O(r^2)$ quasisymmetry

The $O(r^2)$ construction allows triangularity and more accurate quasisymmetry.

$$\mathbf{x}(r,\vartheta,\zeta)=\mathbf{x}_0(\zeta)+X(r,\vartheta,\zeta)\mathbf{n}(\zeta)+Y(r,\vartheta,\zeta)\mathbf{b}(\zeta)+Z(r,\vartheta,\zeta)\mathbf{t}(\zeta)$$

$$X(r,\vartheta,\zeta)=r\left[X_{1c}\cos\vartheta+X_{1s}\sin\vartheta\right]+r^2\left[X_{20}+X_{2c}\cos 2\vartheta+X_{2s}\sin 2\vartheta\right]+O(r^3)$$

Same for Y & Z.

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Same for Y & Z.

- 3 new input parameters: p_2, B_{2c}, B_{2s}.

$$p(r) = p_0 + r^2 p_2 + O(r^4)$$

$$B(r,\vartheta,\varphi) = B_0 + rB_0\bar{\eta}\cos\vartheta + r^2\left[B_{20} + B_{2c}\cos 2\vartheta + B_{2s}\sin 2\vartheta\right] + O(r^3)$$

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- Net loss of 1 degree of freedom. My approach: $B_{20}(\zeta)$ is an output. Need to adjust inputs so $B_{20}(\zeta)\approx$ constant.
- Shafranov shift appears at this order. Matches textbook tokamak result (e.g. *Wesson, Hazeltine & Meiss*):

$$\left(R-R_0-\Delta\right)^2+z^2=r^2, \qquad \Delta=r^2\left(\frac{1}{8R_0}-\frac{\mu_0 p_2 R_0}{2\iota^2 B_0^2}\right)$$

Example of the $O(r^2)$ construction

Inputs: axis shape $R_0(\phi)=1+0.173\cos(2\phi)+0.0168\cos(4\phi)+0.00101\cos(6\phi),$

$$Z_0(\phi)= \quad 0.158\sin(2\phi)+0.0165\sin(4\phi)+0.000985\sin(6\phi),$$

$$I_2=0,\ \ \sigma(0)=0,\ \ \bar{\eta}=0.632,\ \ p_2=0,\ \ B_{2c}=-0.158,\ \ B_{2s}=0,\ \ R/a=10$$

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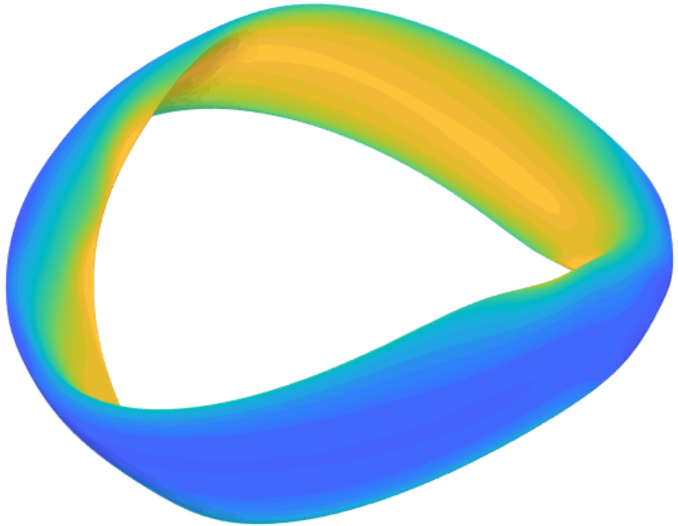
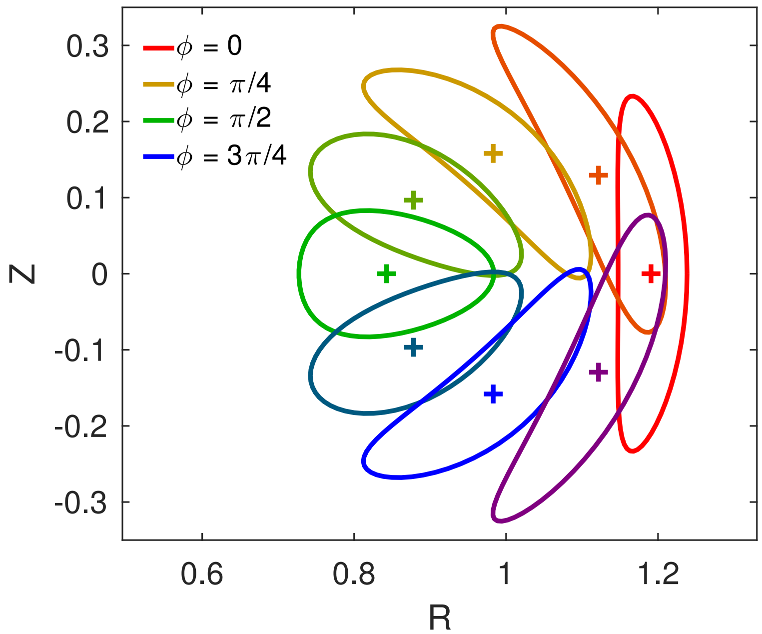
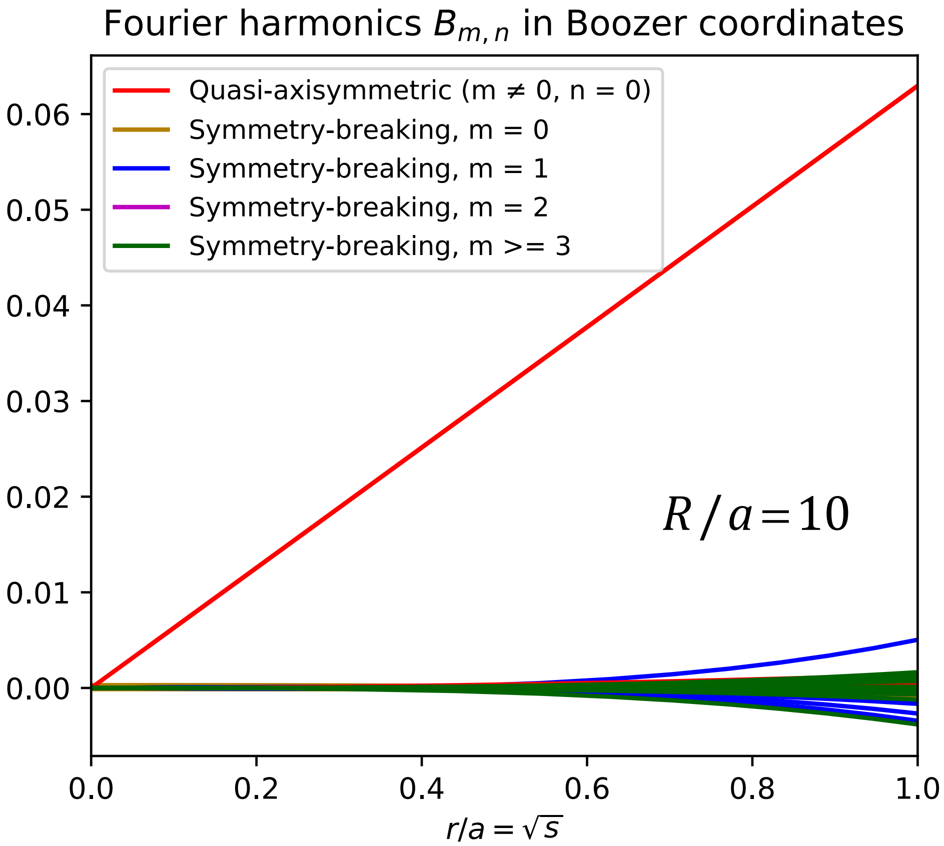
Results: $\iota=0.424$





Quasisymmetry expected for the $O(r^2)$ construction is confirmed by VMEC + BOOZ_XFORM.



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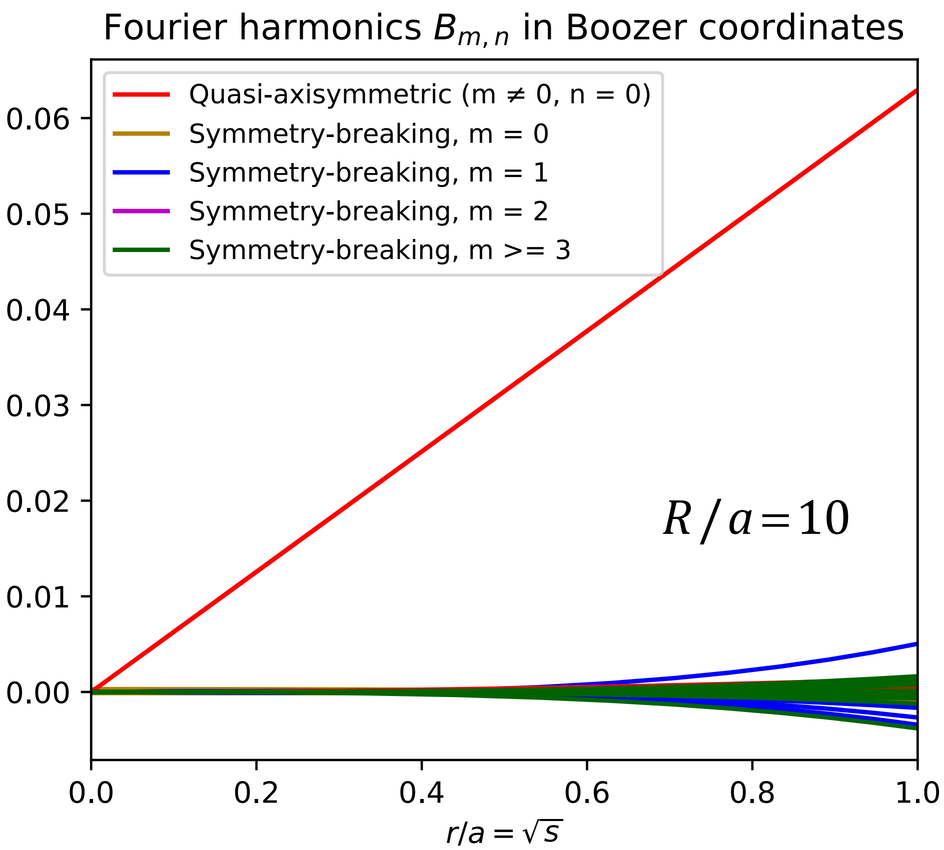

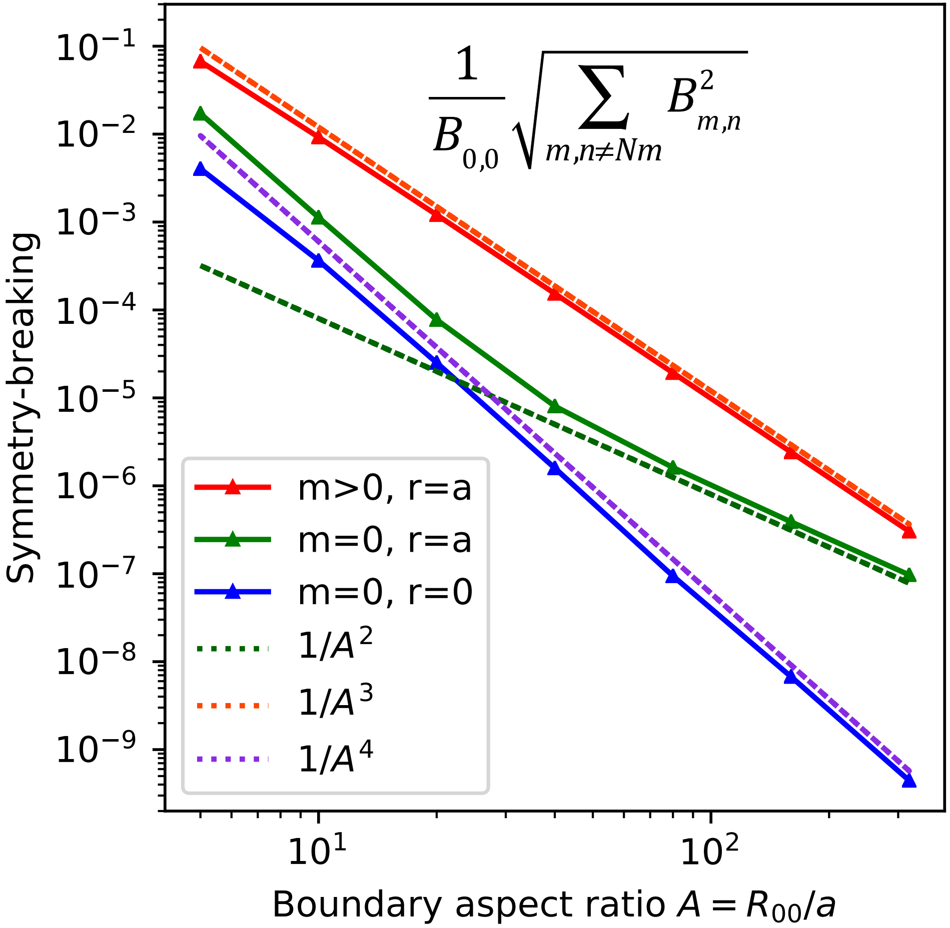

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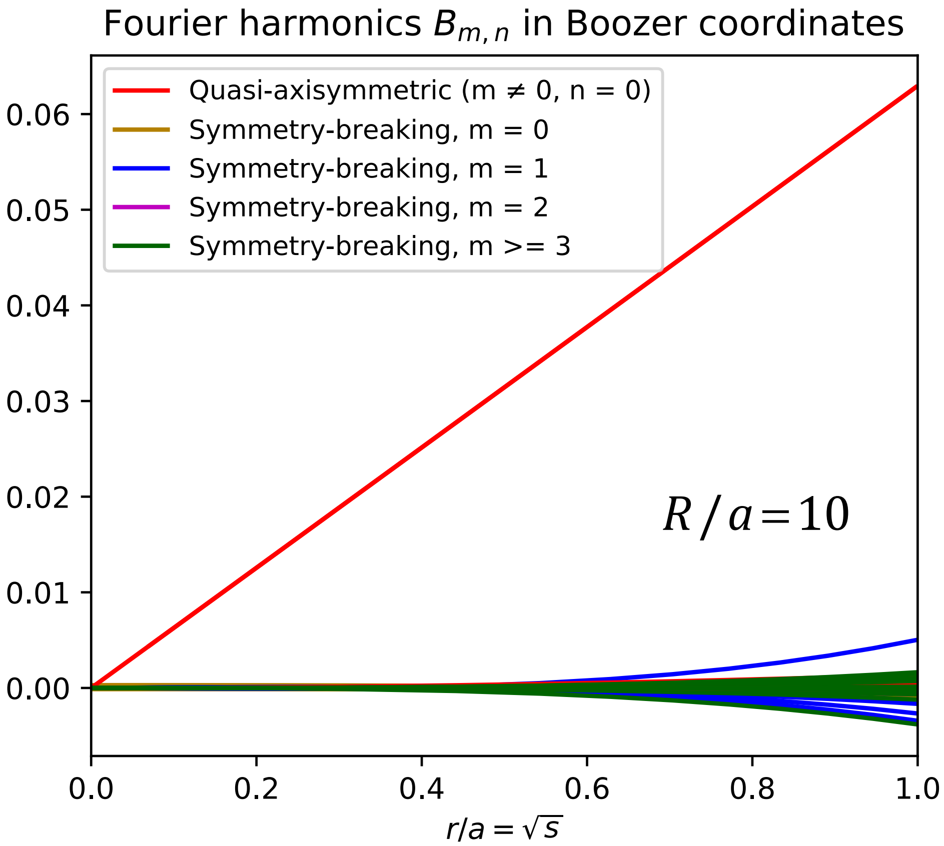
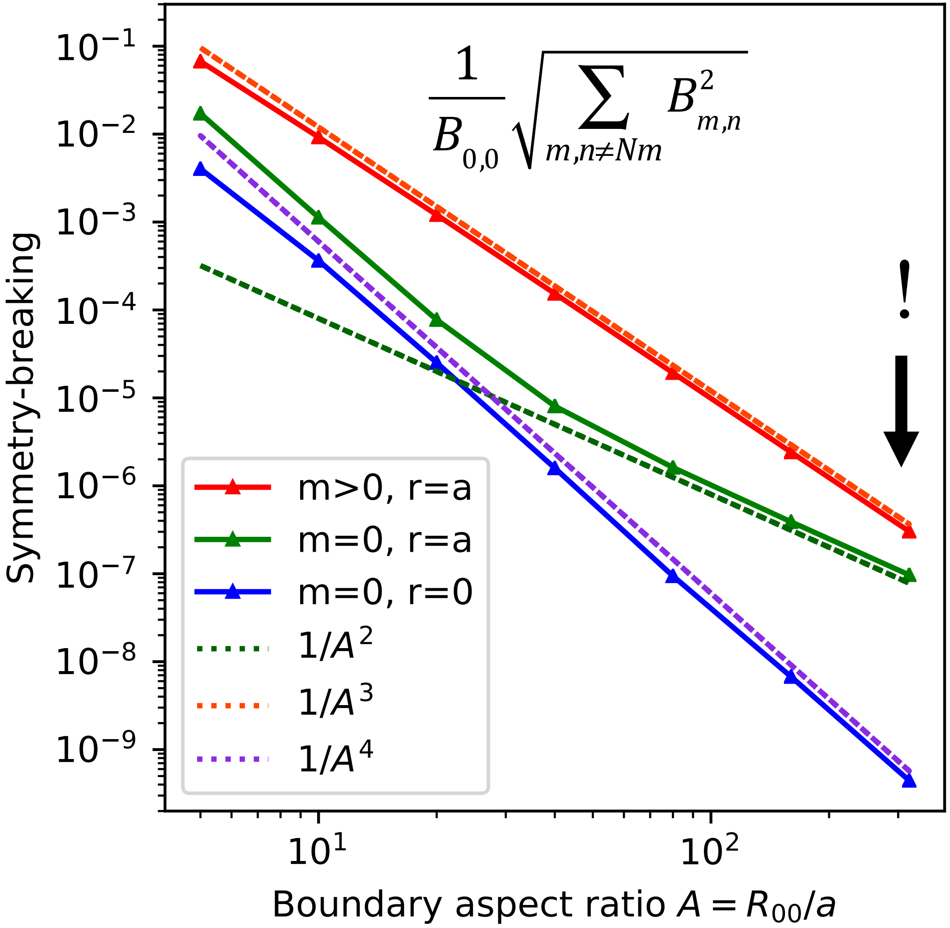




Questions – your input is welcome!

- Does the $O(r^2)$ construction match optimized configurations?
- Are there other ways to extrapolate outward from the axis? E.g. Laplace's equation as initial value problem in r, with regularization.
- It is effective to optimize in the space of axis shapes? (H Mynick).
- For $O(r^2)$, how do you best solve for the axis shape? Can anything be proved about the number or character of $O(r^2)$ solutions?
- Is there an analogous construction to give quasisymmetry at an off-axis surface?

Conclusions

- The equations for quasisymmetric magnetic fields can be solved directly (without optimization) if you expand about the magnetic axis.
- The resulting construction can be useful for generating new initial conditions for optimization.
- We precisely understand the size of the space of magnetic fields that are quasisymmetric near the axis (i.e. to $O(r)$).
- The construction is consistent with configurations obtained by optimization.
- There is hope of definitively identifying all classes of practical quasisymmetric fields (near the axis).

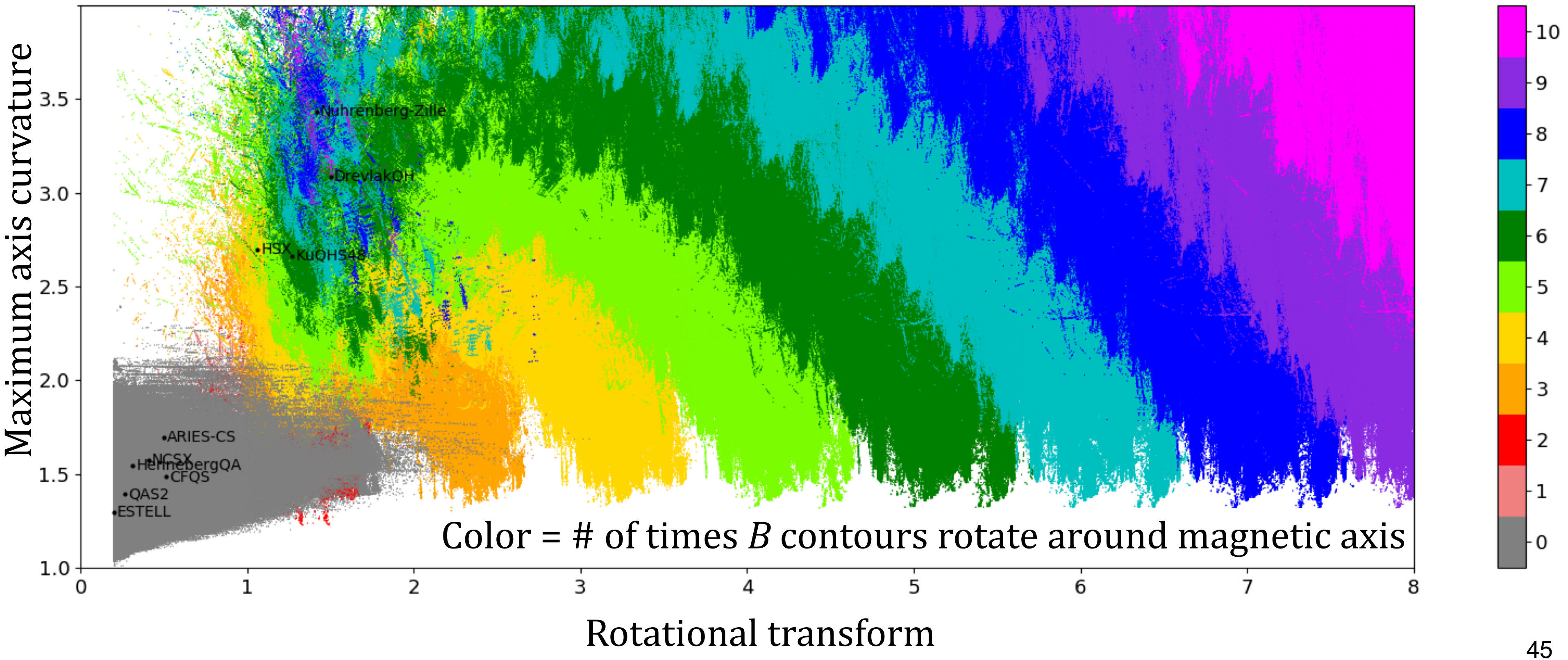
J Plasma Phys (2019)   PPCF (2019)   https://github.com/landreman/quasisymmetry

Extra slides

The fast construction enables brute-force surveys of "all" quasisymmetric fields

Axis shape: $R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(j n_{fp} \phi)$, $\quad Z_0(\phi) = 1 + \sum_{j=1}^{3} Z_j \sin(j n_{fp} \phi)$

2.4×10^8 configurations

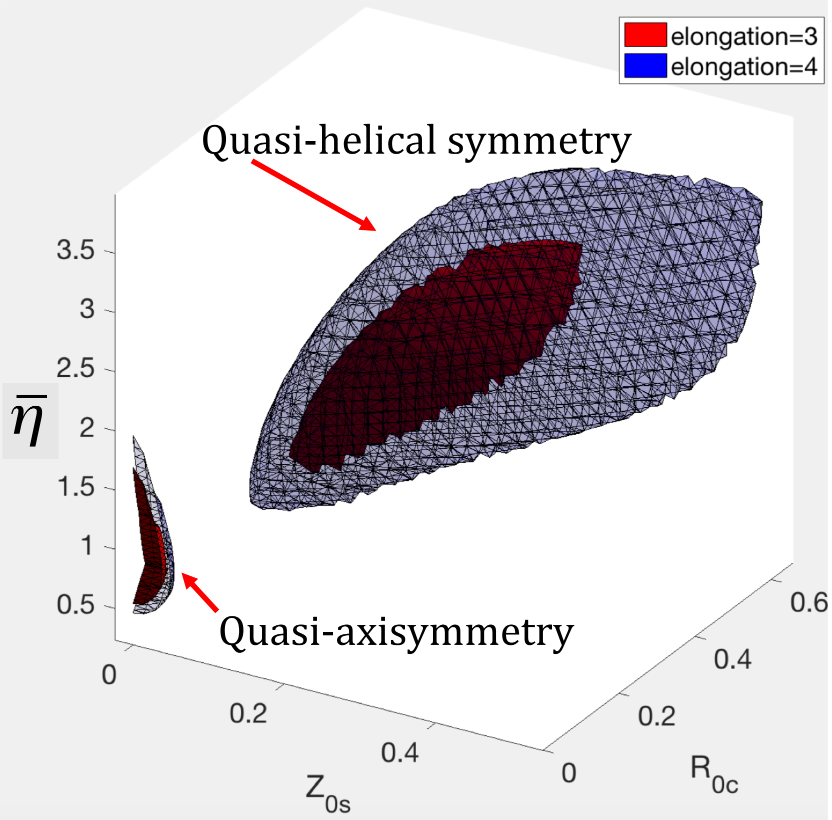


The construction enables fast scans over parameter space.

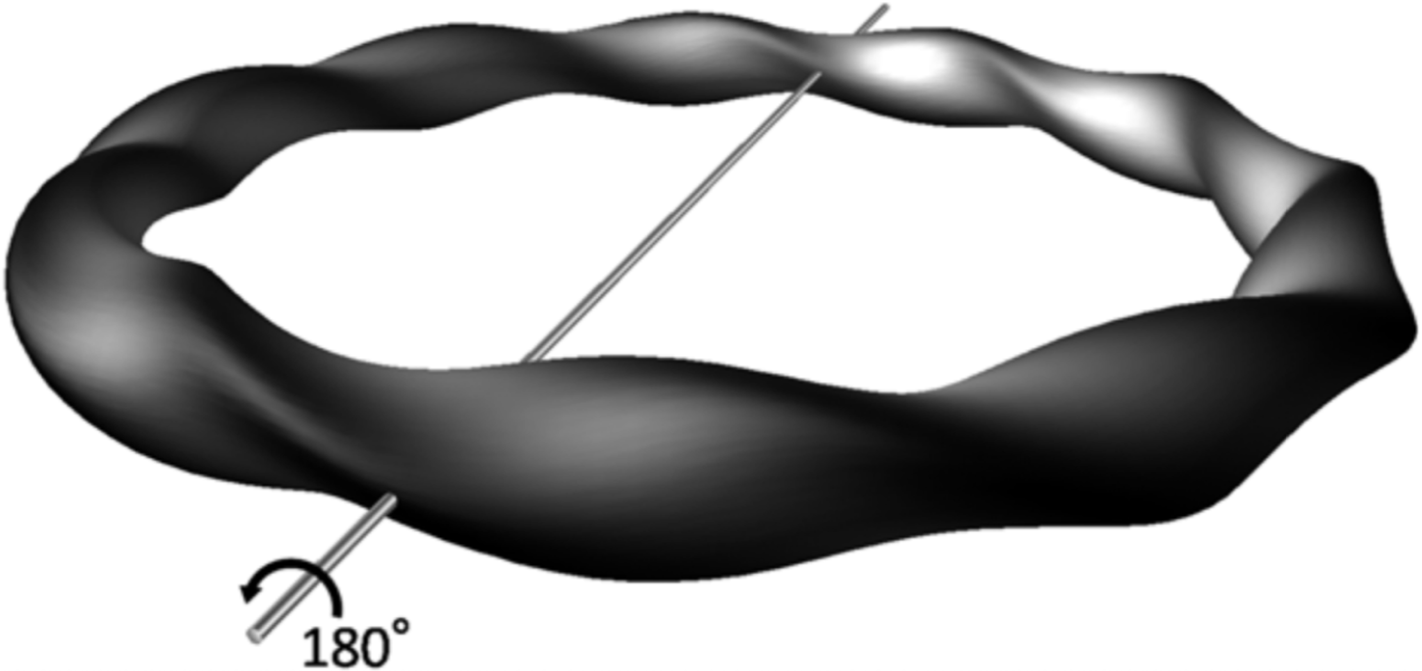
E.g. Scan over $\left(R_{0c}, Z_{0s}, \bar{\eta}\right)$ where magnetic axis shape is

$$R_0(\phi) = 1 + R_{0c}\cos(4\phi)$$
$$Z_0(\phi) = Z_{0s}\sin(4\phi)$$

274,560 solutions generated in <30s on a laptop.



All stellarators built to date have 'stellarator symmetry', which is unrelated to quasisymmetry

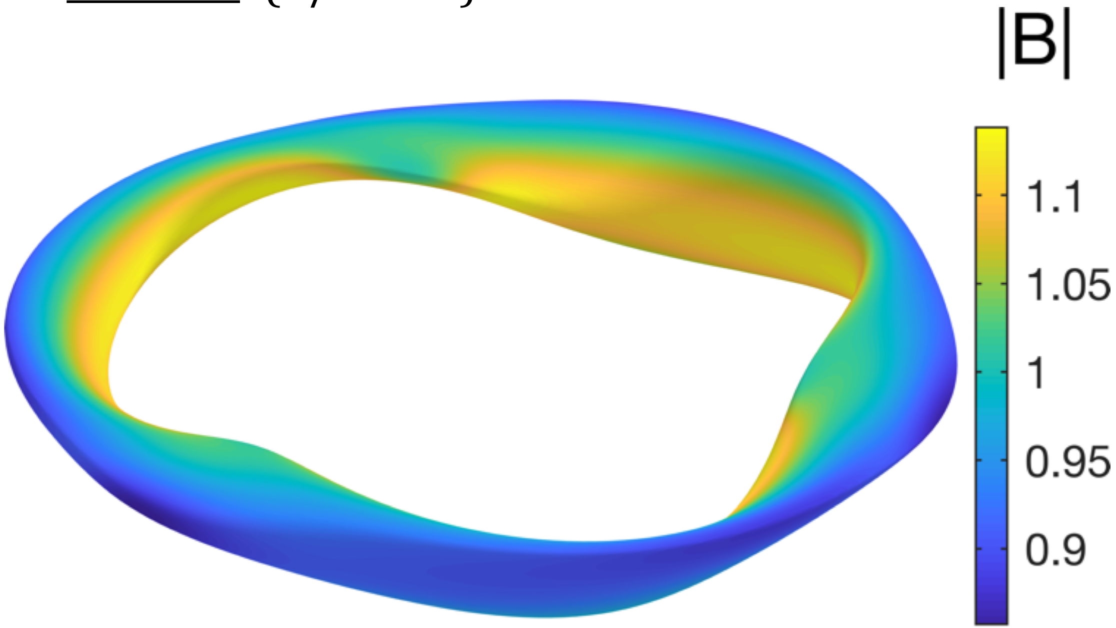


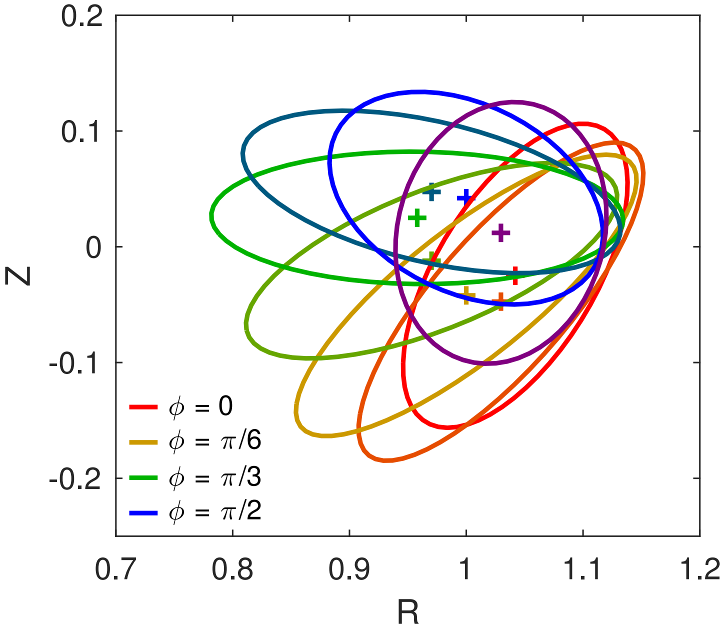
Sugama et al (2011)

You can make a quasi-axisymmetric stellarator without stellarator symmetry

Inputs: axis shape $R_0(\phi) = 1 + 0.042\cos(3\phi)$, $Z_0(\phi) = -0.042\sin(3\phi) - 0.025\cos(3\phi)$, $I_2 = 0$, $\sigma(0) = -0.6$, $\bar{\eta} = -1.1$.

Results: $(R/a = 10)$





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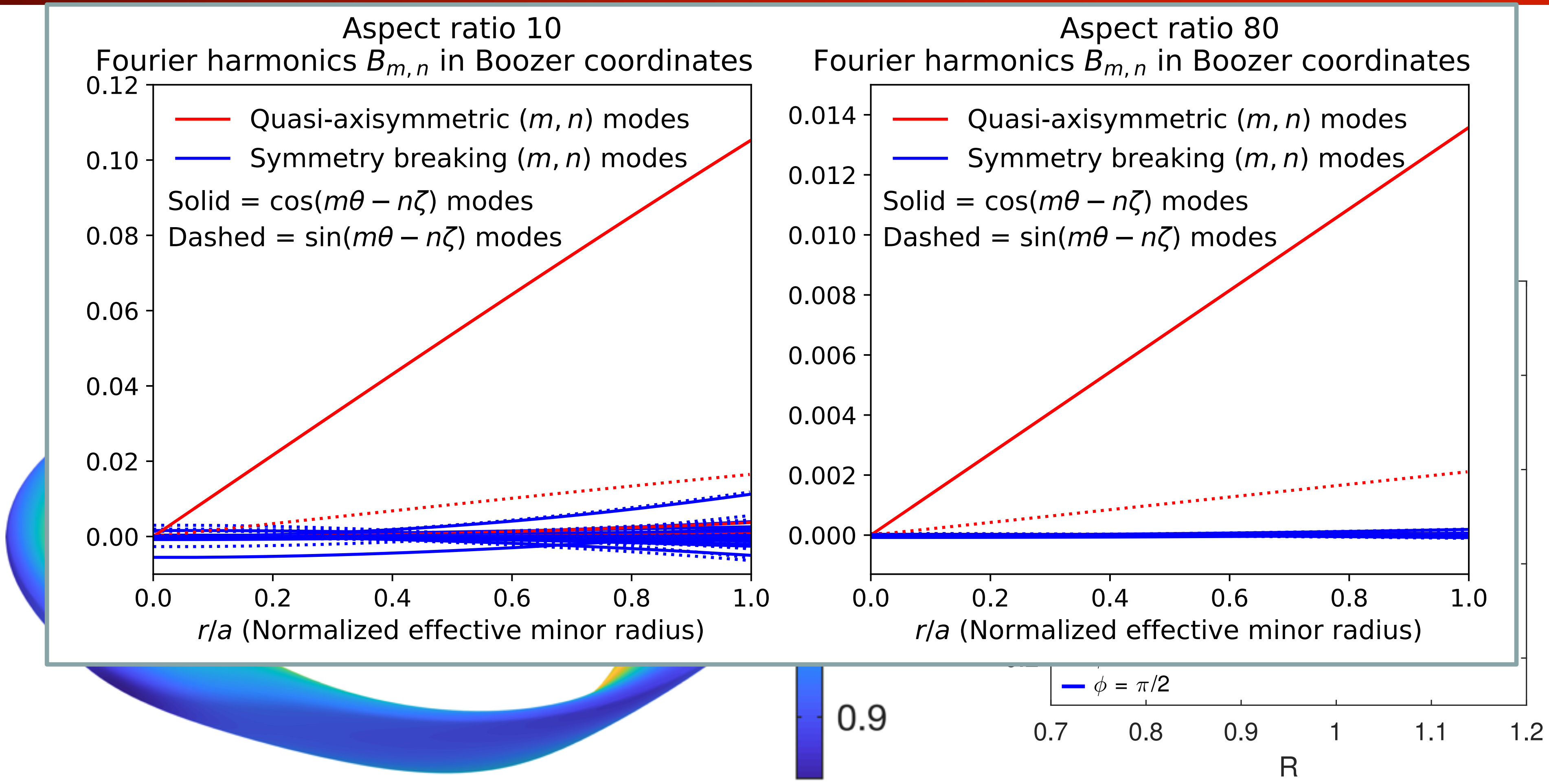
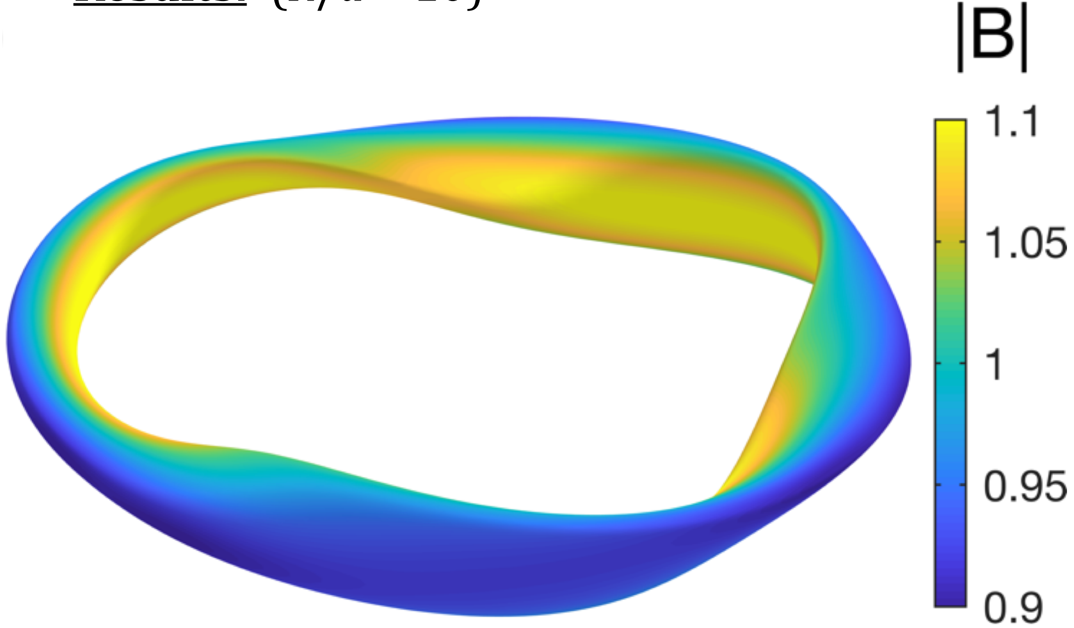


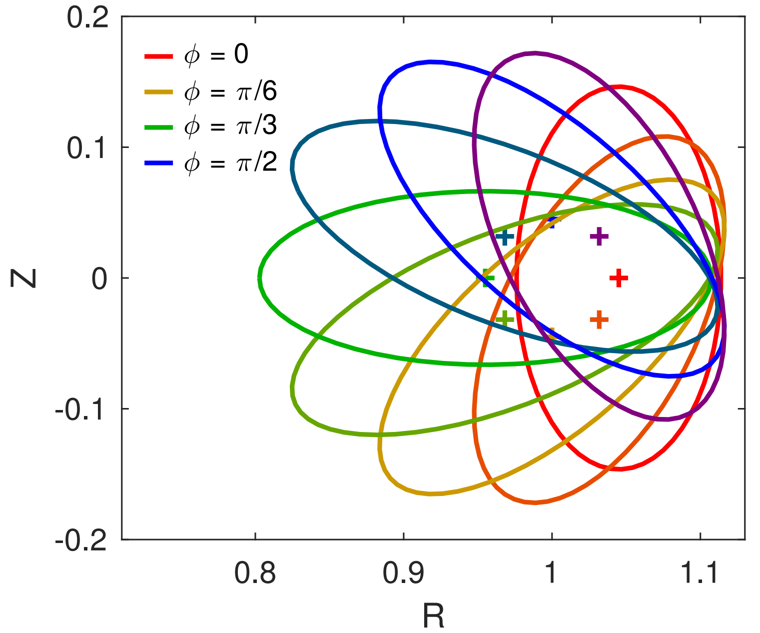
Example construction: quasi-axisymmetry

Inputs: axis shape $R_0(\phi)=1+0.045\cos(3\phi)$, $I_2=0$, $\bar{\eta}=-0.9$.

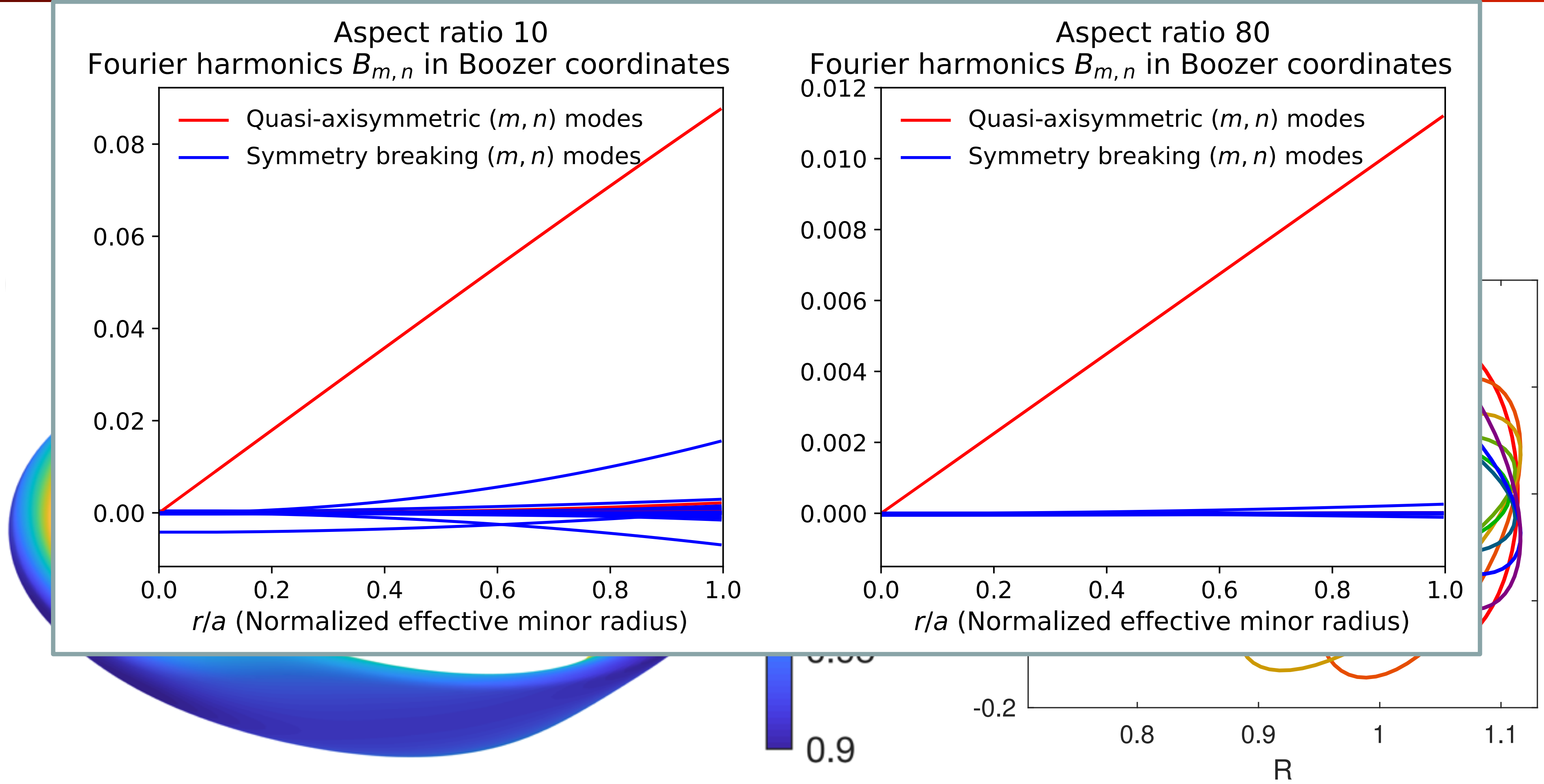
$Z_0(\phi)=-0.045\sin(3\phi)$, $\sigma(0)=0$,

Results: $(R/a = 10)$

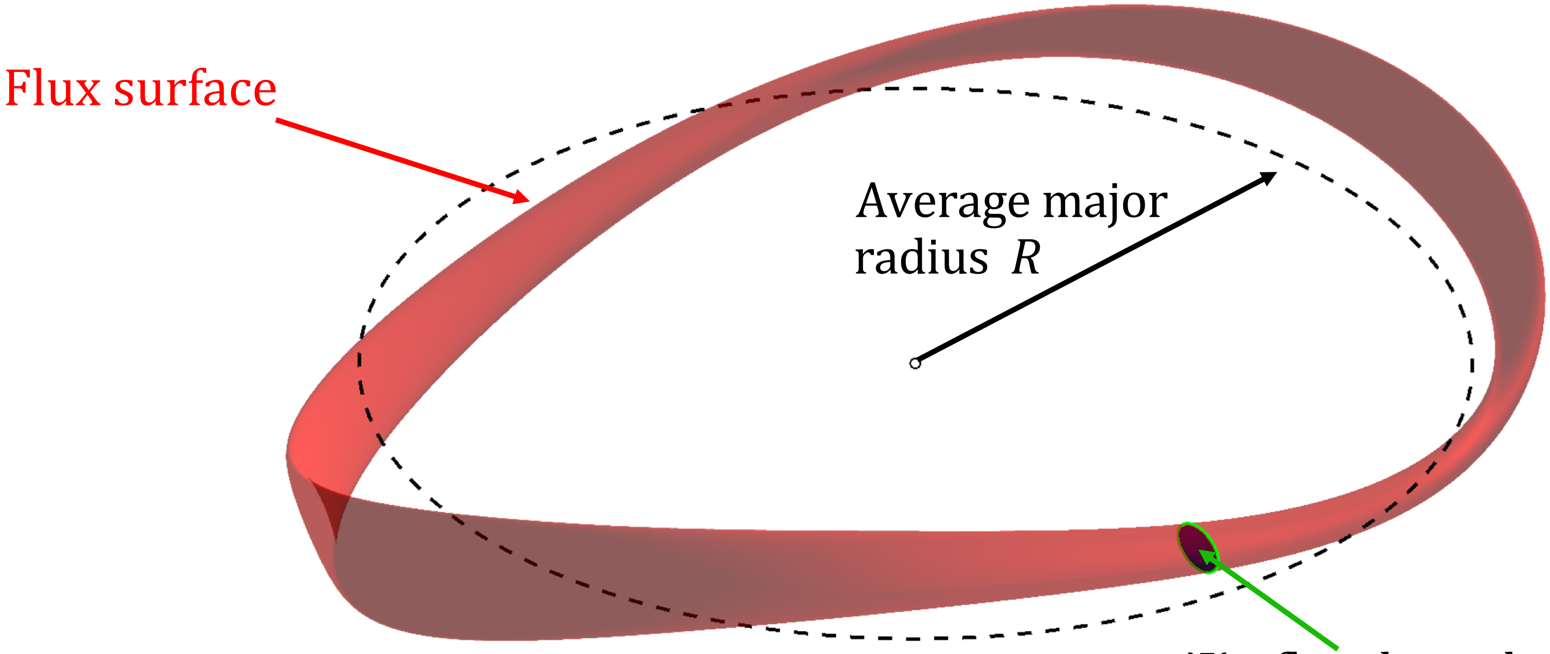




The construction can be verified by comparing to an MHD equilibrium calculation that does not make the *r* expansion.



We will expand in the skinniness of the inner flux surfaces



$\Psi = \text{flux through patch} = \int \mathbf{B} \cdot d\mathbf{a}$

Define effective radius r by $\Psi = \pi r^2 B_{axis}$.

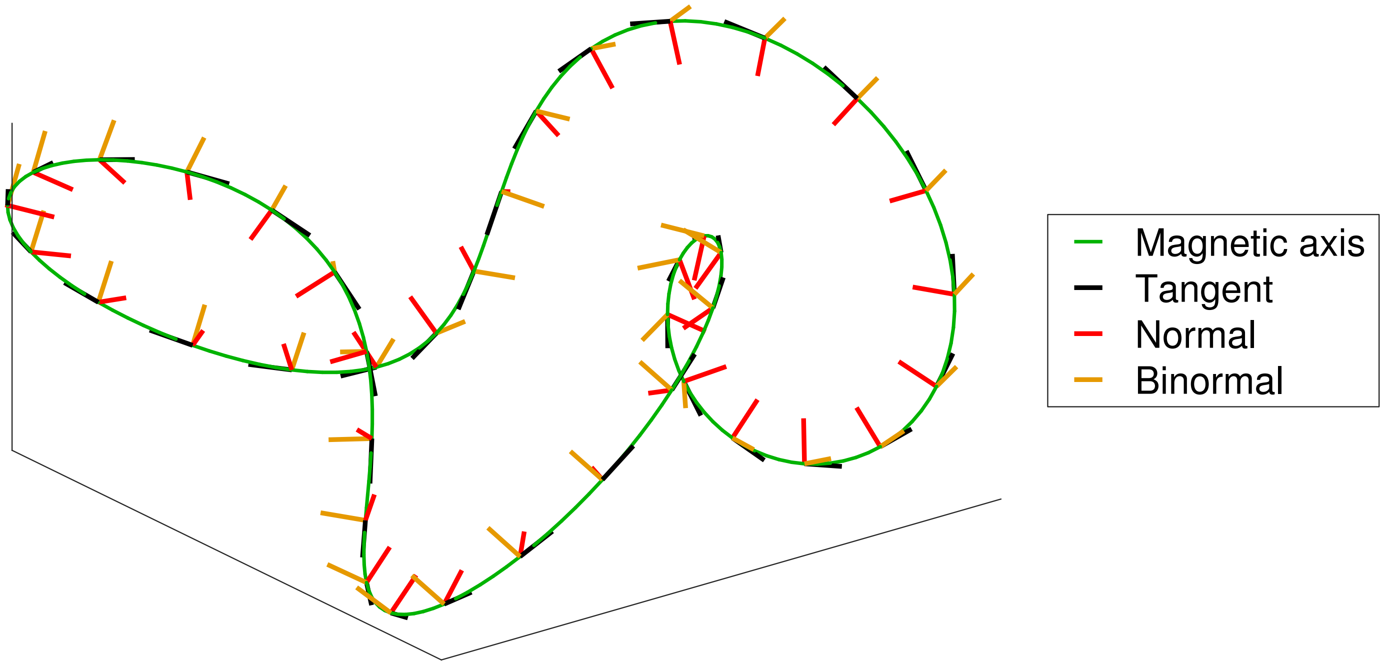
"Aspect ratio" $= \dfrac{R}{r}$ is $\gg 1$

Theory: Expand position vector using Frenet frame, equate 2 representations of B.

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}):\quad \frac{\partial \mathbf{r}_0}{\partial \ell} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{d\ell} = \kappa \mathbf{n}, \qquad \frac{d\mathbf{n}}{d\ell} = -\kappa \mathbf{t} + \tau \mathbf{b}, \qquad \frac{d\mathbf{b}}{d\ell} = -\tau \mathbf{n}$$

$\mathbf{r}_0$ = magnetic axis, κ = curvature, τ = torsion

$\mathbf{t}$ = tangent, $\mathbf{n}$ = normal, $\mathbf{b}$ = binormal

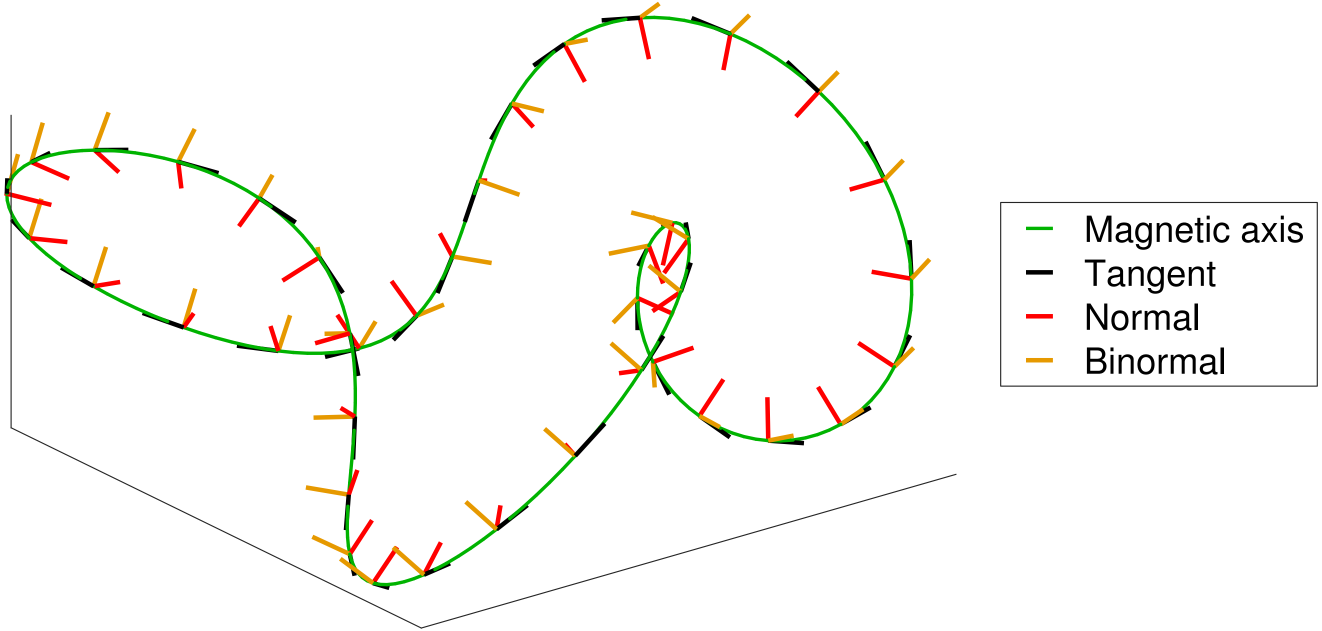


Theory: Write position vector using Frenet frame

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}):\quad \frac{d\mathbf{r}_0}{d\ell}=\mathbf{t},\qquad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n},\qquad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b},\qquad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

$\mathbf{r}_0$ = magnetic axis, κ = curvature, τ = torsion, $\mathbf{t}$ = tangent, $\mathbf{n}$ = normal, $\mathbf{b}$ = binormal

$$\mathbf{r}(r,\theta,\zeta)=\mathbf{r}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta)$$



Theory: Write position vector using Frenet frame, expand in small r = (flux)$^{1/2}$

Frenet frame $(\mathbf{t},\mathbf{n},\mathbf{b})$: $\frac{d\mathbf{r}_0}{d\ell}=\mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n}, \quad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b}, \quad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$

$\mathbf{r}_0=$ magnetic axis, $\kappa=$ curvature, $\tau=$ torsion, $\mathbf{t}=$ tangent, $\mathbf{n}=$ normal, $\mathbf{b}=$ binormal

$$\begin{aligned}
\mathbf{r}(r,\theta,\zeta) &= \mathbf{r}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta) \\
&= \mathbf{r}_0(\zeta)+rX_{1c}(\zeta)\cos\theta\,\mathbf{n}(\zeta)+r\left[Y_{1s}(\zeta)\sin\theta+Y_{1c}(\zeta)\cos\theta\right]\mathbf{b}(\zeta)+O(r^2)
\end{aligned}$$

Using magnetohydrodynamic equilibrium $(\mathbf{J}\times\mathbf{B}=\nabla p)$

Theory: Write position vector using Frenet frame, expand in small r = (flux)$^{1/2}$

Frenet frame $(\mathbf{t},\mathbf{n},\mathbf{b})$: $\frac{d\mathbf{r}_0}{d\ell}=\mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n}, \quad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b}, \quad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$

$\mathbf{r}_0=$ magnetic axis, $\kappa=$ curvature, $\tau=$ torsion, $\mathbf{t}=$ tangent, $\mathbf{n}=$ normal, $\mathbf{b}=$ binormal

$$\mathbf{r}(r,\theta,\zeta)=\mathbf{r}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta)$$

$$=\mathbf{r}_0(\zeta)+rX_{1c}(\zeta)\cos\theta\,\mathbf{n}(\zeta)+r\left[Y_{1s}(\zeta)\sin\theta+Y_{1c}(\zeta)\cos\theta\right]\mathbf{b}(\zeta)+O(r^2)$$

$$X_{1c}(\zeta)=\frac{\bar{\eta}}{\kappa(\zeta)}, \qquad Y_{1s}(\zeta)=\frac{\kappa(\zeta)}{\bar{\eta}}, \qquad Y_{1c}(\zeta)=\frac{\sigma(\zeta)\kappa(\zeta)}{\bar{\eta}}$$

Toroidal angle $\zeta\propto$ arclength, $\bar{\eta}=$ constant: $B=B_0\left[1+r\bar{\eta}\cos(\theta-N\varphi)+O(r^2)\right]$

$$\frac{d\sigma}{d\zeta}+\iota\left[\frac{\bar{\eta}^4}{\kappa^4}+1+\sigma^2\right]-2\frac{\bar{\eta}^2}{\kappa^2}\left[I_2-\tau\right]=0$$

$I_2=$ current density

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$$\mathbf{r}_0 = \text{magnetic axis}, \qquad \kappa = \text{curvature}, \qquad \tau = \text{torsion}$$

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$\mathbf{r}_0 =$ magnetic axis, $\kappa =$ curvature, $\tau =$ torsion

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$$X(r,\theta,\zeta)=r\left[X_{1s}(\zeta)\sin\theta+X_{1c}(\zeta)\cos\theta\right]+O(r^2).$$ Same for Y, Z.

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$$X(r,\theta,\zeta) = r\left[X_{1s}(\zeta)\sin\theta + X_{1c}(\zeta)\cos\theta\right] + O(r^2). \quad \text{Same for } Y, Z.$$

$$\mathbf{B} = B_r \nabla r + B_\theta \nabla\theta + B_\zeta \nabla\zeta, \qquad \mathbf{B} = \nabla\psi\times\nabla\theta + \iota\nabla\zeta\times\nabla\psi$$

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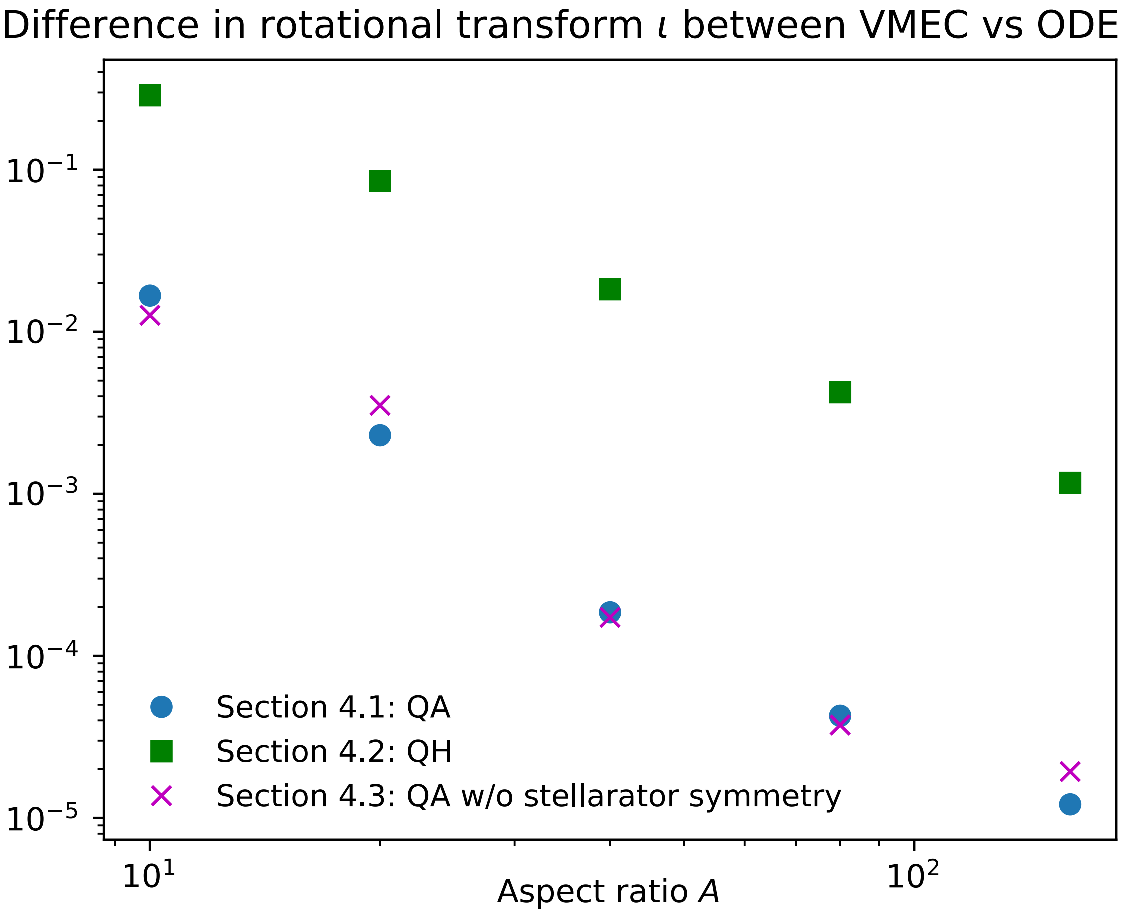
$$\mathbf{B} = B_r \nabla r + B_\theta \nabla \theta + B_\zeta \nabla \zeta, \qquad \mathbf{B} = \nabla \psi \times \nabla \theta + \iota \nabla \zeta \times \nabla \psi$$

$$\text{Dual relations:} \quad \nabla r = \left[\frac{\partial \mathbf{r}}{\partial r} \cdot \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \zeta}\right]^{-1} \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \zeta}, \quad \text{cyclic permutations.}$$

The rotational transform computed by VMEC converges to the value computed by the Garren-Boozer approach.



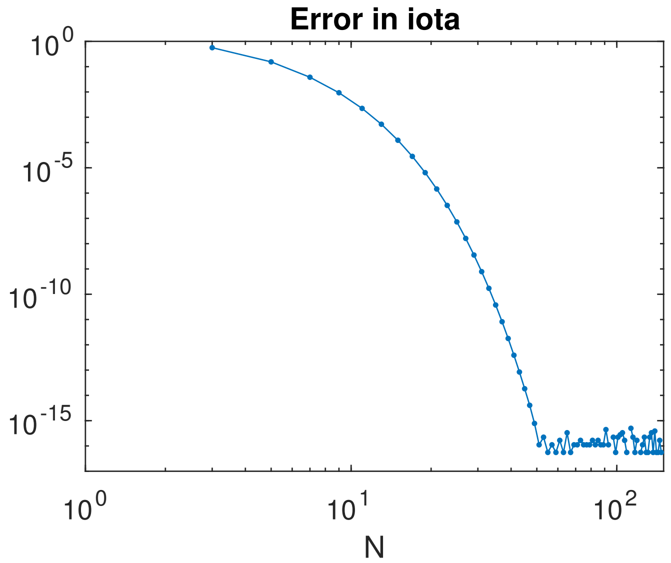
The ODE is solved with spectral accuracy using pseudospectral discretization + Newton iteration

Uniform grid in ϕ with N points: $\phi_1 = 0,\ \phi_2 = 2\pi/(Nn_{fp}),\ ...,\ \phi_N = 2\pi(N-1)/(Nn_{fp})$.

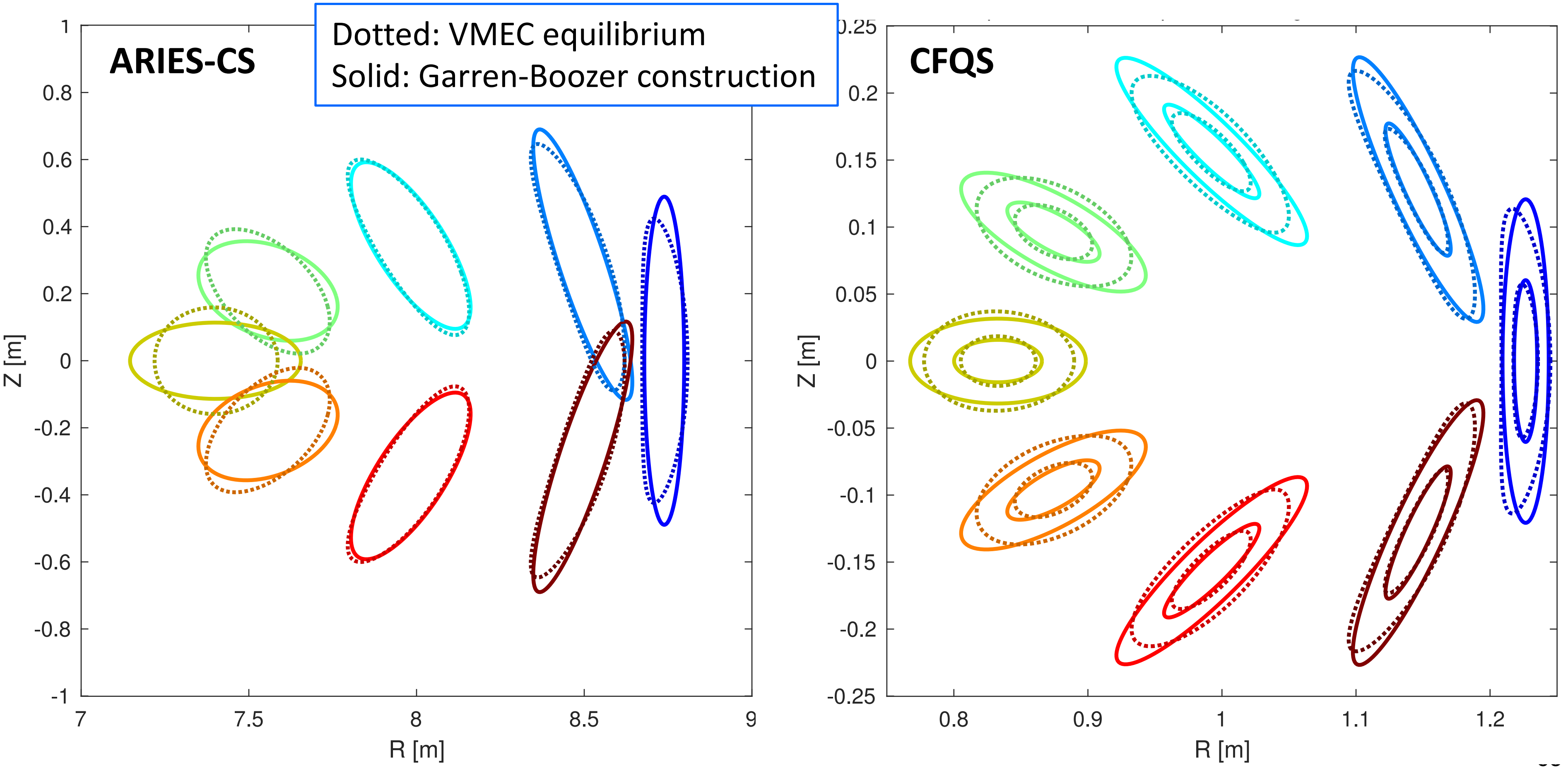
Vector of N unknowns: $\left(\iota,\ \sigma(\phi_2),\ \sigma(\phi_3),\ ...,\ \sigma(\phi_N)\right)^T$

N equations: impose ODE at $\phi_1,\ ...,\ \phi_N$.

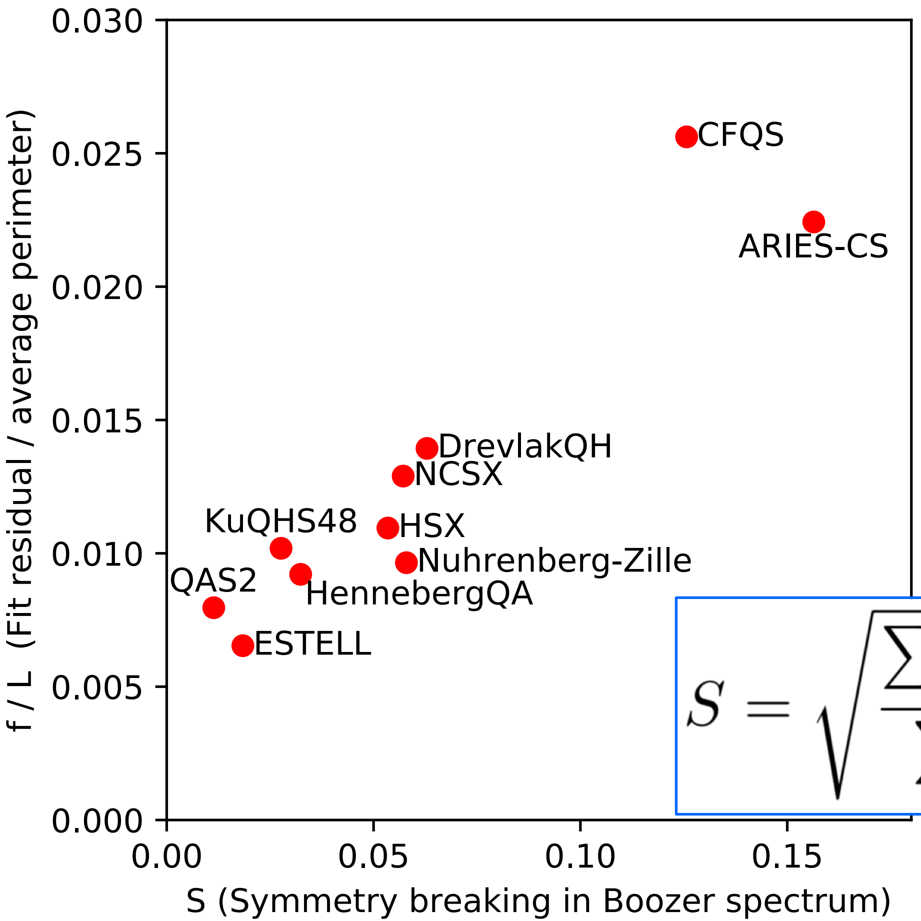
$\frac{d\sigma}{d\phi} \rightarrow D\sigma$ where D is a pseudospectral differentiation matrix.



Of 10 configurations examined, the fit is less good for 2



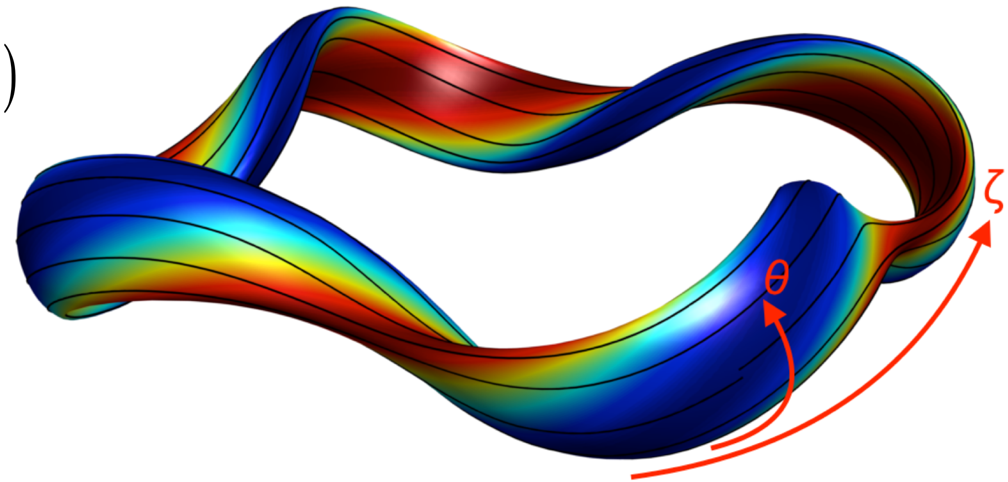
The configurations with relatively poor fits can be explained by their larger symmetry-breaking



$$S = \sqrt{\frac{\sum_{m\neq 0, n \neq mN} B_{m,n}^2}{\sum_{m,n=mN} B_{m,n}^2}}$$

The conventional approach to finding quasisymmetric fields works but has shortcomings

Want magnetic field strength B
to have quasisymmetry: $B = B(r,\ \theta - N\zeta)$



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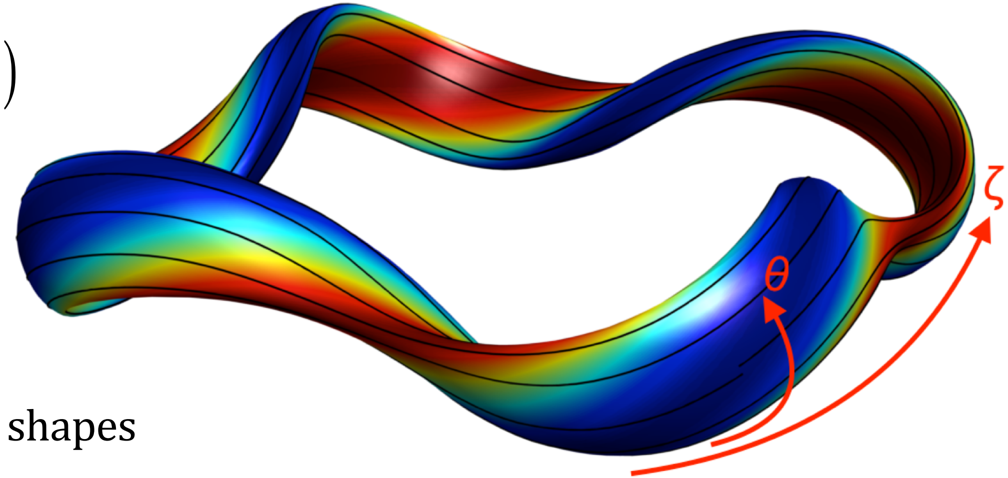

$$\min_X f(X)$$

Parameter space: $X =$ toroidal boundary shapes

Objective: $f = \sum_{m,n \neq Nm} B_{m,n}^2(r_0)$ where $B(r,\theta,\zeta) = \sum_{m,n} B_{m,n}(r)\exp(im\theta - in\zeta)$

The conventional approach to finding quasisymmetric fields works but has shortcomings

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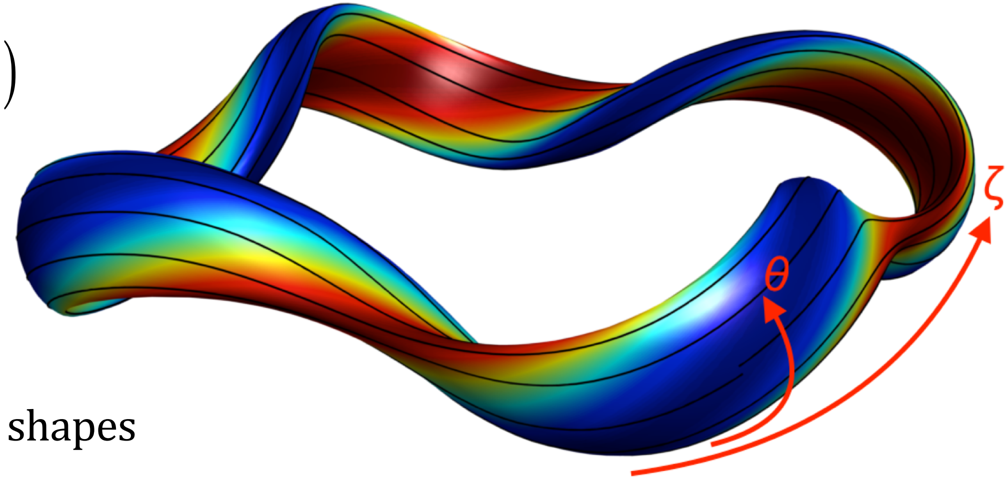

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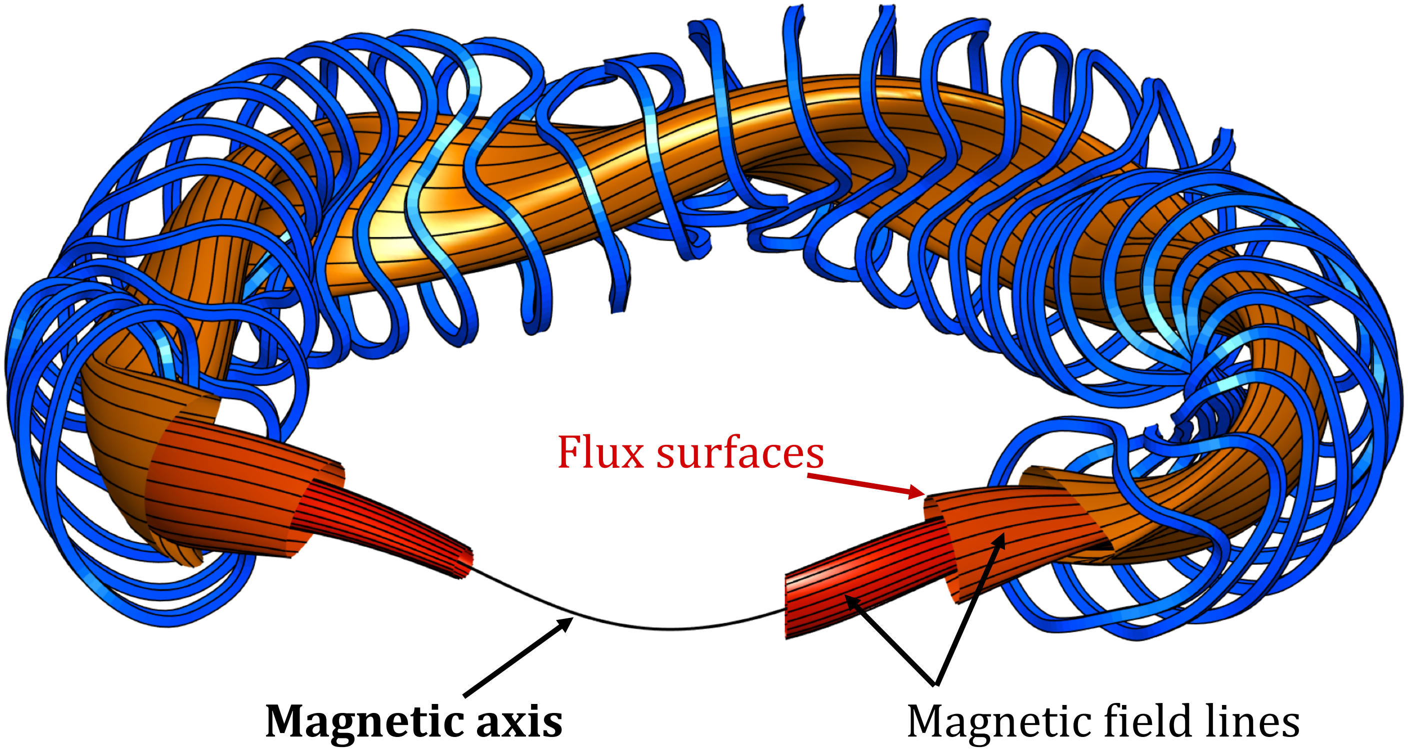
- Computationally expensive.
- What is the size & character of the solution space?
- Result depends on initial condition, so cannot be sure you've found all solutions.

Alternative: expand equations near the magnetic axis



Mercier (1964), Lortz & Nührenberg (1976), Garren & Boozer (1991)

The size of the space of fields that are quasisymmetric to O(r) can be precisely understood.

Given $P(\zeta) > 0$, $Q(\zeta)$, and $\sigma(0)$, with $P(\zeta)$ and $Q(\zeta)$

2π-periodic, bounded, and integrable, a solution to

$$\frac{d\sigma}{d\zeta} + \iota\left(P + \sigma^2\right) + Q = 0 \qquad (1)$$

is a pair $\{\iota,\ \sigma(\zeta)\}$ solving (1) where $\sigma(\zeta)$ is 2π-periodic.

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Theorem: A solution exists and it is unique.

ML, Sengupta, and Plunk (2019). Probably an earlier reference?

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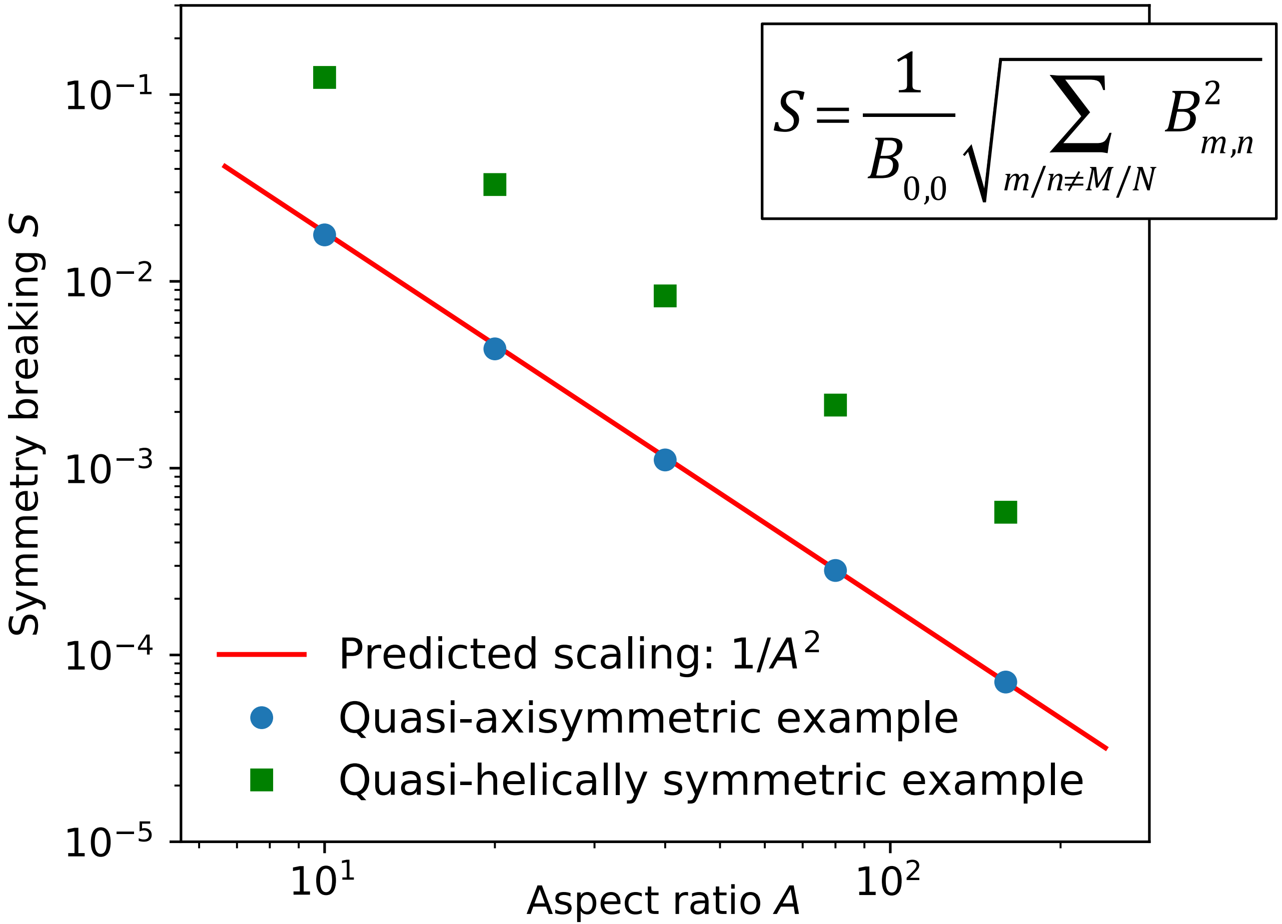
<u>Theorem:</u> A solution exists and it is unique.

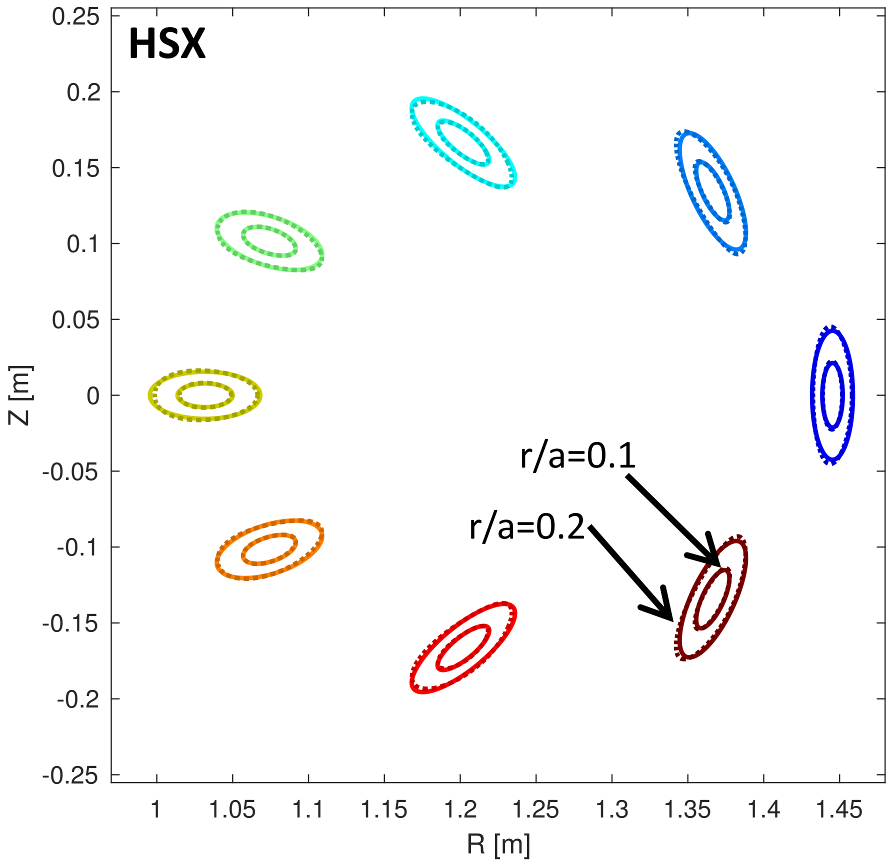
$\Rightarrow$ Numerical solution is very robust.

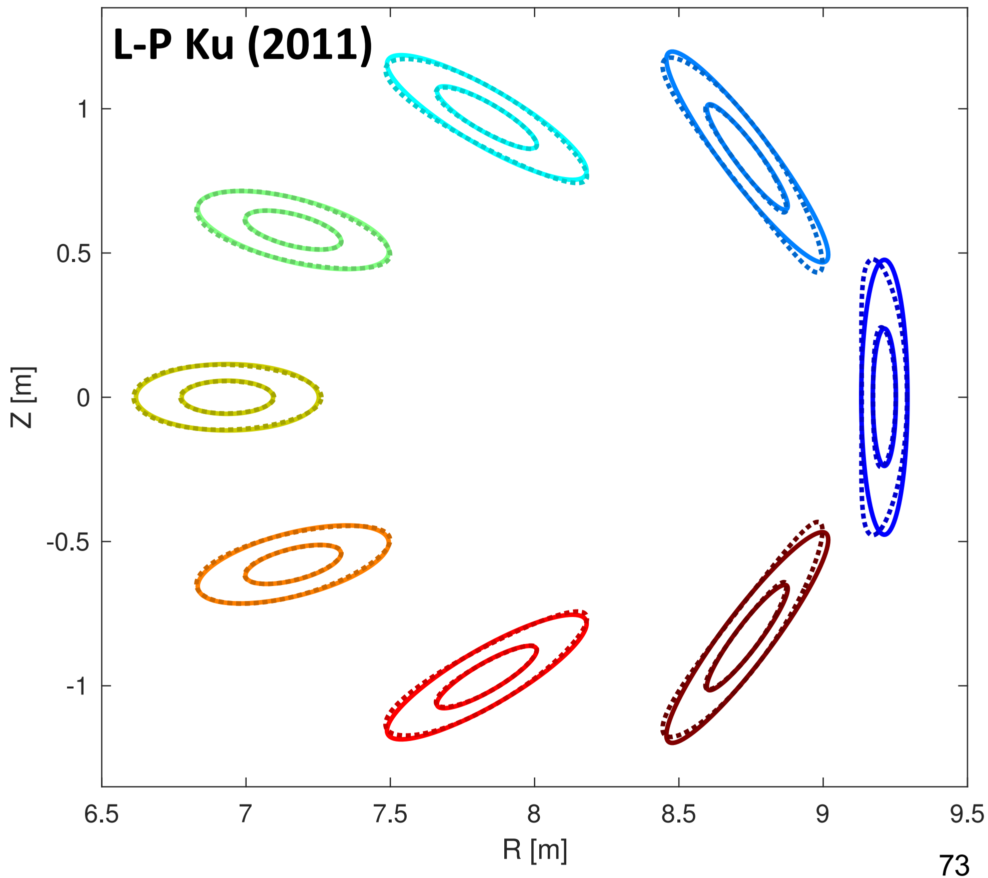
The symmetry-breaking Fourier amplitudes scale as predicted.



Quasi-helically symmetric configurations

Dotted: VMEC equilibrium
Solid: Garren-Boozer construction





Quasi-axisymmetric configurations

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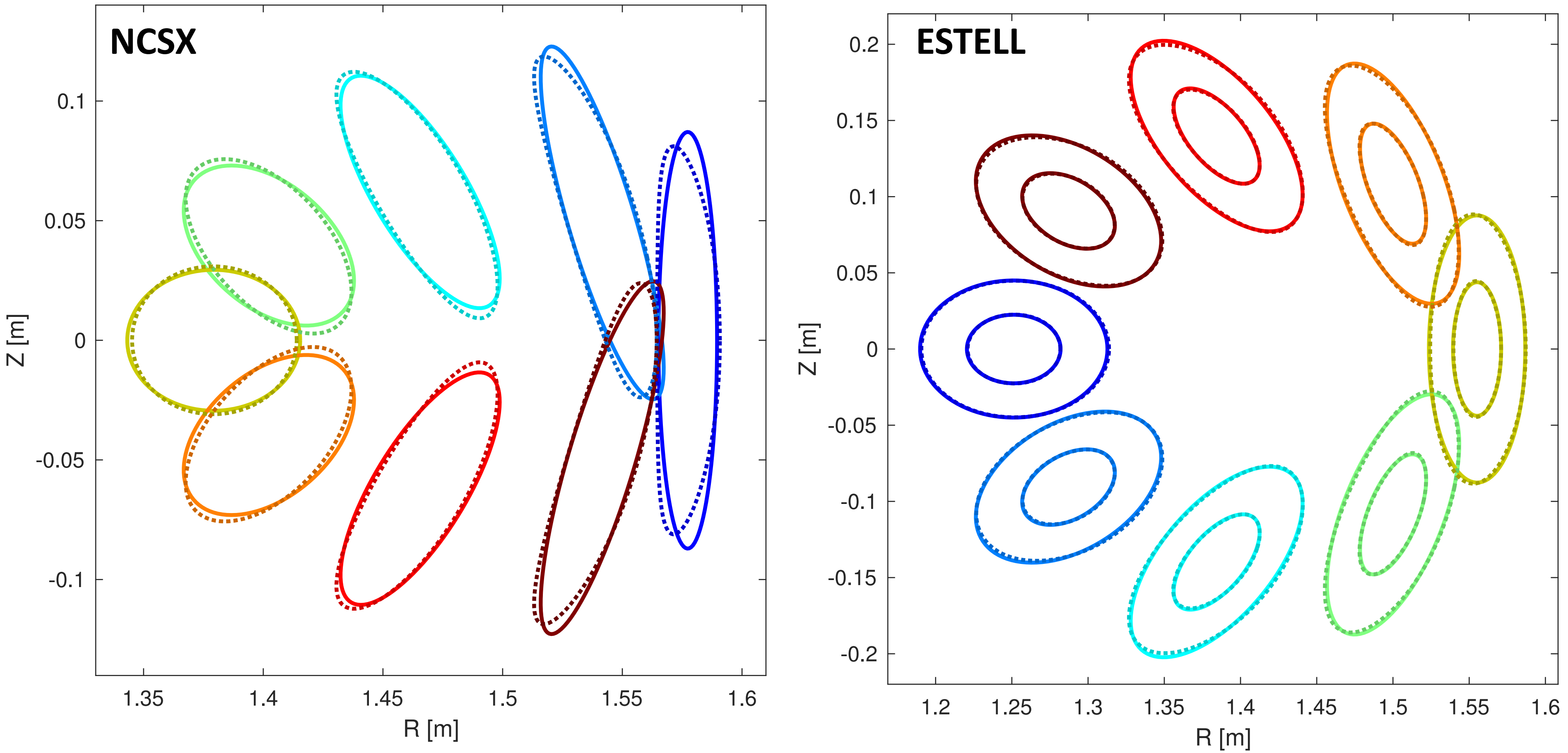
Solid: Garren-Boozer construction



Omnigenity is a weaker confinement condition than quasisymmetry.

Definition of omnigenity: The radial drift has a time average of 0 for all particles.

$$\oint \left(\mathbf{v}_d \cdot \nabla r \right) dt = 0 \quad \forall \text{ magnetic moments \& energies.}$$

- J Cary & S Shasharina, *Physics of Plasmas* **4**, 3323 (1997).
- J Cary & S Shasharina, *Physical Review Letters* **78**, 674 (1997).
- P Helander & J Nührenberg, *Plasma Physics and Controlled Fusion* 51, 055004 (2009).
- M Landreman & P J Catto, *Physics of Plasmas* **19**, 056103 (2012).

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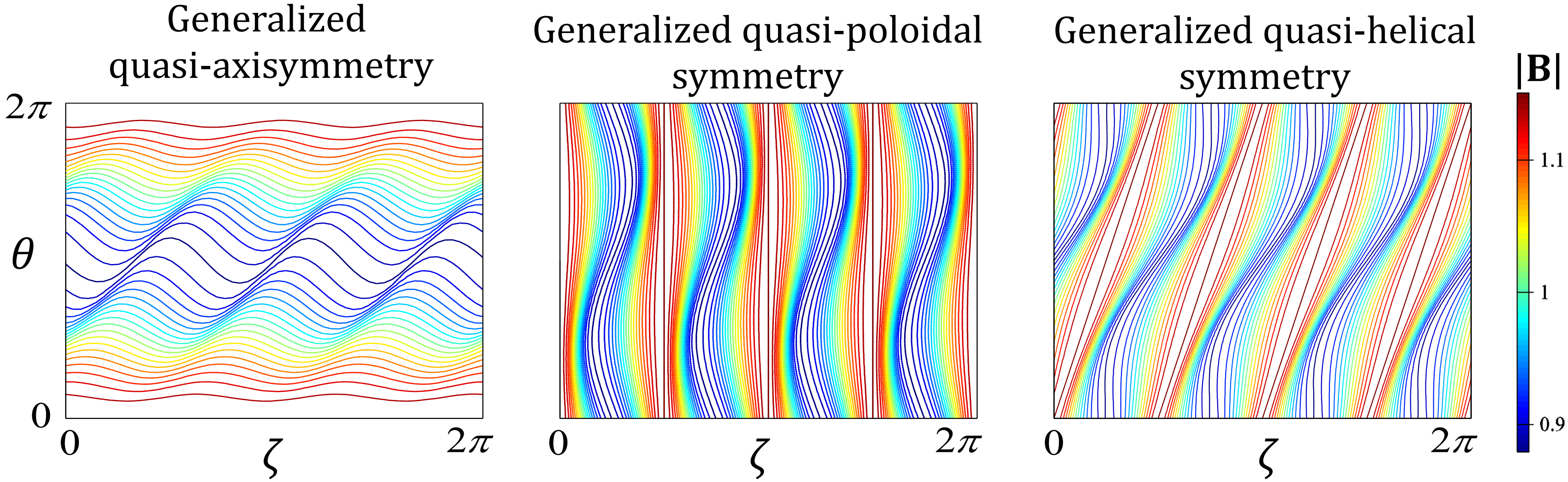
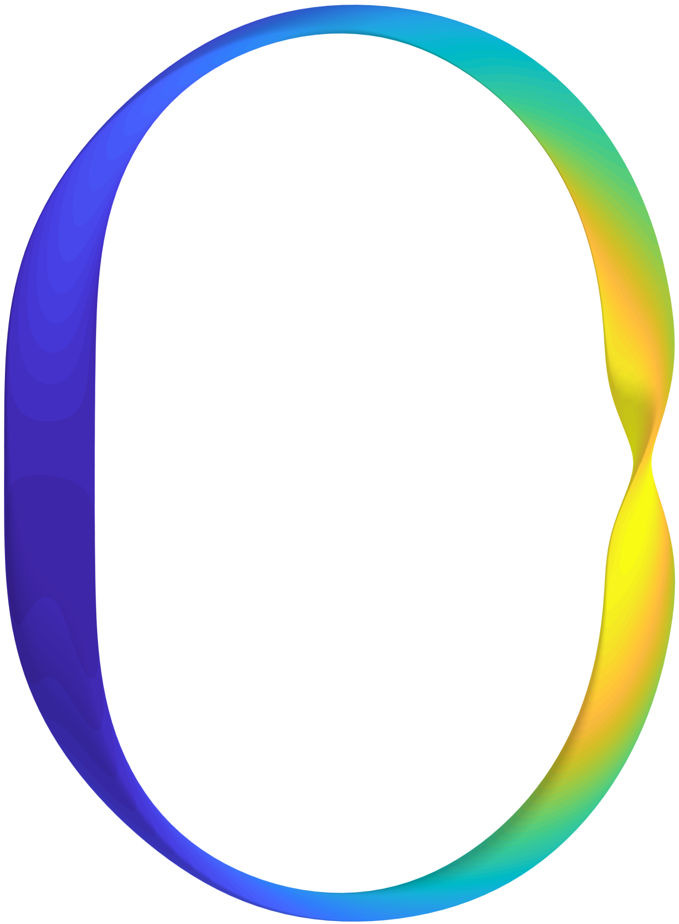
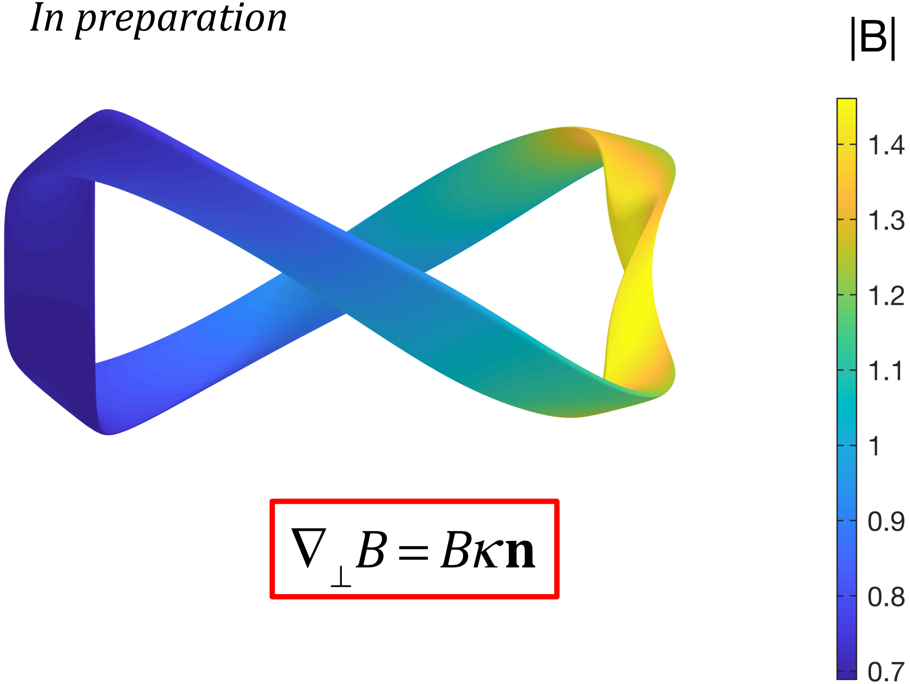


The near-axis analysis can be generalized to construct omnigenous configurations



G G Plunk, ML, and P Helander,
In preparation



$$\nabla_{\perp} B = B\kappa \mathbf{n}$$

Quasi-poloidal symmetry is not possible near the axis, but omnigenity is.

Outline

- Construction for $O(r)$ quasisymmetry
  - Theory, & the number of solutions
  - Numerical results
  - Comparison to "real experiments"
  - The landscape of solutions
- Extensions
  - Omnigenity
  - $O(r^2)$ quasisymmetry

Extending the construction to higher order is tricky

- We can only “half-specify” the axis shape:
  - A curve like the axis is given by 2 functions, e.g. {curvature, torsion} or $\{R(\boldsymbol{\phi}), Z(\boldsymbol{\phi})\}$.
  - At $O(r)$, (# unknowns)-(# equations)=2 so we can specify (almost) any axis. But at $O(r^2)$, (# unknowns)-(# equations)=1 so we cannot.

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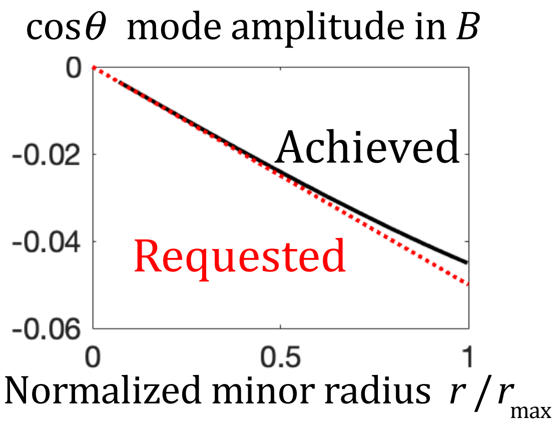
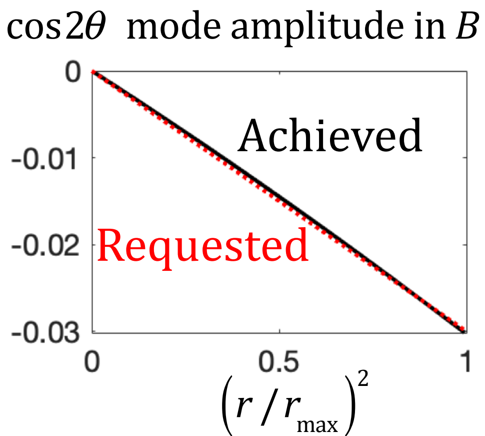
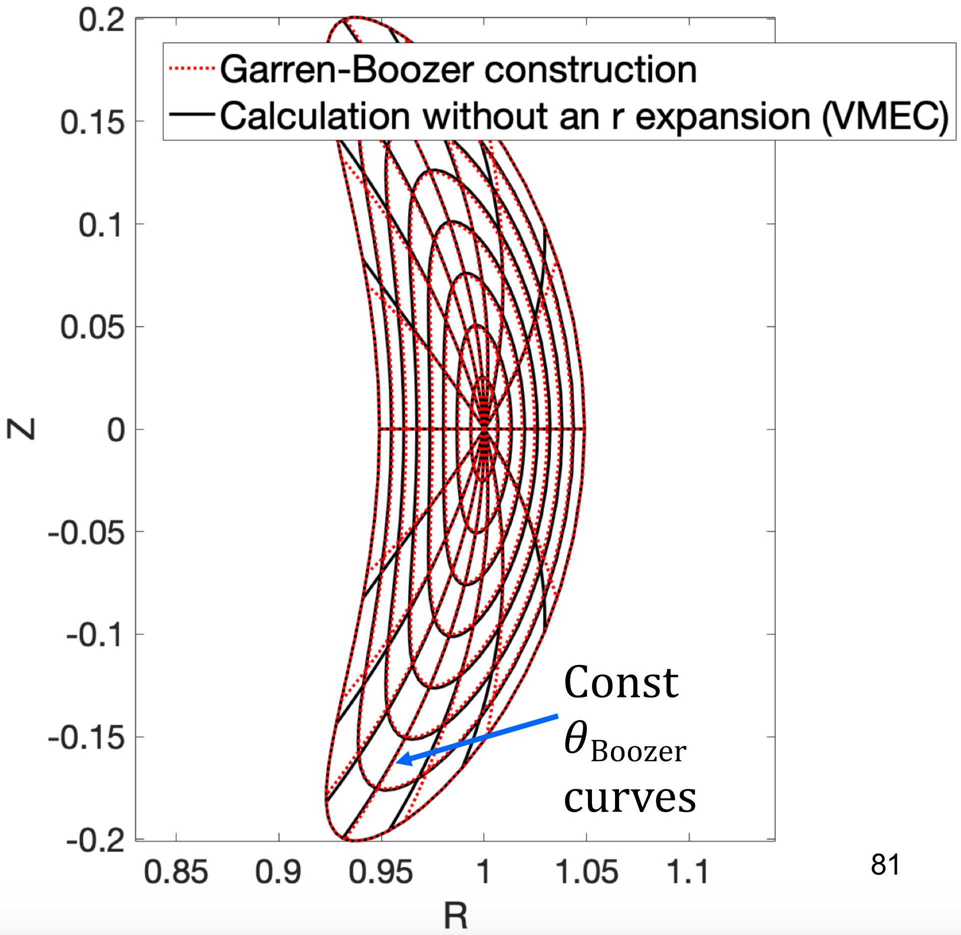
  - At $O(r)$, (# unknowns)-(# equations)=2 so we can specify (almost) any axis. But at $O(r^2)$, (# unknowns)-(# equations)=1 so we cannot.
- No existence & uniqueness theorem for solutions (yet).
- Magnetic shear (variation of rotational transform) does not appear until $O(r^3)$.

We are working to extend the construction to $O(r^2)$, enabling greater shaping

Axisymmetric example:







We now have a recipe for generating quasisymmetric VMEC input files: Set r to a small finite value a.

Inputs:

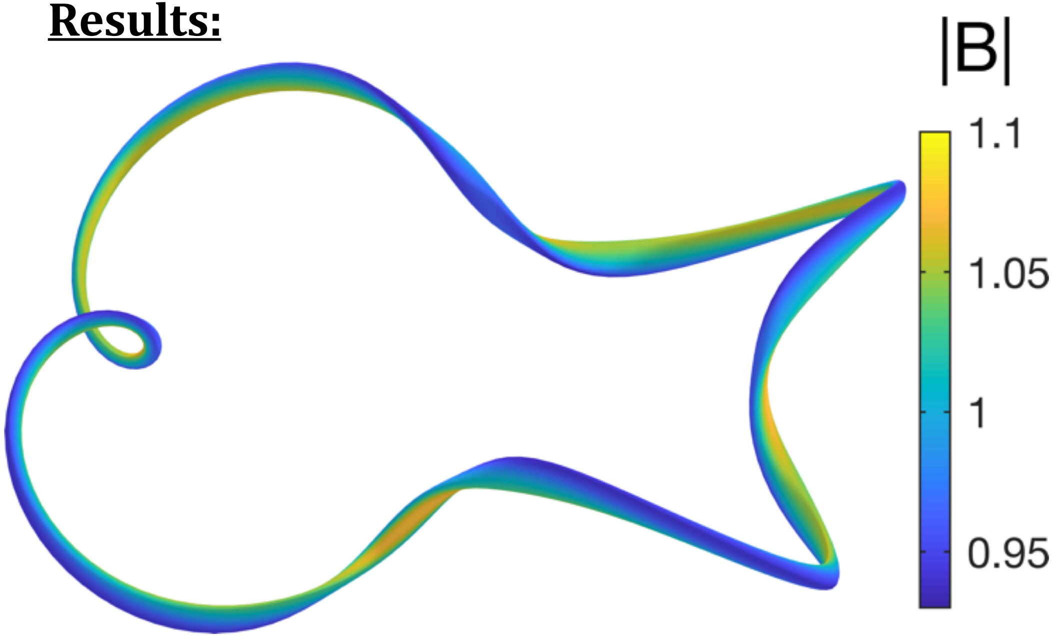
axis shape $R_0(\phi) = 1 + 0.265\cos(4\phi)$,

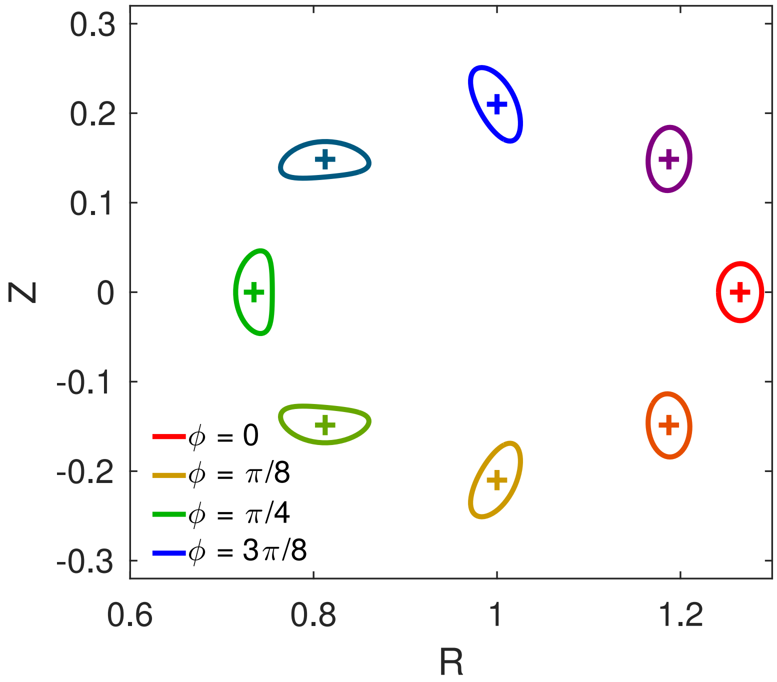
$Z_0(\phi) = -0.21\sin(4\phi)$,

$I_2 = 0$, $\bar{\eta} = -2.25$,

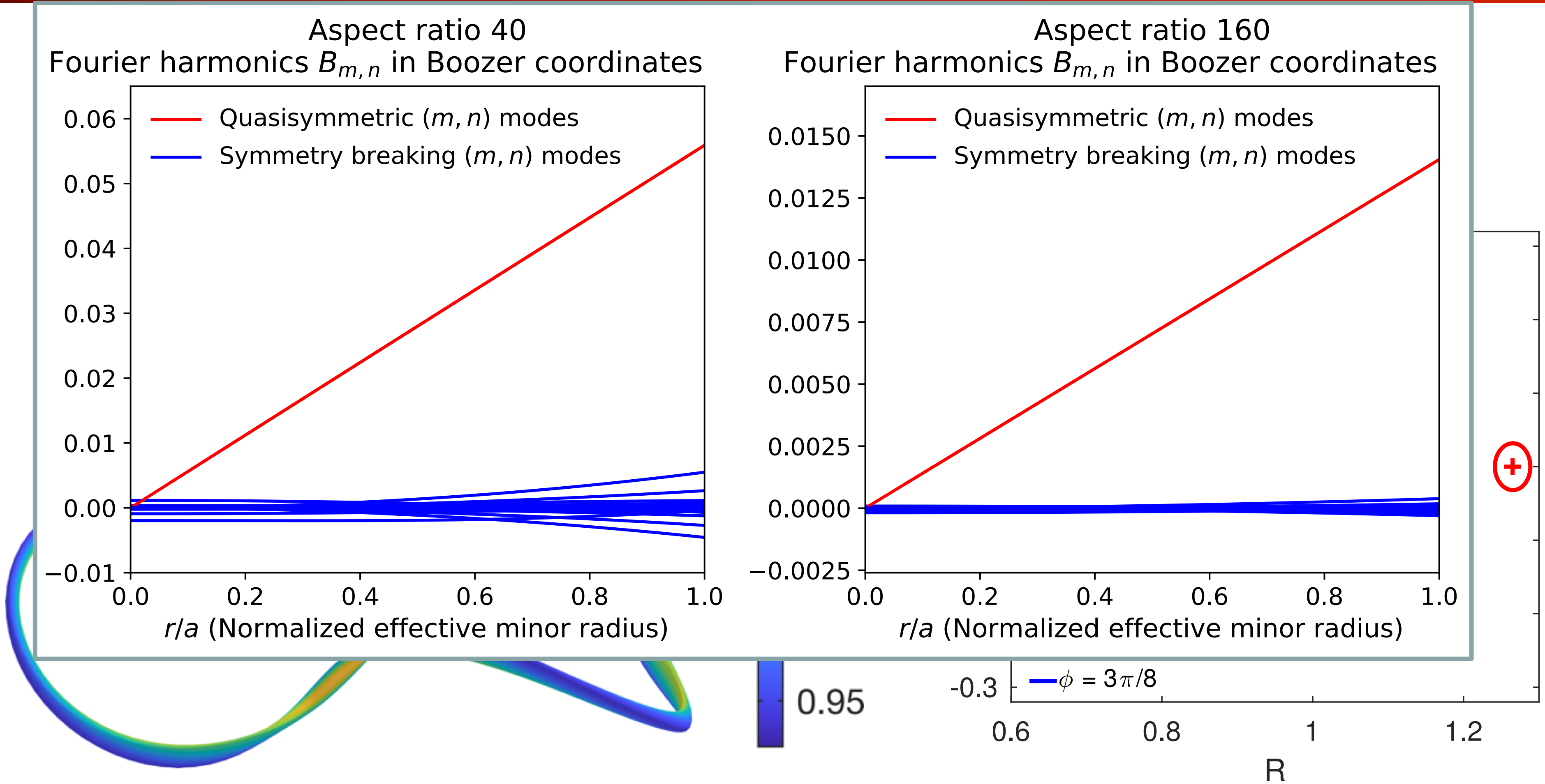
$\sigma(0) = 0$, $R/a = 40$.

Results:





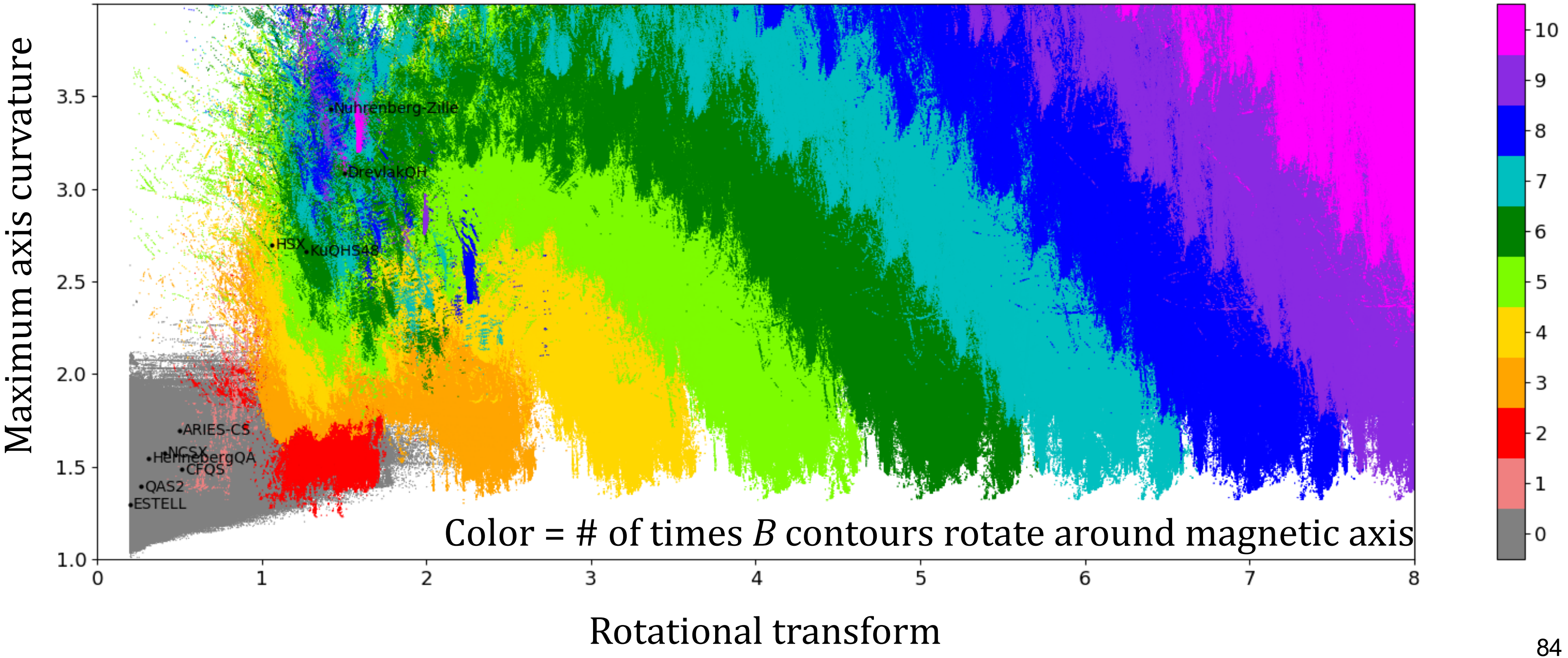
The construction can be verified by comparing to VMEC + BOOZ_XFORM.



The fast construction enables brute-force surveys of ”all” quasisymmetric fields

Axis shape: $R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(j n_{fp} \phi), \quad Z_0(\phi) = 1 + \sum_{j=1}^{3} Z_j \sin(j n_{fp} \phi)$

$2.4\text{x}10^8$ configurations



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