Why are some flux surface shapes hard to make?
(with coils far from the plasma)

Matt Landreman, Per Helander
Outline

• Background
  – Coils should be far from the plasma
  – Concave flux surface shapes seem hard to make
  – Ill-posedness

• Possible analytical approaches
  – Local perspective
  – 2D case: complex analysis

• Applications & questions
In a reactor, must fit ~ 1.5m “blanket” between plasma and coils to absorb neutrons.

But at fixed plasma shape & size, coils shapes become impractical if they are too far away:

*Coils offset a uniform distance from W7-X plasma:*

- 25cm separation
- 50cm separation
- 65cm separation
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So we must scale everything up:
So insights into Laplace’s equation could directly reduce cost of fusion!

Between plasma and the coils, the equations are just

\[ B = \nabla \Phi, \quad \Delta \Phi = 0. \]

Or, Biot-Savart law:

\[ B(r) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]

\[ \mu_0 = \text{a constant} = 1.26 \times 10^{-6} \text{ N} / \text{A}^2, \quad I = \text{coil current} \]
The small plasma-to-coil separation has been a headache for W7-X

“Lesson 1: A lack of generous margins, clearances and reasonable tolerance levels implies an unnecessary increase of the complexity and leads to late design changes. This has a strong impact on schedule, budget, man-power and potentially sours the relationship to funding bodies.”

Klinger et al, Fusion Engineering & Design (2013)
Coils farther from the plasma would reduce ripple, hence improve confinement & reduce # of coils needed.
It is consistently found that it is hard to get coils far from concave flux surface shapes. Why?
Concave shapes are hard to make in 2D too

E.g. ITER

Peanut plasma shape

H plasma shape

Divertor plasma shape

Landreman & Boozer, Phys Plasmas (2016)
Yet, curvature *per se* is not hard to achieve

Vacuum $\mathbf{B}$ from 2 straight wires:

So a field line’s radius of curvature is not a good measure of distance to coils.
Calculating the currents that produce a given B is an ill-posed problem

2 very different coil shapes can produce nearly the same B in the confinement region.

Extrapolating B outward from the plasma is like treating Laplace’s eq as an initial value problem:

\[
\frac{\partial^2 \Phi}{\partial z^2} = -\frac{\partial \Phi}{\partial x^2} - \frac{\partial \Phi}{\partial y^2}
\]

Nearly cancel
Calculating the currents that produce a given $B$ is an ill-posed problem

2 very different coil shapes can produce nearly the same $B$ in the confinement region.

So we probably can’t prove \textit{precisely} where the coils must be. But can we say something about the \textit{best-case scenario}? e.g. “There must be a coil within some distance $d$”? 
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**Local approach: extrapolate from a point**

\[ \mathbf{B} = \nabla \Phi, \quad \Delta \Phi = 0. \]

Given \( \mathbf{B} \) and its first few derivatives at a point \( P \), can we compute some number \( k \) such that \( \mathbf{B} \) (or \( \Phi \)) must grow at least as fast as \( \sim \exp(k \, d) \), where \( d \) is the distance from \( P \)?

Or,

Given \( \mathbf{B} \) and its first few derivatives at \( P \), what is the minimum distance to a singularity in \( \mathbf{B} \) (or \( \Phi \))?
Probably a gradient scale length of $\mathbf{B}$ indicates its “complexity”, but which scale length is most meaningful?

$\nabla \mathbf{B}$ for a vacuum field contains 4 independent scale lengths:
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$\nabla \mathbf{B}$ for a vacuum field contains 4 independent scale lengths:

$\mathbf{B} = \nabla \Phi$ so $\nabla \mathbf{B} = \nabla \nabla \Phi$ is a symmetric $3 \times 3$ matrix $\Rightarrow$ 6 degrees of freedom.

$$\nabla \mathbf{B} = \begin{pmatrix}
\Phi_{,xx} & \Phi_{,xy} & \Phi_{,xz} \\
\Phi_{,xy} & \Phi_{,yy} & \Phi_{,yz} \\
\Phi_{,xz} & \Phi_{,yz} & \Phi_{,zz}
\end{pmatrix}$$

-1 since $0 = \nabla \cdot \mathbf{B}$, -1 since coordinate system can be rotated to make one vanish.
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$-1$ since $0 = \nabla \cdot B$, $-1$ since coordinate system can be rotated to make one vanish.

Example set of 4 independent inverse scale lengths:

$$\frac{B \cdot \nabla B}{B^2}, \quad \frac{\|\nabla B\|}{B}, \quad \frac{1}{B} \sqrt{\langle \nabla B \rangle : \langle \nabla B \rangle}, \quad \frac{\langle \nabla B \rangle \cdot \langle \nabla B \rangle \cdot \langle \nabla B \rangle}{\|\nabla B\|^2 B}$$

where $B = |B|$.  


In a world consisting of an infinite straight wire, $|\nabla B|/B$ and $\sqrt{(\nabla B)\cdot(\nabla B)/(2B^2)}$ equal $1/R$, the inverse distance to the wire, and $(\nabla B)/B$ points to the wire.
Some of these local measures of $B$ complexity have appealing properties.

In a world consisting of an infinite straight wire, $|\nabla B| / B$ and $\sqrt{(\nabla B) : (\nabla B) / (2B^2)}$ equal $1/R$, the inverse distance to the wire, and $(\nabla B) / B$ points to the wire.

For $\Phi(x, y, z) = A \exp(kz) \sin(kx)$ with constants $A$ and $k$, then $|\nabla B| / B$ and $\sqrt{(\nabla B) : (\nabla B) / (2B^2)}$ give the exponentiation scale length $k$ and $(\nabla B) / B$ points in the direction of exponential growth.
Some of these local measures seem to identify the problematic regions:

\[
\frac{\nabla B}{B} \quad \frac{n \cdot \nabla B}{B} \quad \sqrt{\left(\nabla B\right) : \left(\nabla B\right)} \quad 2B^2
\]
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In 2D, complex analysis is illuminating

P Helander, “Extension-of-B.pdf”

Kerner, Pfirsch, & Tasso, Nuclear Fusion 12, 433 (1972)
Write
\[ B = B_z \hat{z} + \nabla \psi \times \hat{z} = B_z \hat{z} + \nabla \phi, \quad B_z = \text{constant} \]

Then
\[ \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \Rightarrow w(x + iy) = \phi(x, y) + i\psi(x, y) \text{ is analytic.} \]

and \( w \) can be found by conformal mapping.

Example: consider a semicircle “carved out” of the plasma. Conformal mapping to upper half plane by
\[ w(z) = z + \frac{1}{z} \]

In general, if \( f \) is a real analytic function
\[ \phi(x, y) = \Re f[w(x + iy)] \]

There is always a singularity (a coil) within the upper unit semi-circle.
Field lines:

Figure 1: Magnetic field lines corresponding to the choice $f(w) = w$. Figure 2: Magnetic field lines corresponding to the choice $f(w) = 1/(1 + w^2)$.

Note
- Location of singularities depends both on boundary shape and on boundary data (B).
- Depending on boundary data, singularities can be arbitrarily close to the boundary

Is there some upper bound on the distance to the nearest singularity depending only on the boundary shape?
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Along the *magnetic axis* of a quasisymmetric field, we have equations for $\mathbf{B}$ and $\nabla \mathbf{B}$.

*Garren & Boozer (1991), my talk Friday*

Given $\mathbf{B}$ and $\nabla \mathbf{B}$ along this closed curve, how best can we exclude infeasible solutions?

E.g. $|\mathbf{b} \cdot \nabla \mathbf{b}| < \text{threshold}$, or $(\nabla \mathbf{B} : \nabla \mathbf{B}) / B^2 < \text{threshold}$?
Questions

• Is there an illuminating & quick-to-compute measure of why some $B$ configurations are hard to make with distant coils?

• Is it possible to prove rigorously that concave flux surface shapes require a coil nearby?

• Is principal curvature the relevant quantity, or something else?

• Is there a way to generalize the complex variables approach from 2D to 3D?

• Given $B$ and $\nabla B$ along the magnetic axis, what is the best estimate for “complexity” of coils or $B$ a finite distance from axis?
Extra slides
To increase plasma-coil separation for given plasma shape, either coil complexity or device size increases.

In a reactor, must fit ~ 1.5m “blanket” between plasma and coils to absorb neutrons.

Coils offset a uniform distance from W7-X plasma:

- 25cm separation
- 50cm separation
- 75cm separation

Must scale everything up:
It is consistently found that it is hard to get coils far from concave flux surface shapes. Why?

Coil winding surface
Plasma surface

NCSX
W7-X
Plasma-coil distance is a crucial quantity for viability of a stellarator reactor.

“Being the most influential parameter for the stellarator’s size and cost, $\Delta_{\text{min}}$ [minimum plasma-coil distance] optimization was crucial to the overall design.”

Calculating the currents that produce a given B is an ill-posed problem

Given data for B on the plane z = 0, assuming B is a vacuum field, what is B off the plane?

Like initial-value problem with Laplace's eq:

$$\frac{\partial^2 \Phi}{\partial z^2} = -\frac{\partial \Phi^2}{\partial x^2} - \frac{\partial \Phi^2}{\partial y^2}$$

Tiny short-wavelength changes to initial data grow exponentially:

$$B(x, y, z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \exp(ik_x x + ik_y y) \left[ \left( \ldots \right) \exp\left(z \sqrt{k_x^2 + k_y^2}\right) + \left( \ldots \right) \exp\left(-z \sqrt{k_x^2 + k_y^2}\right) \right]$$

So we probably can’t say precisely where the coils must be. But can we say something about the best-case scenario?, e.g. “There must be a coil within some distance d”?
Calculating the currents that produce a given $B$ is an ill-posed problem

For any $\varepsilon > 0$, there exist two coil shapes that differ to $O(1)$ yet produce a difference $< \varepsilon$ in $B$ in the confinement region.

Biot-Savart Law:

$$B(r) = \frac{\mu_0 I}{4\pi} \int_{\text{coil}} \frac{dr' \times (r - r')}{|r - r'|^3}$$

Nearly cancel
Calculating the currents that produce a given $B$ is an ill-posed problem.

Given data for $B$ on the plane $z = 0$, assuming $B$ is a vacuum field, what is $B$ off the plane?

At $z = 0$, $\Phi = \sin(x) \exp(z)$ is hard to distinguish from

$$\Phi = \sin(x) \exp(z) + e^{-100} \sin(100x) \exp(100z)$$

Like initial-value problem with Laplace's eq:

$$\frac{\partial^2 \Phi}{\partial z^2} = -\frac{\partial \Phi}{\partial x^2} - \frac{\partial \Phi}{\partial y^2}$$

So we probably can’t say precisely where the coils must be. But can we say something about the best-case scenario?, e.g. “There must be a coil within some distance $d$”?
• Call a fast coil code (NESCOIL or REGCOIL) for each iteration of the plasma shape, and penalize coil complexity (NCSX, ROSE).
• Penalize negative principal curvature (A Bader)
Some of these local measures of B complexity have appealing properties.

Let $B = |B|$.  

In a world consisting of an infinite straight wire, $|\nabla B|/B$ and $\sqrt{\left(\nabla B\right)\cdot\left(\nabla B\right)/(2B^2)}$ equal $1/R$, the inverse distance to the wire, and $(\nabla B)/B$ points to the wire.

For $\Phi = A\exp\left(k_x x + k_y y + k_z z\right) + c.c.$ with complex constants $\{A, k_x, k_y, k_z\}$ such that $k_x^2 + k_y^2 + k_z^2 = 0$, then $|\nabla B|/B$ and $\sqrt{\left(\nabla B\right)\cdot\left(\nabla B\right)/(2B^2)}$ give the exponentiation scale $\frac{1}{2}\sqrt{(k_x + k_x^*)^2 + (k_y + k_y^*)^2 + (k_z + k_z^*)^2}$ and $(\nabla B)/B$ points in the direction of exponential growth.