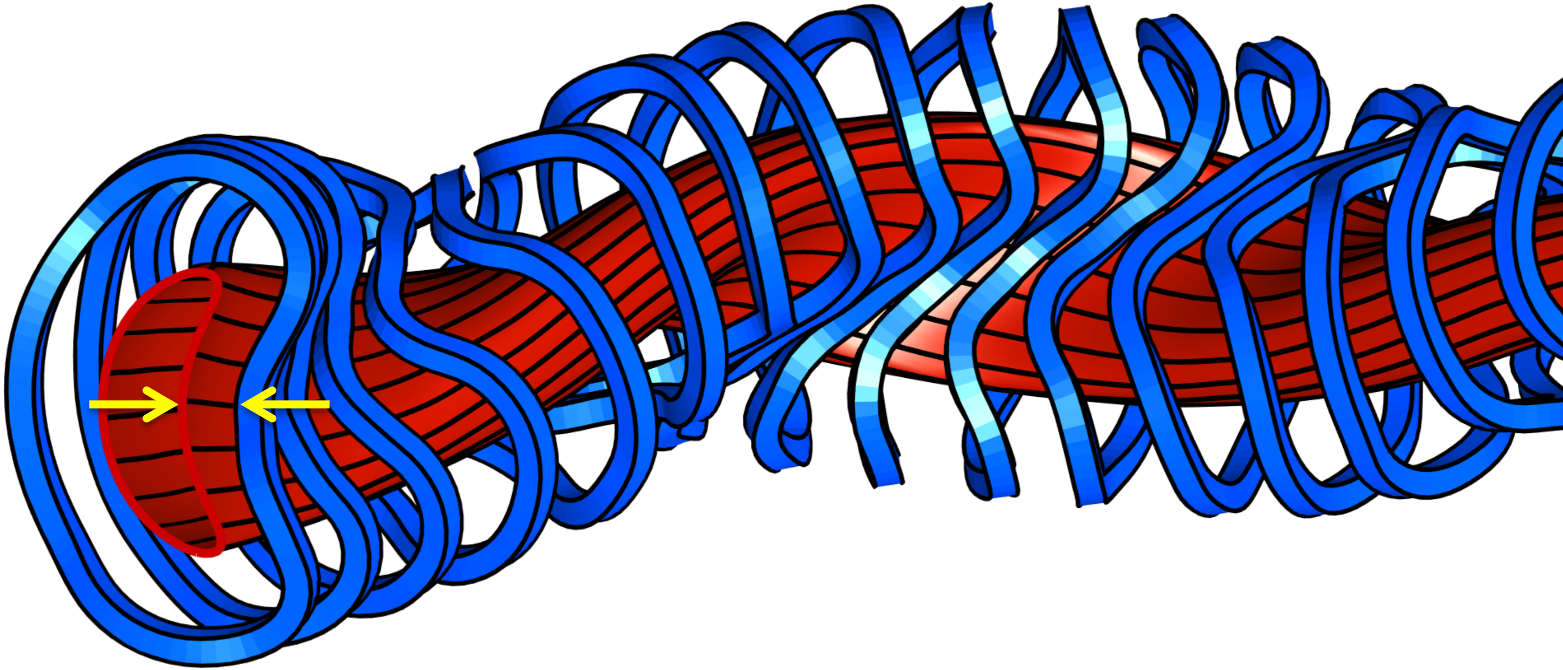
Why are some flux surface shapes hard to make?

(with coils far from the plasma)



Matt Landreman, Per Helander

Outline

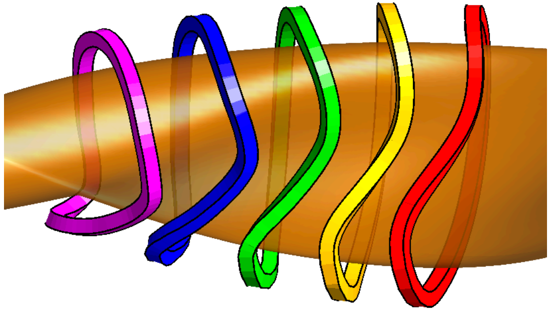
- Background
  - Coils should be far from the plasma
  - Concave flux surface shapes seem hard to make
  - Ill-posedness
- Possible analytical approaches
  - Local perspective
  - 2D case: complex analysis
- Applications & questions

In a reactor, must fit ~ 1.5m "blanket" between plasma and coils to absorb neutrons

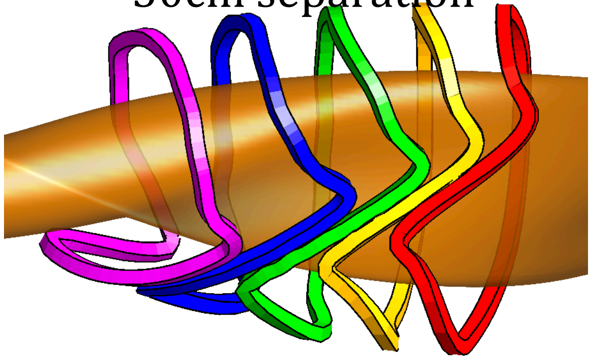
But at fixed plasma shape & size, coils shapes become impractical if they are too far away:

Coils offset a uniform distance from W7-X plasma:

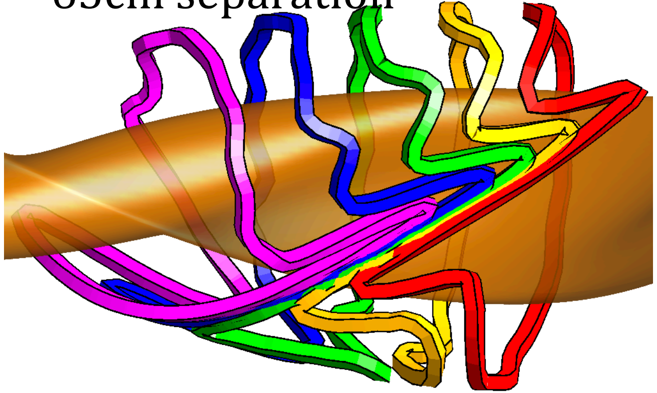
25cm separation



50cm separation



65cm separation



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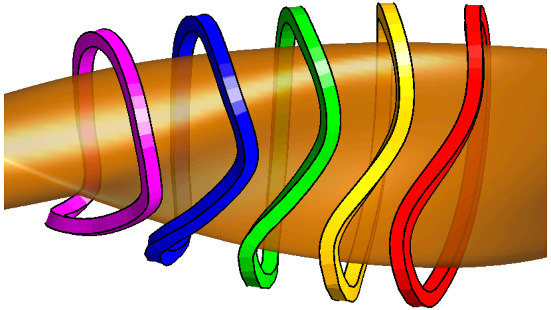

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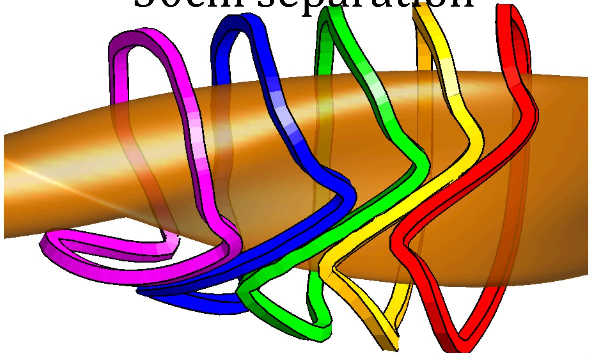

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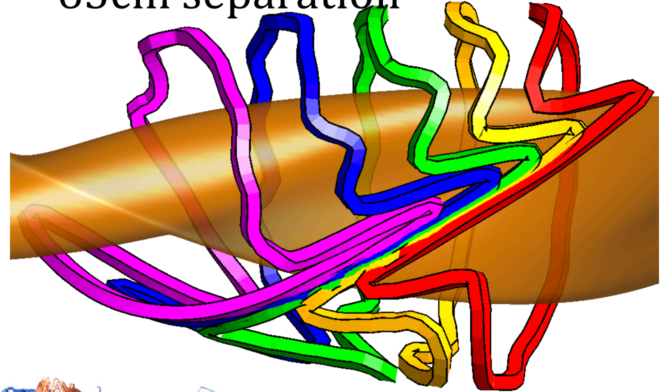
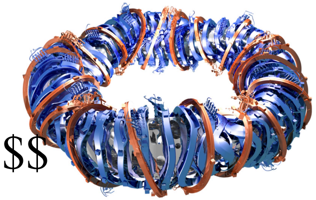
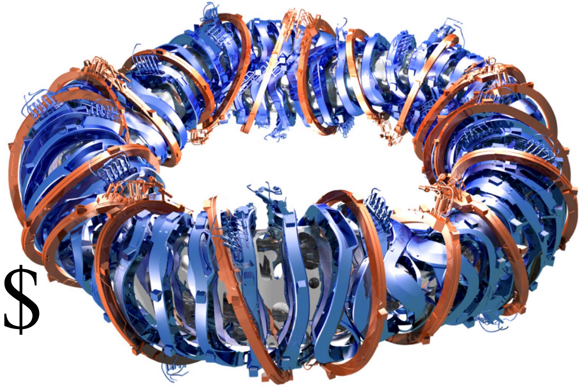


So we must scale everything up:

$$$



$$$



So insights into Laplace's equation could directly reduce cost of fusion!

Between plasma and the coils, the equations are just

$$\mathbf{B} = \nabla\Phi, \qquad \Delta\Phi = 0.$$

Or, Biot-Savart law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

$\mu_0 = \text{a constant} = 1.26 \times 10^{-6} N/A^2, \qquad I = \text{coil current}$

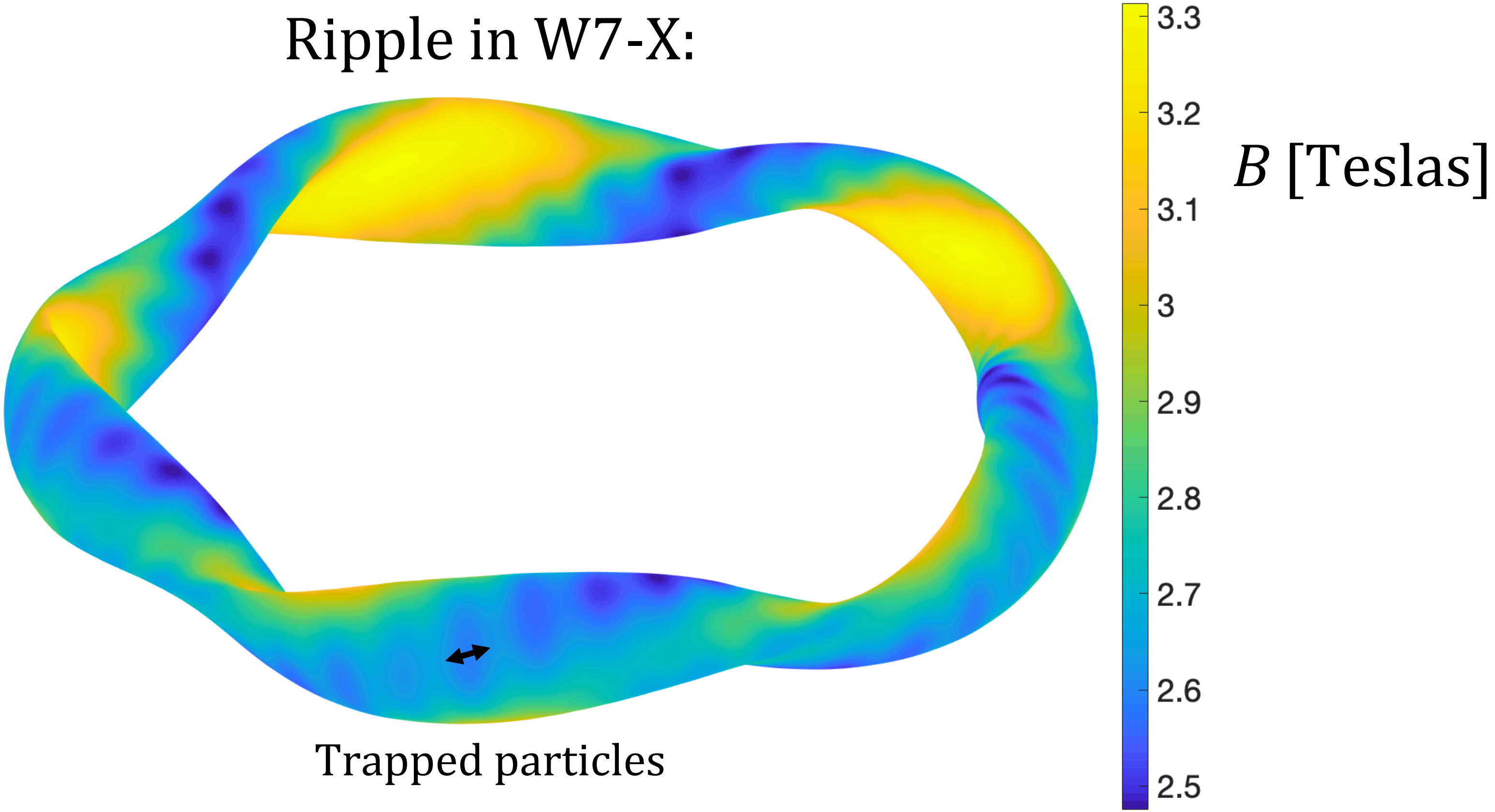
The small plasma-to-coil separation has been a headache for W7-X

"Lesson 1: A lack of generous margins, clearances and reasonable tolerance levels implies an unnecessary increase of the complexity and leads to late design changes. This has a strong impact on schedule, budget, man-power and potentially sours the relationship to funding bodies."

Klinger et al, Fusion Engineering & Design (2013)

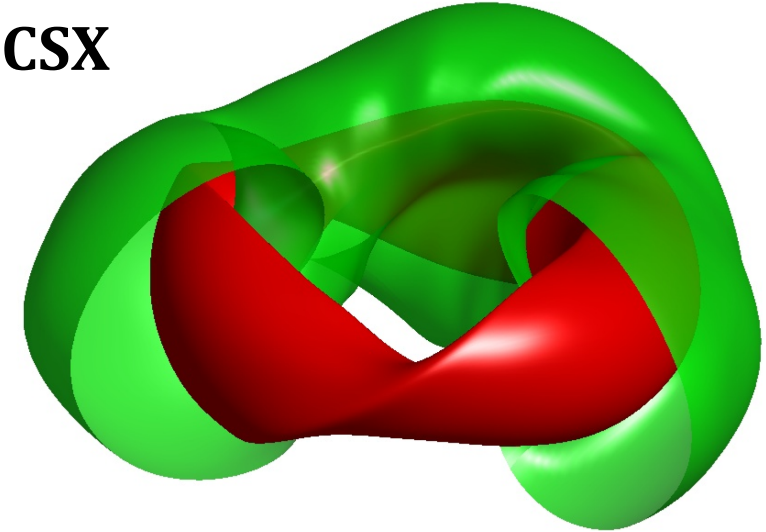
Coils farther from the plasma would reduce ripple, hence improve confinement & reduce # of coils needed

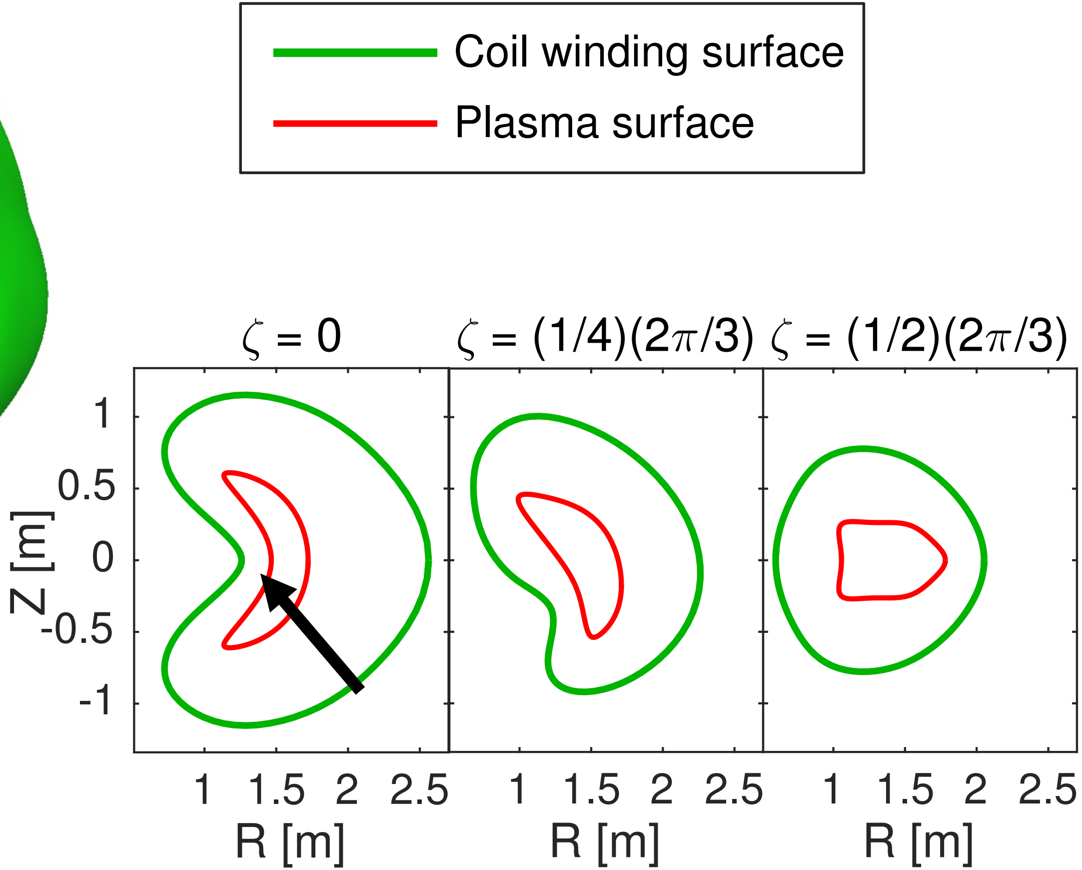
Ripple in W7-X:



It is consistently found that it is hard to get coils far from concave flux surface shapes. Why?

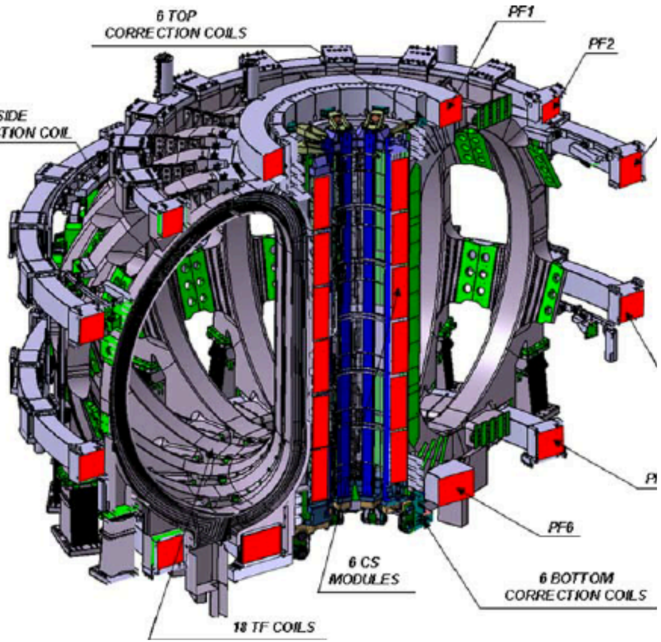
NCSX

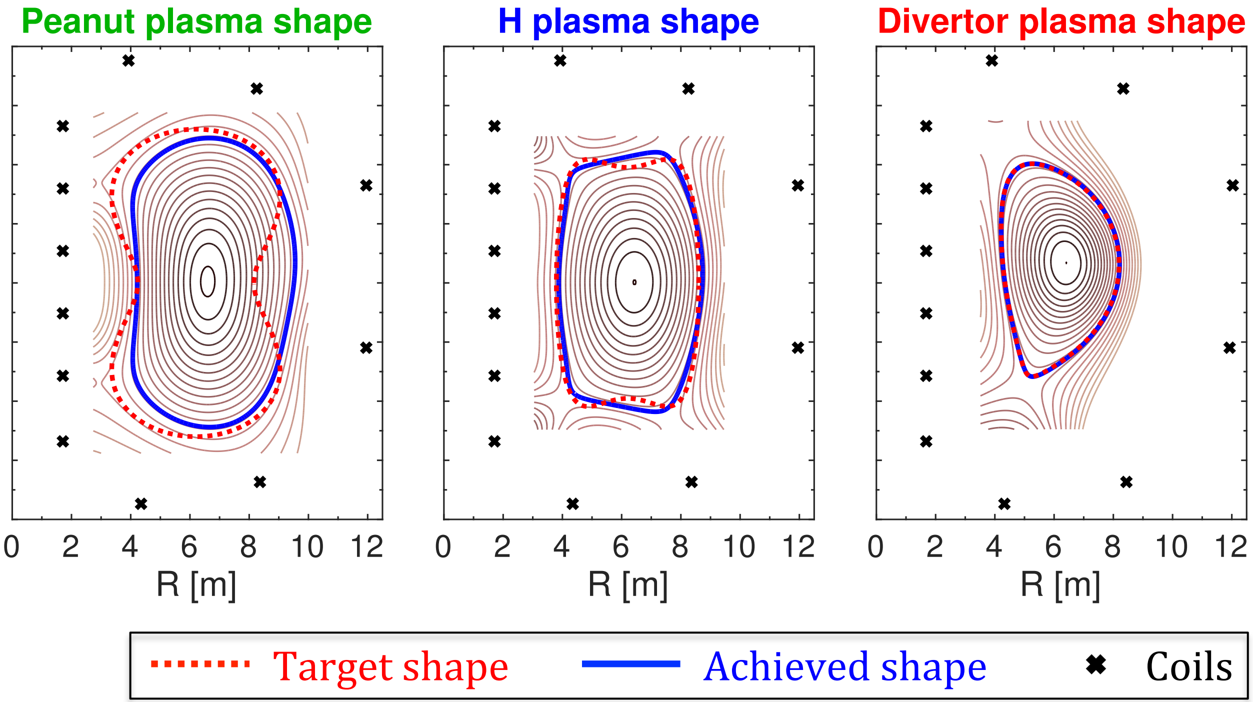




Concave shapes are hard to make in 2D too

E.g. ITER

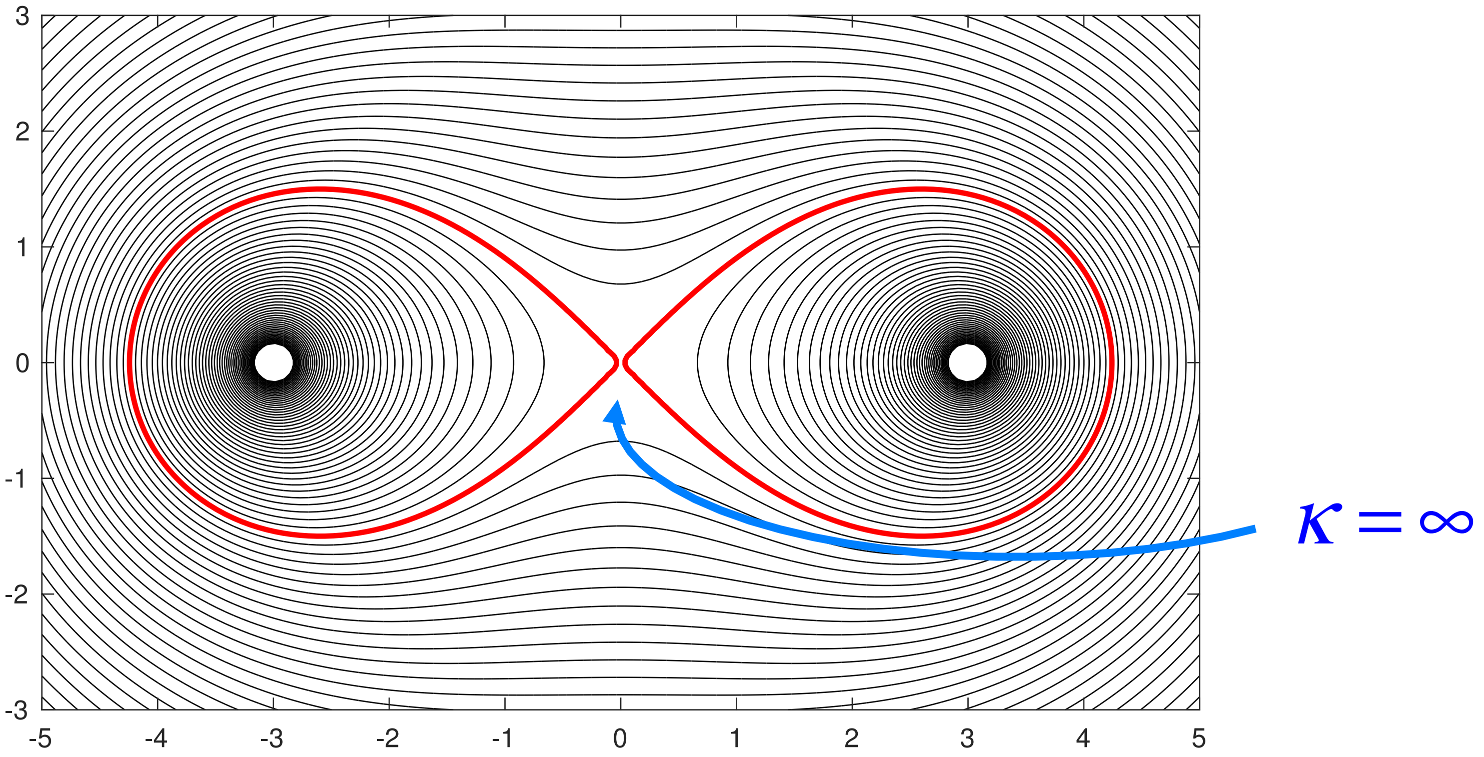




Landreman & Boozer, Phys Plasmas (2016)

Yet, curvature *per se* is not hard to achieve

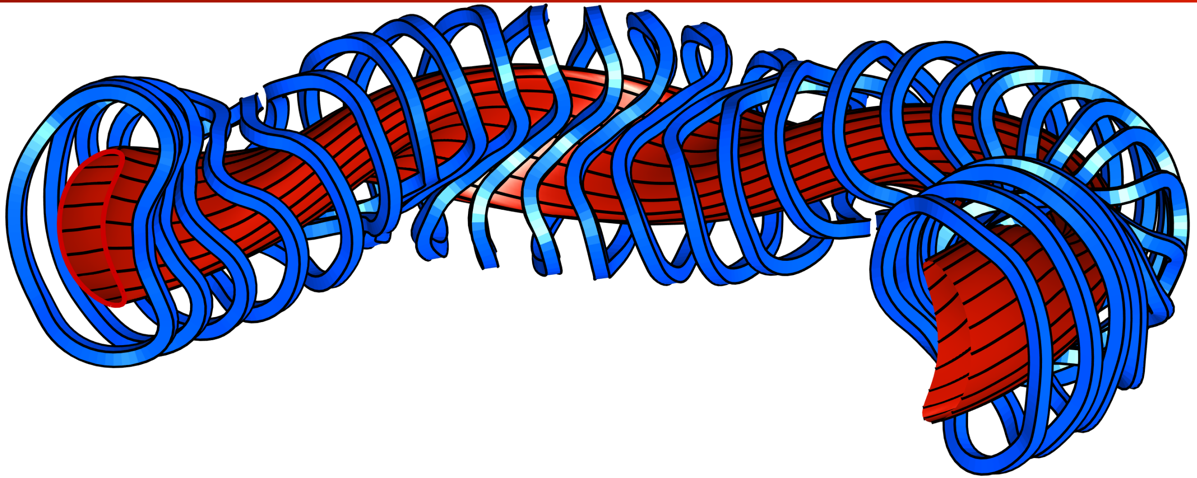
Vacuum **B** from 2 straight wires:



So a field line's radius of curvature is not a good measure of distance to coils.

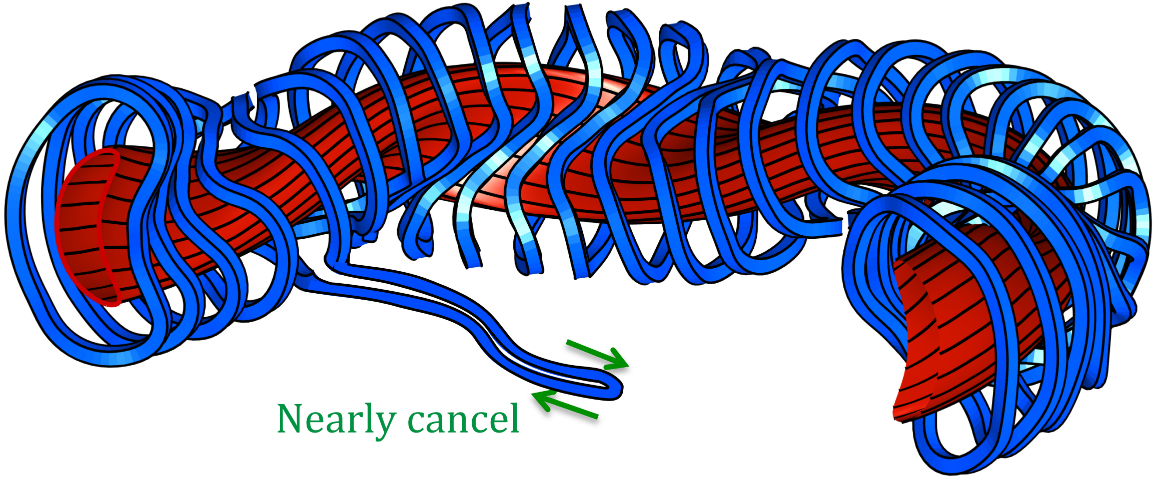
Calculating the currents that produce a given B is an ill-posed problem

2 very different coil shapes can produce nearly the same **B** in the confinement region.



Extrapolating **B** outward from the plasma is like treating Laplace's eq as an initial value problem:

$$\frac{\partial^2 \Phi}{\partial z^2} = -\frac{\partial \Phi^2}{\partial x^2} - \frac{\partial \Phi^2}{\partial y^2}$$



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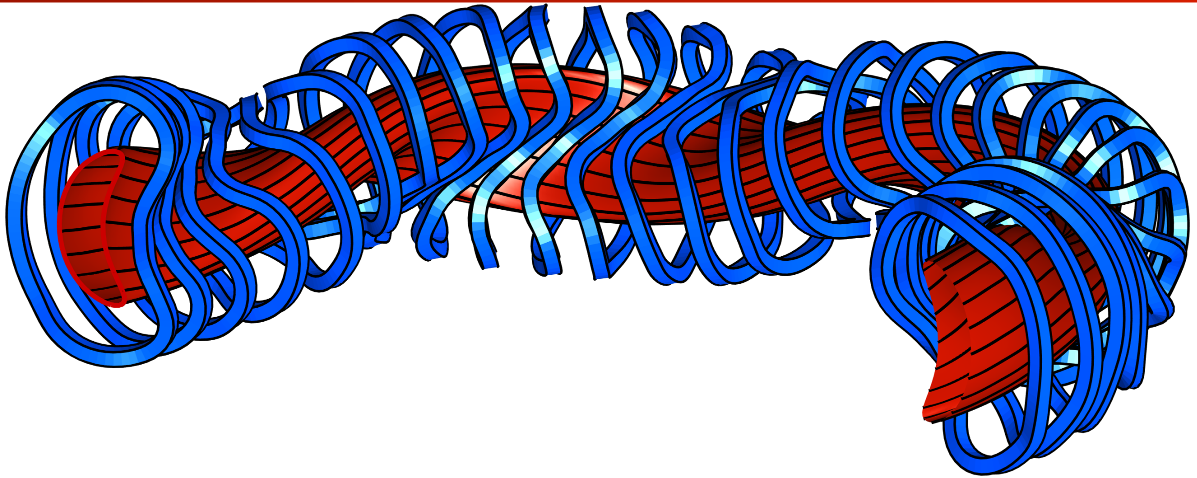
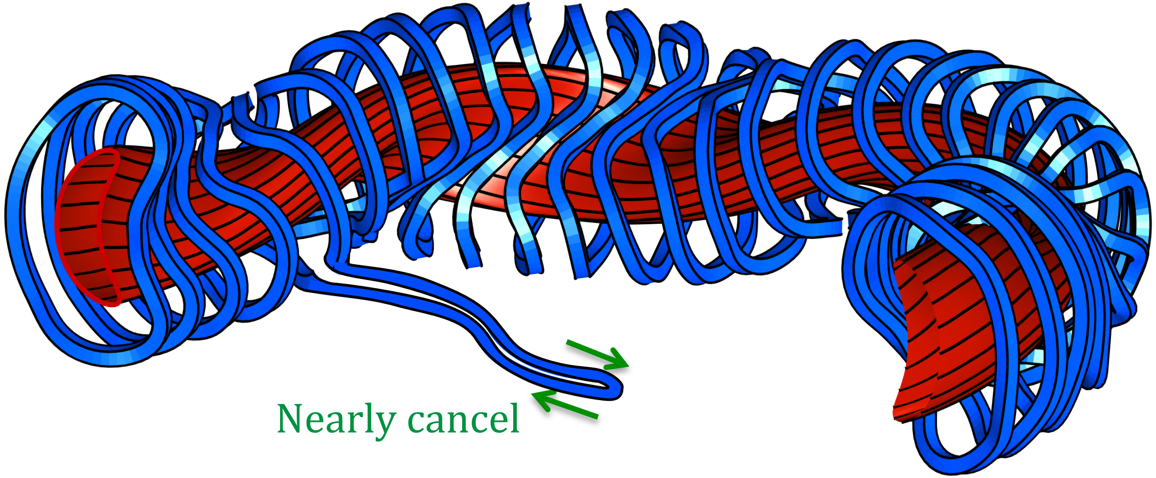


So we probably can't prove *precisely* where the coils must be. But can we say something about the *best-case scenario*?, e.g. "There must be a coil within some distance *d*"?



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Local approach: extrapolate from a point

$$\mathbf{B} = \nabla\Phi, \qquad \Delta\Phi = 0.$$

Given $\mathbf{B}$ and its first few derivatives at a point P, can we compute some number k such that $\mathbf{B}$ (or $\boldsymbol{\Phi}$) must grow at least as fast as $\sim \exp(k\, d)$, where d is the distance from P?

Or,

Given $\mathbf{B}$ and its first few derivatives at P, what is the minimum distance to a singularity in $\mathbf{B}$ (or $\boldsymbol{\Phi}$) ?

Probably a gradient scale length of **B** indicates its "complexity", but which scale length is most meaningful?

$\nabla \mathbf{B}$ for a vacuum field contains **4** independent scale lengths:

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$\nabla\mathbf{B}$ for a vacuum field contains **4** independent scale lengths:

$\mathbf{B}=\nabla\Phi$ so $\nabla\mathbf{B}=\nabla\nabla\Phi$ is a symmetric 3×3 matrix $\Rightarrow$ 6 degrees of freedom.

$$\nabla\mathbf{B}=\begin{pmatrix} \Phi_{,xx} & \Phi_{,xy} & \Phi_{,xz} \\ \Phi_{,xy} & \Phi_{,yy} & \Phi_{,yz} \\ \Phi_{,xz} & \Phi_{,yz} & \Phi_{,zz} \end{pmatrix}$$

-1 since $0=\nabla\cdot\mathbf{B}$, $\quad -1$ since coordinate system can be rotated to make one vanish.

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Example set of 4 independent inverse scale lengths:

$$\frac{\mathbf{B}\cdot\nabla B}{B^2},\quad \frac{|\nabla B|}{B},\quad \frac{1}{B}\sqrt{(\nabla\mathbf{B}):(\nabla\mathbf{B})},\quad \frac{(\nabla B)\cdot(\nabla\mathbf{B})\cdot(\nabla B)}{|\nabla B|^2 B}$$

where $B=|\mathbf{B}|$.

Some of these local measures of **B** complexity have appealing properties

In a world consisting of an infinite straight wire, $\left|\nabla B\right|/B$ and $\sqrt{\left(\nabla \mathbf{B}\right):\left(\nabla \mathbf{B}\right)/\left(2B^2\right)}$ equal $1/R$, the inverse distance to the wire, and $\left(\nabla B\right)/B$ points to the wire.

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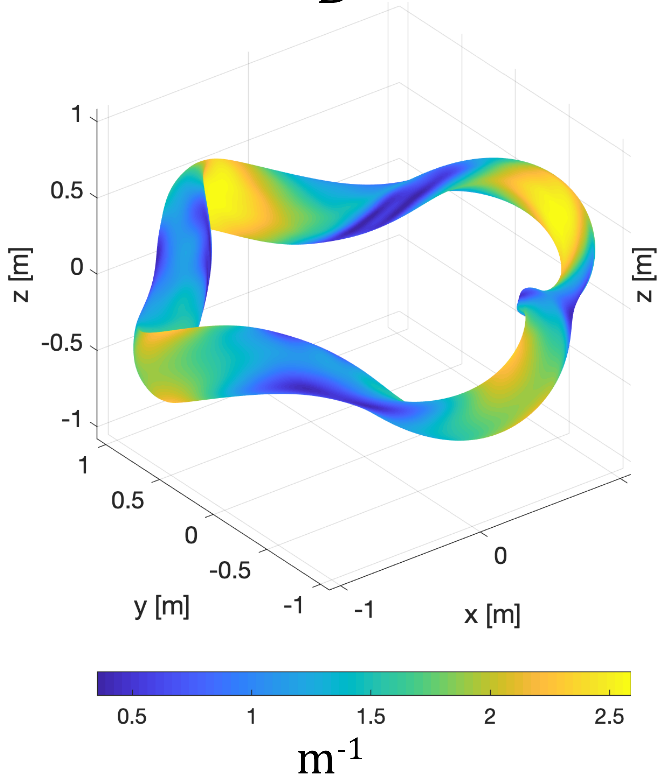
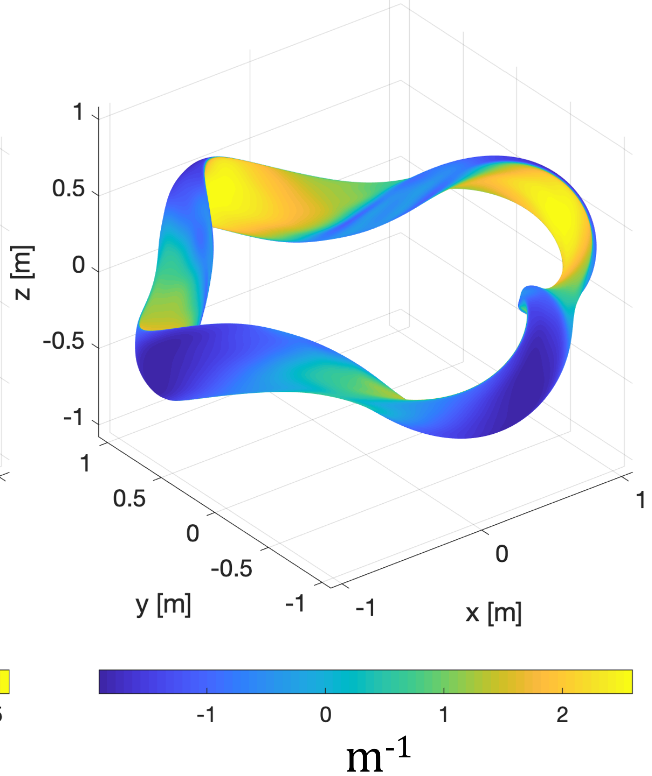
For $\Phi\left(x,y,z\right)=A\exp\left(kz\right)\sin\left(kx\right)$ with constants A and k, then $\left|\nabla B\right|/B$ and $\sqrt{\left(\nabla \mathbf{B}\right):\left(\nabla \mathbf{B}\right)/\left(2B^2\right)}$ give the exponentiation scale length k and $\left(\nabla B\right)/B$ points in the direction of exponential growth.

Some of these local measures seem to identify the problematic regions

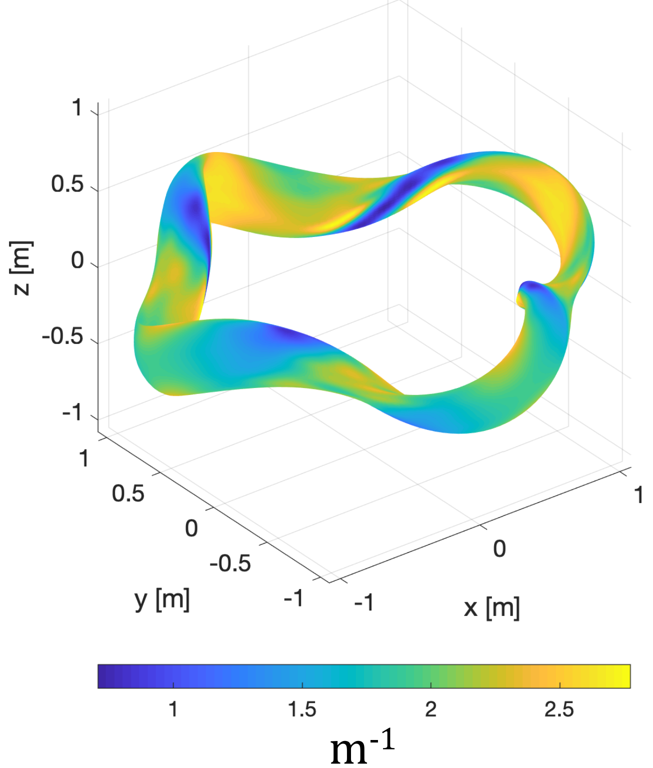
$$\frac{|\nabla B|}{B}$$



$$\frac{\mathbf{n}\cdot\nabla B}{B}$$



$$\sqrt{\frac{(\nabla\mathbf{B}):(\nabla\mathbf{B})}{2B^2}}$$



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In 2D, complex analysis is illuminating

P Helander, “Extension-of-B.pdf”

Kerner, Pfirsch, & Tasso, Nuclear Fusion **12**, 433 (1972)

Write

$$\mathbf{B} = B_z \hat{\mathbf{z}} + \nabla\psi \times \hat{\mathbf{z}} = B_z \hat{\mathbf{z}} + \nabla\phi, \qquad B_z = \text{constant}$$

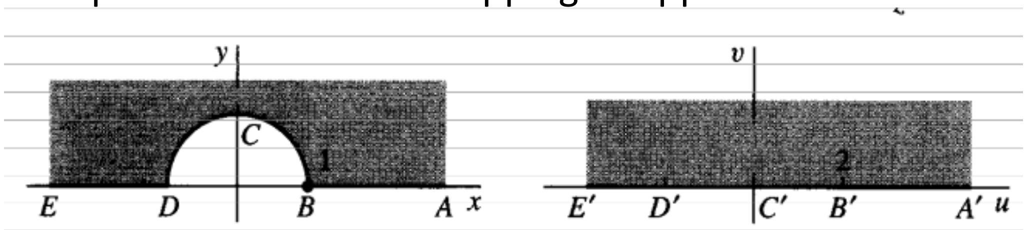
Then

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad \Rightarrow \quad w(x+iy) = \phi(x,y) + i\psi(x,y) \text{ is analytic.}$$

and *w* can be found by conformal mapping.

Example: consider a semicircle “carved out“ of the plasma. Conformal mapping to upper half plane by

$$w(z) = z + \frac{1}{z}$$



Interior of semi-circle DCB -> lower half plane

$$w(0) = \infty$$

In general, if *f* is a real analytic function

$$\phi(x,y) = \Re f[w(x+iy)]$$

There is always a singularity (a coil) within the upper unit semi-circle.

Field lines:

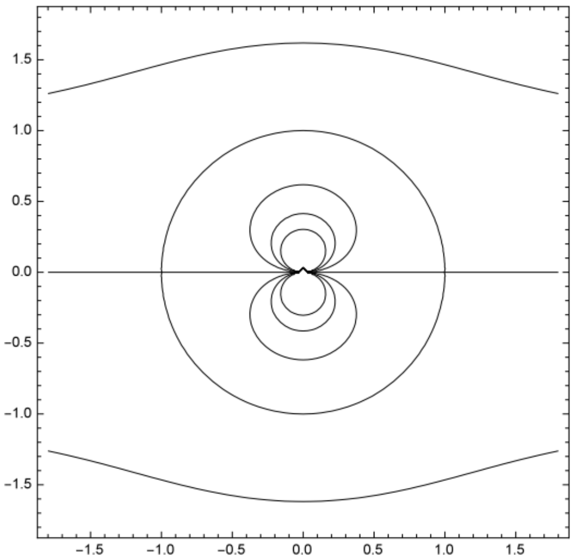


Figure 1: *Magnetic field lines corresponding to the choice* $f(w) = w$.

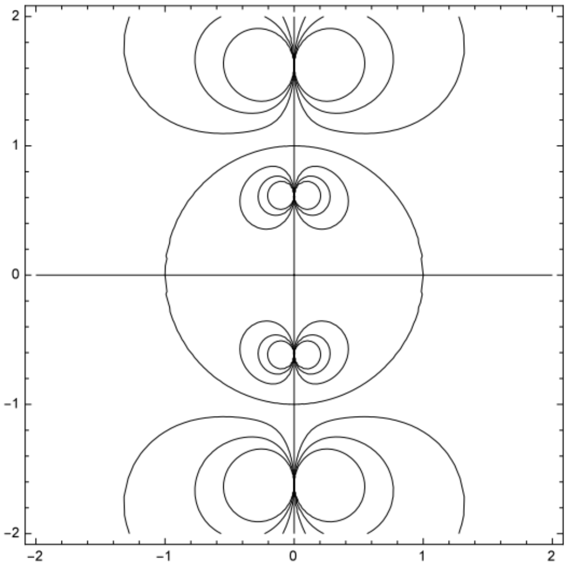


Figure 2: *Magnetic field lines corresponding to the choice* $f(w) = 1/(1 + w^2)$.

Note

- Location of singularities depends both on boundary shape and on boundary data (**B**).
- Depending on boundary data, singularties can be arbitrarily close to the boundary

Is there some upper bound on the distance to the nearest singularity depending only on the boundary shape?

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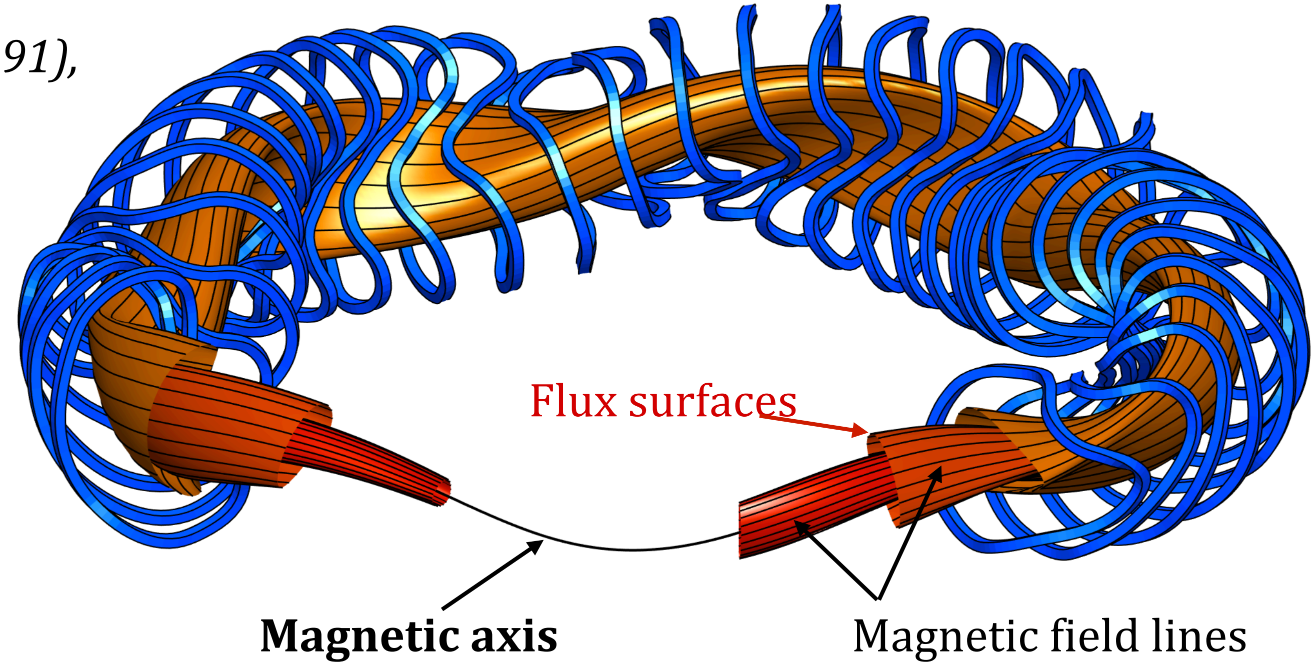
Application: finding easy-to-make near-axis quasisymmetric fields

Along the *magnetic axis* of a quasisymmetric field, we have equations for **B** and **∇B**.

Garren & Boozer (1991),
my talk Friday



Given **B** and **∇B** along this closed curve, how best can we exclude infeasible solutions?

$$\text{E.g. } \left|\mathbf{b}\cdot\nabla\mathbf{b}\right| < \text{threshold}, \quad \text{or} \quad \left(\nabla\mathbf{B}:\nabla\mathbf{B}\right)/B^2 < \text{threshold?}$$

Questions

- Is there an illuminating & quick-to-compute measure of why some **B** configurations are hard to make with distant coils?
- Is it possible to prove rigorously that concave flux surface shapes require a coil nearby?
- Is principal curvature the relevant quantity, or something else?
- Is there a way to generalize the complex variables approach from 2D to 3D?
- Given **B** and **∇B** along the magnetic axis, what is the best estimate for "complexity" of coils or **B** a finite distance from axis?

Extra slides

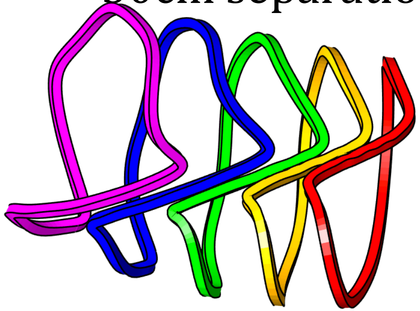
To increase plasma-coil separation for given plasma shape, either coil complexity or device size increases.

In a reactor, must fit ~ 1.5m “blanket” between plasma and coils to absorb neutrons.

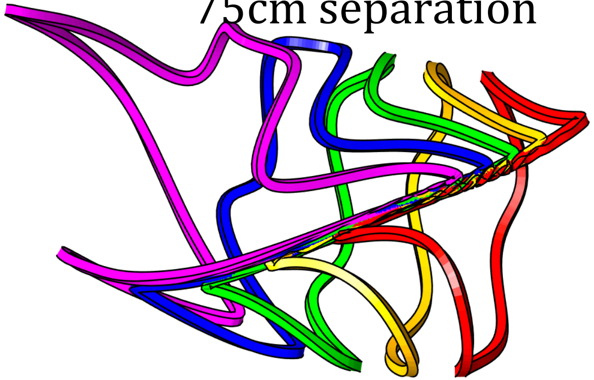
Coils offset a uniform distance from W7-X plasma:

25cm separation

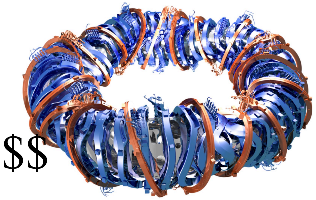
50cm separation



75cm separation



Must scale everything up:



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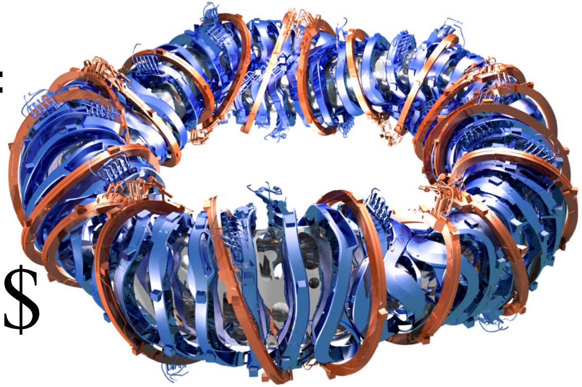

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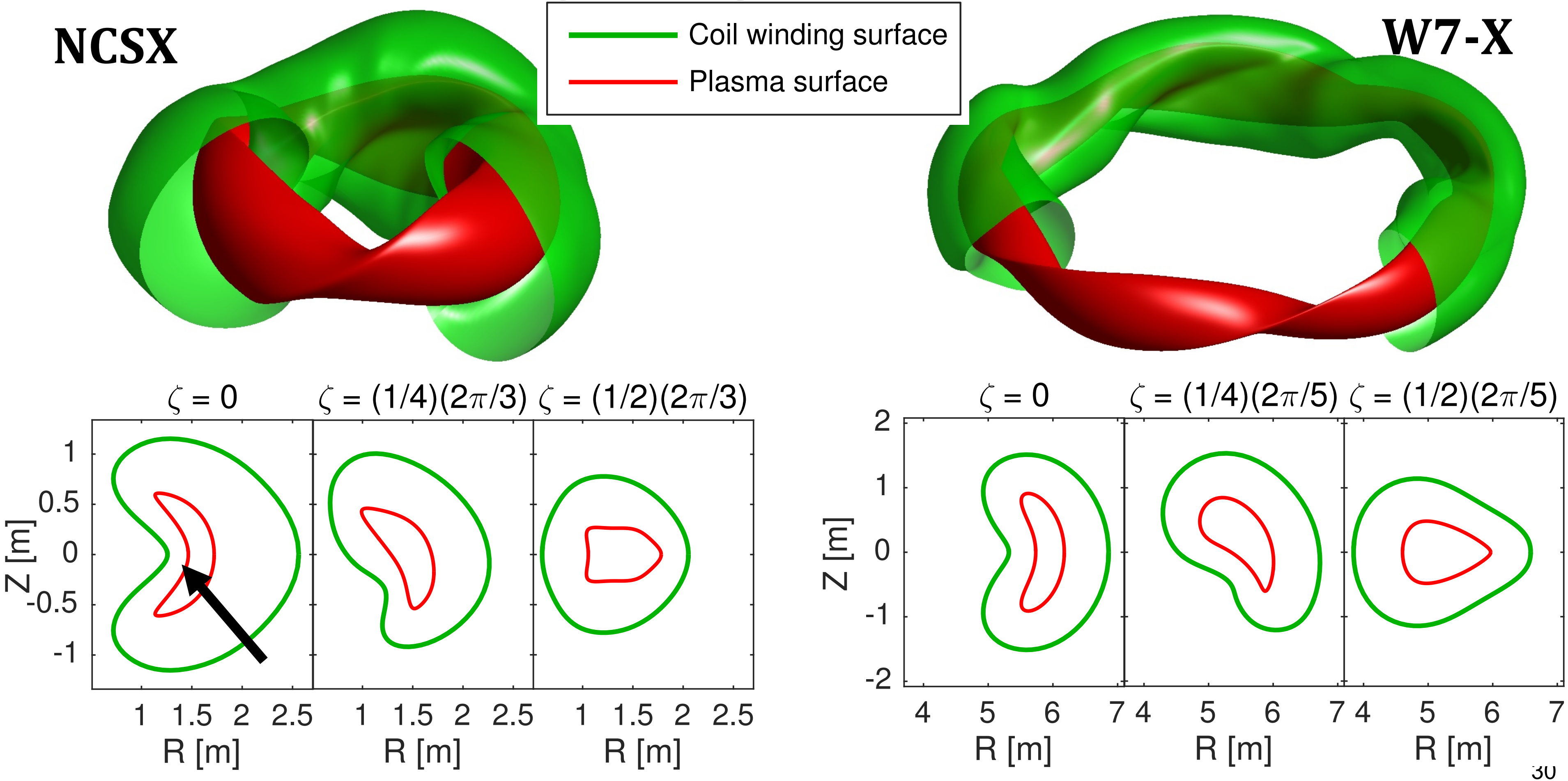
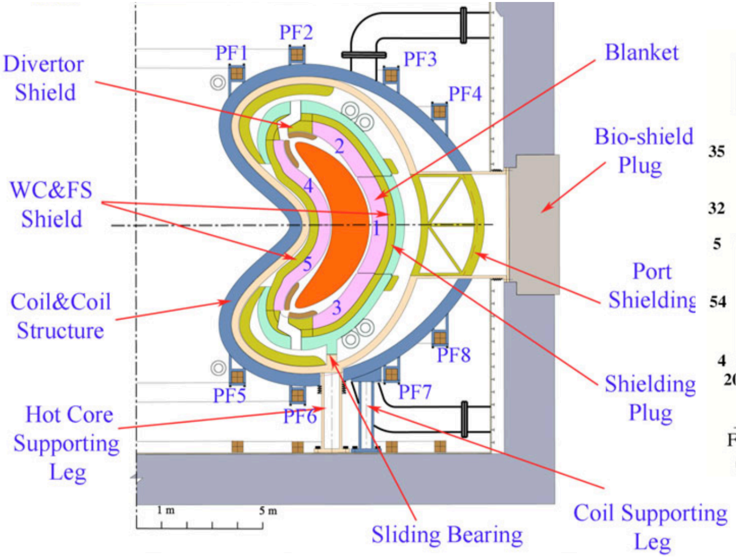
It is consistently found that it is hard to get coils far from concave flux surface shapes. Why?

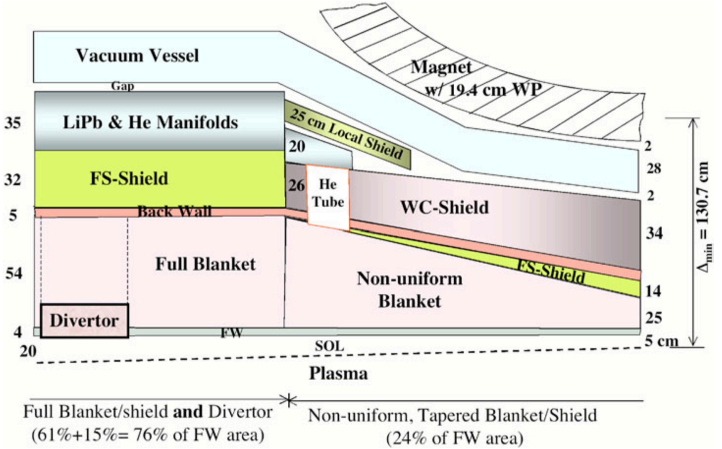


Plasma-coil distance is a crucial quantity for viability of a stellarator reactor.

"Being the most influential parameter for the stellarator's size and cost, Δ_{min} [minimum plasma-coil distance] optimization was crucial to the overall design."

ARIES-CS study, El-Guebaly et al, *Fusion Sci Tech* (2008)





Calculating the currents that produce a given B is an ill-posed problem

Given data for **B** on the plane z = 0, assuming **B** is a vacuum field, what is **B** off the plane?

Like initial-value problem with Laplace's eq: $\frac{\partial^2 \Phi}{\partial z^2} = -\frac{\partial \Phi^2}{\partial x^2} - \frac{\partial \Phi^2}{\partial y^2}$

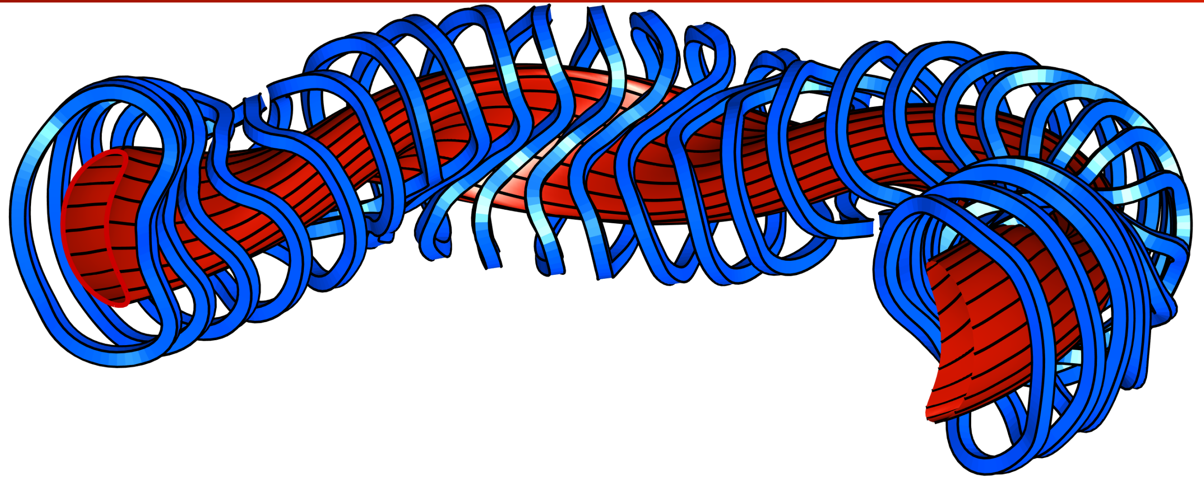
Tiny short-wavelength changes to initial data grow exponentially:

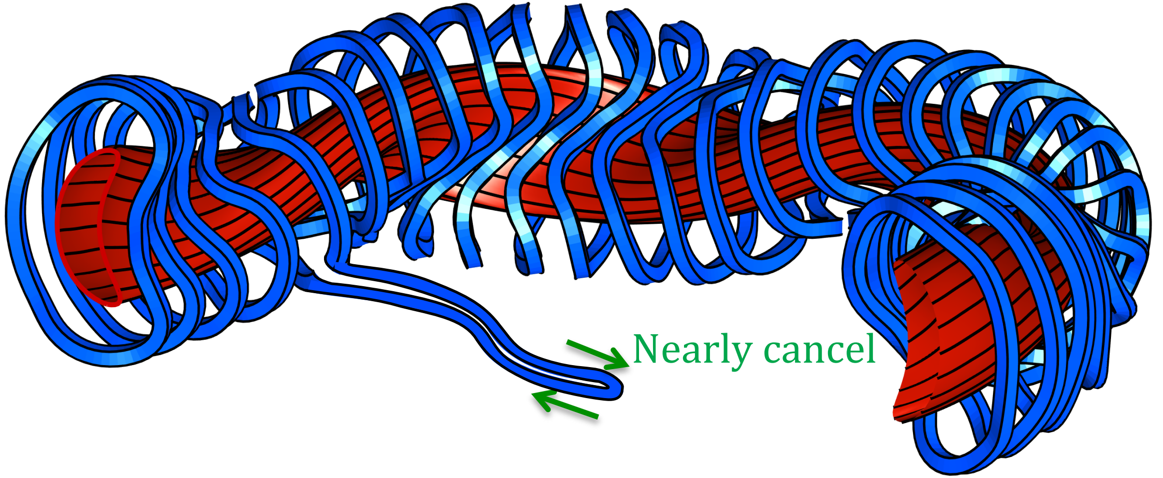
$$\mathbf{B}(x,y,z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \exp(ik_x x + ik_y y)\left[\left(\quad .... \quad \right) \exp\left(z\sqrt{k_x^2 + k_y^2} \right) + \left(\quad .... \quad \right) \exp\left(-z\sqrt{k_x^2 + k_y^2} \right) \right]$$

So we probably can't say precisely where the coils must be. But can we say something about the *best-case scenario*?, e.g. "There must be a coil within some distance *d*"?

Calculating the currents that produce a given B is an ill-posed problem

For any $\boldsymbol{\varepsilon} > 0$, there exist two coil shapes that differ to O(1) yet produce a difference $< \boldsymbol{\varepsilon}$ in **B** in the confinement region.





Biot-Savart Law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\text{coil}} \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

Calculating the currents that produce a given B is an ill-posed problem

Given data for **B** on the plane $z = 0$, assuming **B** is a vacuum field, what is **B** off the plane?

At $z=0$, $\Phi=\sin(x)\exp(z)$ is hard to distinguish from

$$\Phi=\sin(x)\exp(z)+e^{-100}\sin(100x)\exp(100z)$$

Like initial-value problem with Laplace's eq: $\dfrac{\partial^2\Phi}{\partial z^2}=-\dfrac{\partial\Phi^2}{\partial x^2}-\dfrac{\partial\Phi^2}{\partial y^2}$

So we probably can't say *precisely* where the coils must be. But can we say something about the *best-case scenario*?, e.g. "There must be a coil within some distance *d*"?

- Call a fast coil code (NESCOIL or REGCOIL) for each iteration of the plasma shape, and penalize coil complexity (NCSX, ROSE).
- Penalize negative principal curvature (A Bader)

Some of these local measures of B complexity have appealing properties

Let $B = |\mathbf{B}|$.

In a world consisting of an infinite straight wire, $|\nabla B|/B$ and $\sqrt{(\nabla\mathbf{B}):(\nabla\mathbf{B})/(2B^2)}$ equal $1/R$, the inverse distance to the wire, and $(\nabla B)/B$ points to the wire.

For $\Phi = A\exp(k_x x + k_y y + k_z z) + c.c.$ with complex constants $\{A, k_x, k_y, k_z\}$ such that $k_x^2 + k_y^2 + k_z^2 = 0$, then $|\nabla B|/B$ and $\sqrt{(\nabla\mathbf{B}):(\nabla\mathbf{B})/(2B^2)}$ give the exponentiation scale $\frac{1}{2}\sqrt{(k_x + k_x^*)^2 + (k_y + k_y^*)^2 + (k_z + k_z^*)^2}$ and $(\nabla B)/B$ points in the direction of exponential growth.