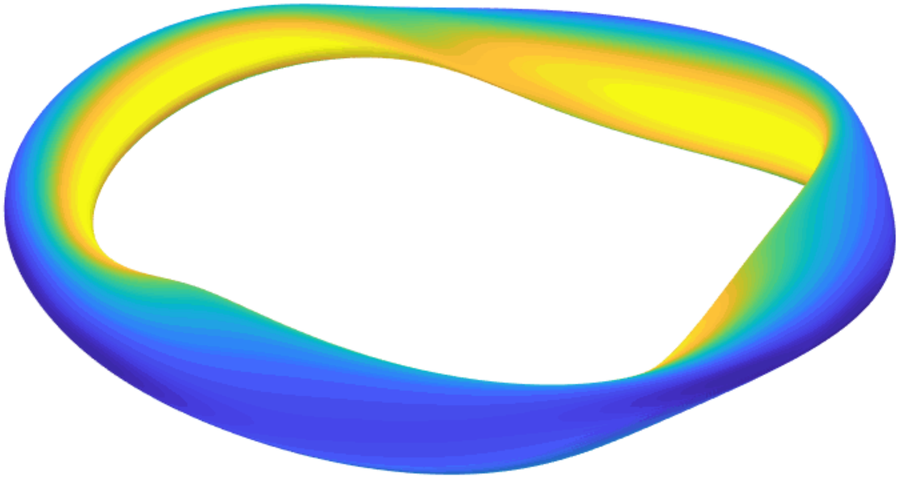
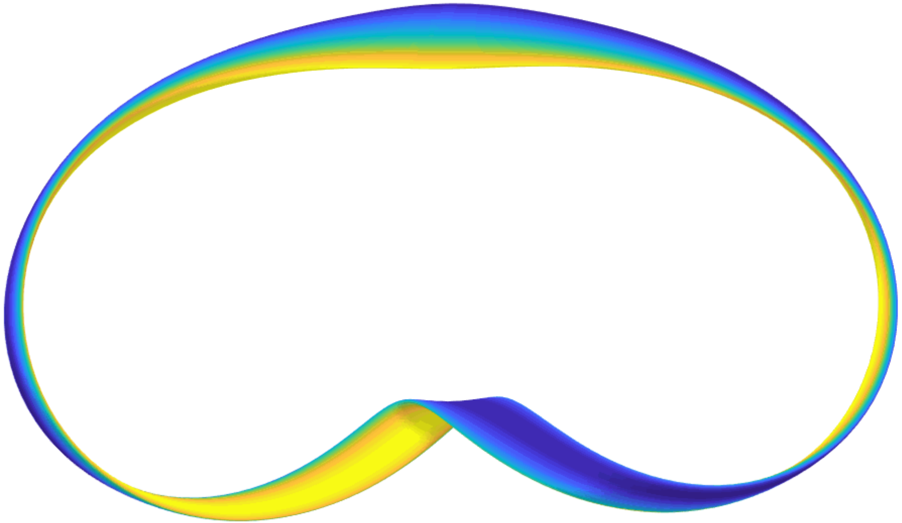
Constructing magnetic fields with hidden symmetry





Matt Landreman[1]

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1. University of Maryland 2. NYU 3. Max Planck Institute for Plasma Physics

J Plasma Phys (2019) arXiv:1809.10246 https://github.com/landreman/quasisymmetry

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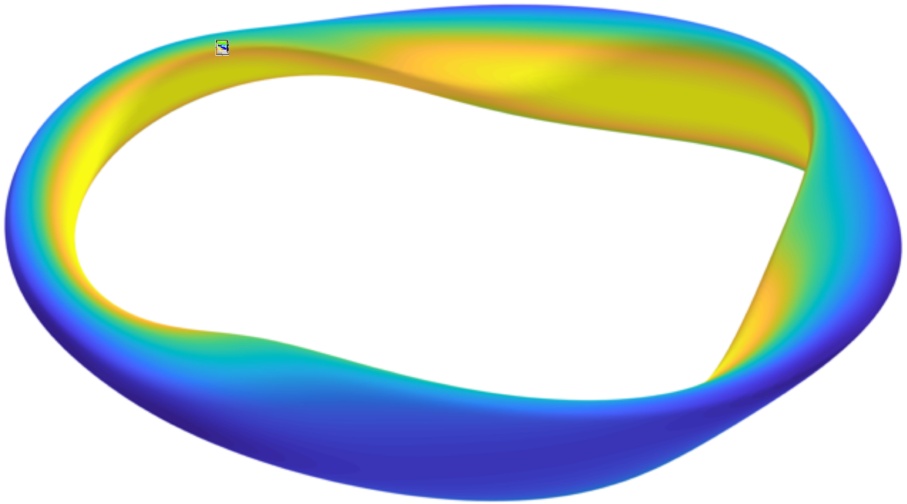

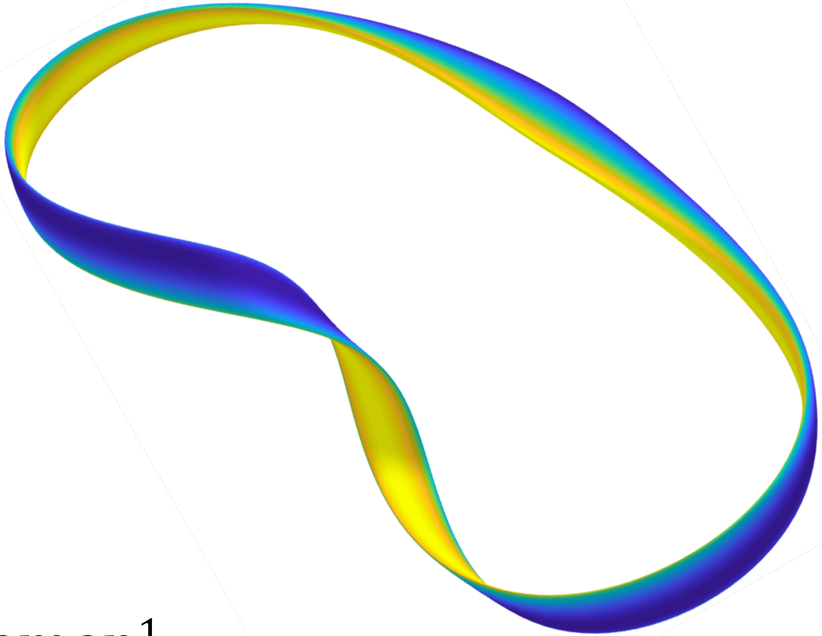

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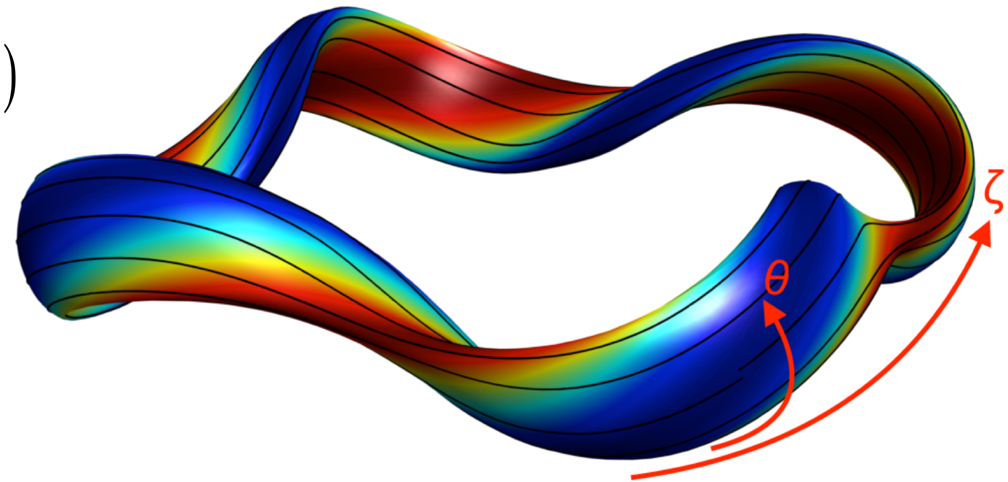
The conventional approach to finding quasisymmetric fields works but has shortcomings

Want magnetic field strength B
to have quasisymmetry: $B = B(r,\, \theta - N\zeta)$



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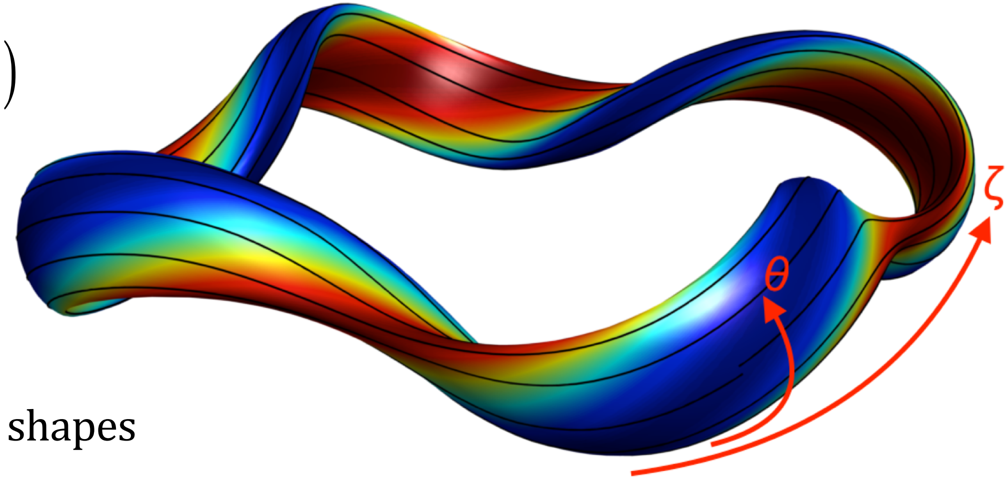

$$\min_X f\left(X\right)$$

Parameter space: $X =$ toroidal boundary shapes

Objective: $f = \sum_{m,n \neq Nm} B_{m,n}^2\left(r_0\right)$ where $B\left(r,\theta,\zeta\right) = \sum_{m,n} B_{m,n}\left(r\right)\exp\left(im\theta - in\zeta\right)$

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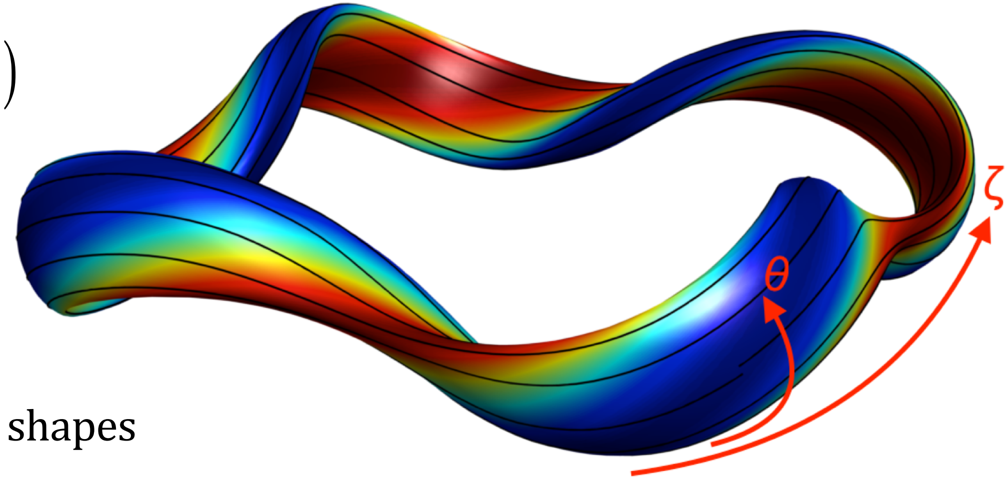

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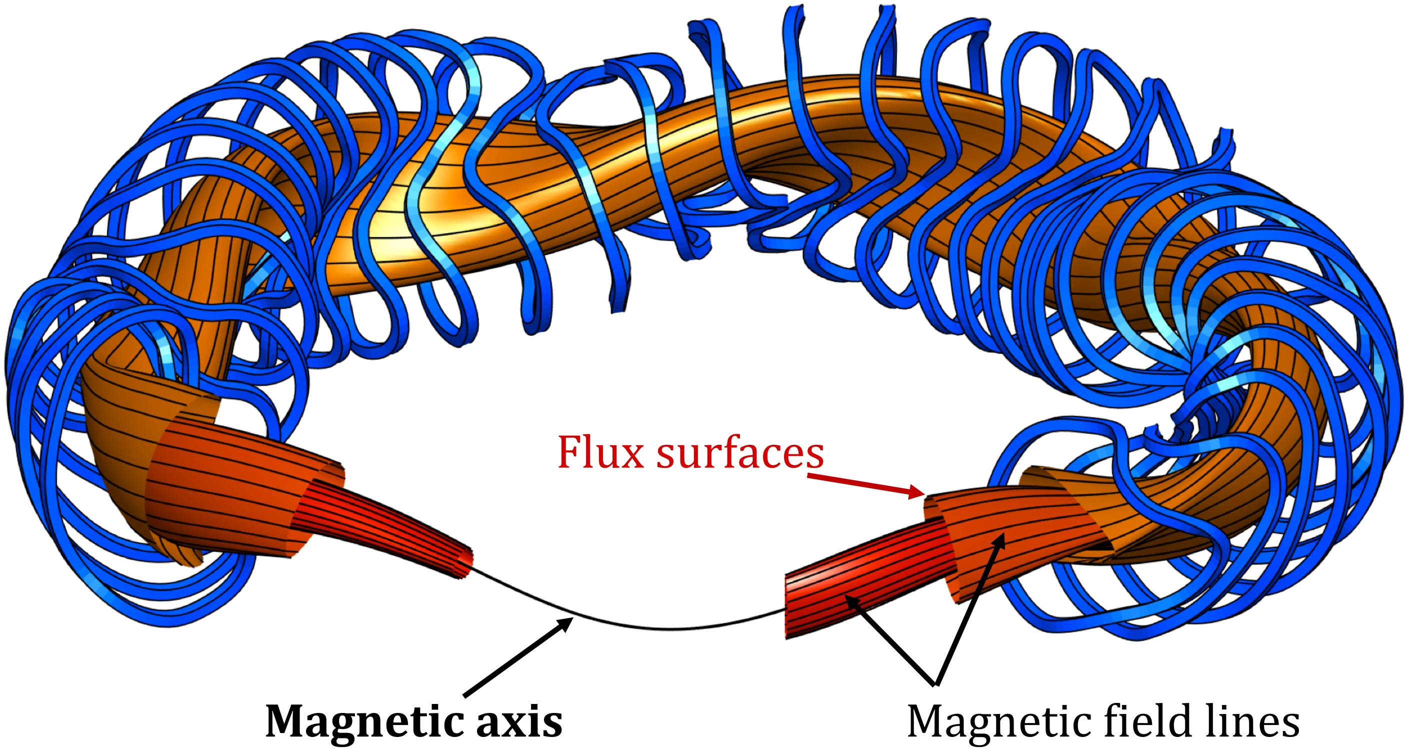
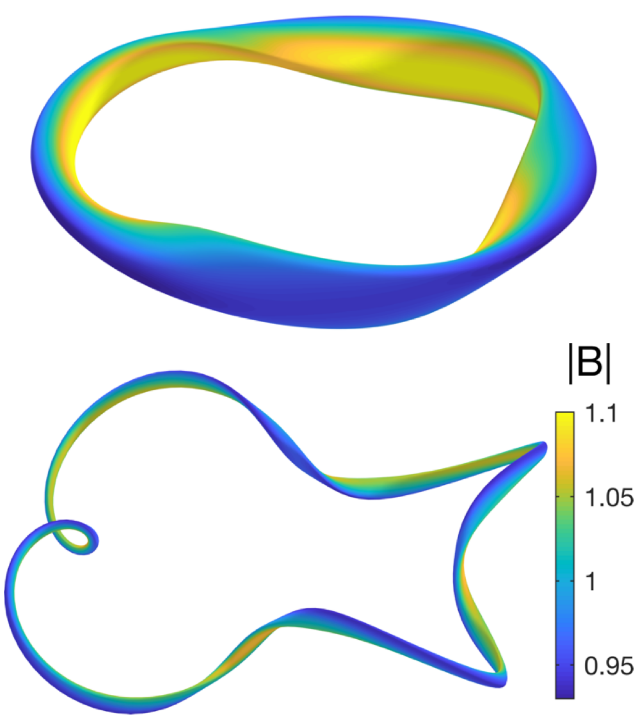
- Computationally expensive.
- What is the size & character of the solution space?
- Result depends on initial condition, so cannot be sure you’ve found all solutions.

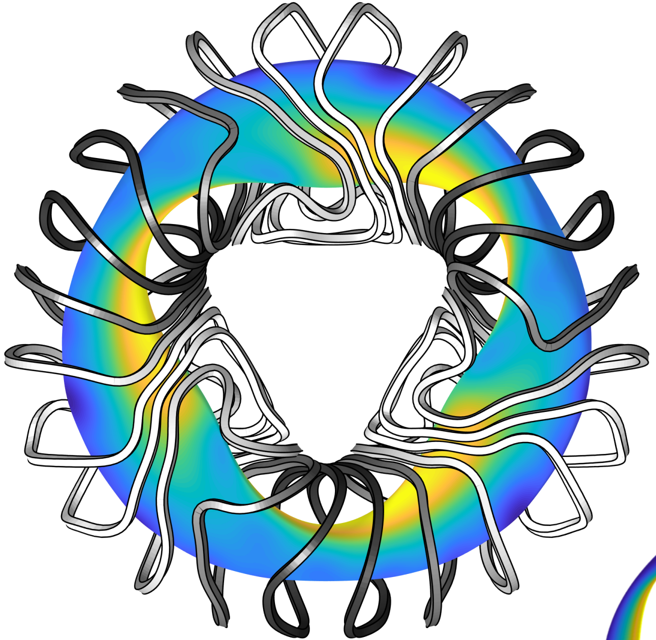
Alternative: expand equations near the magnetic axis

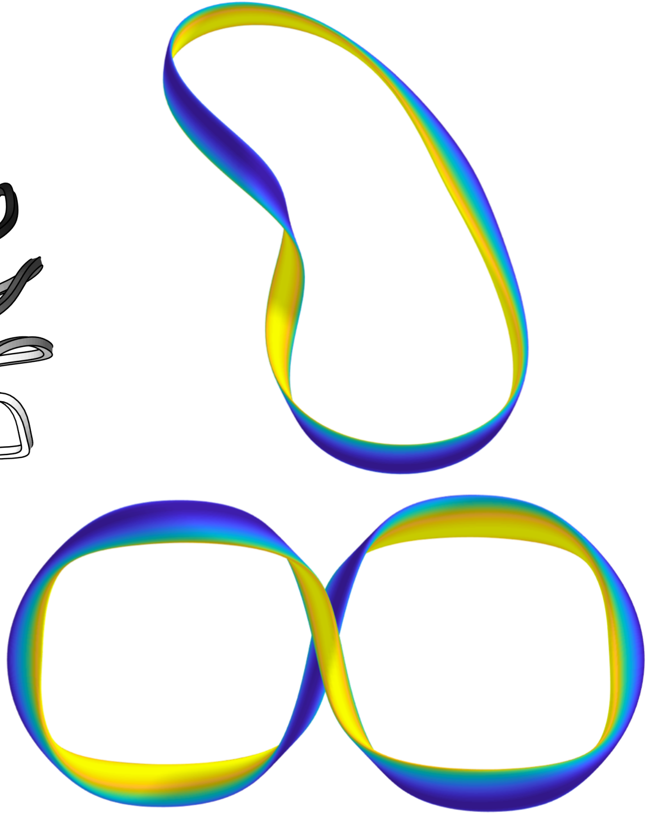


Mercier (1964), Lortz & Nührenberg (1976), Garren & Boozer (1991)

Using the near-axis expansion, we can directly construct quasisymmetric B fields







Goals:
- Speed.
- Understand the solution space.
- Generate good initial conditions for optimization.

Outline

- Construction for $O(r)$ quasisymmetry
  - Theory, & the number of solutions
  - Numerical results
  - Comparison to “real experiments”
  - The landscape of solutions
- Extensions
  - Omnigenity
  - $O(r^2)$ quasisymmetry

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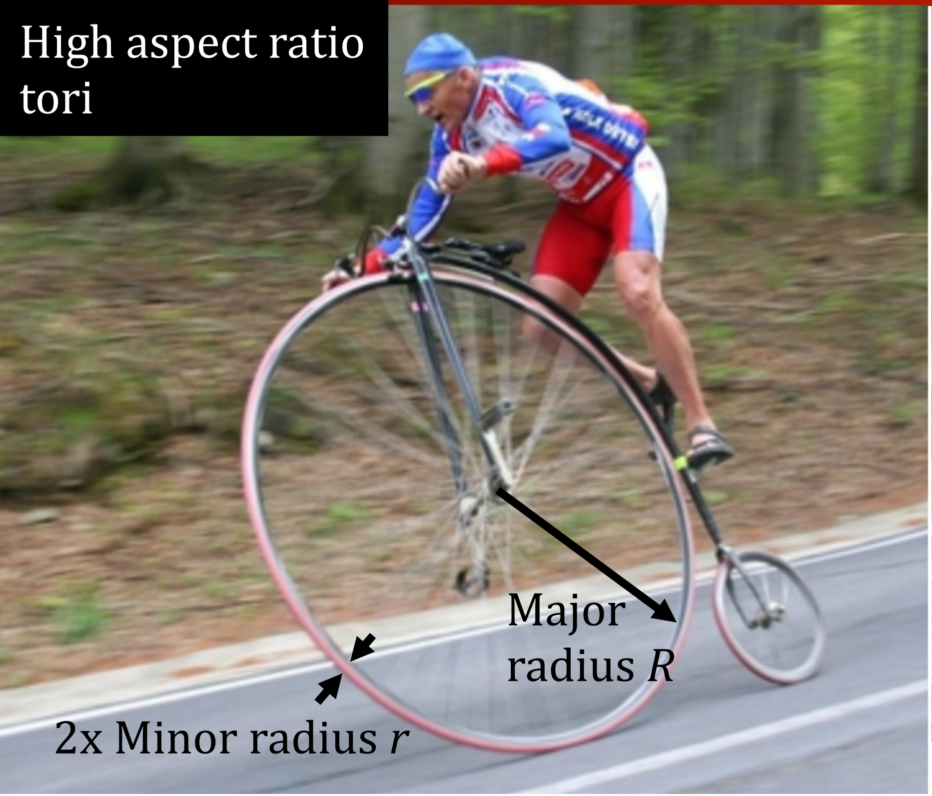
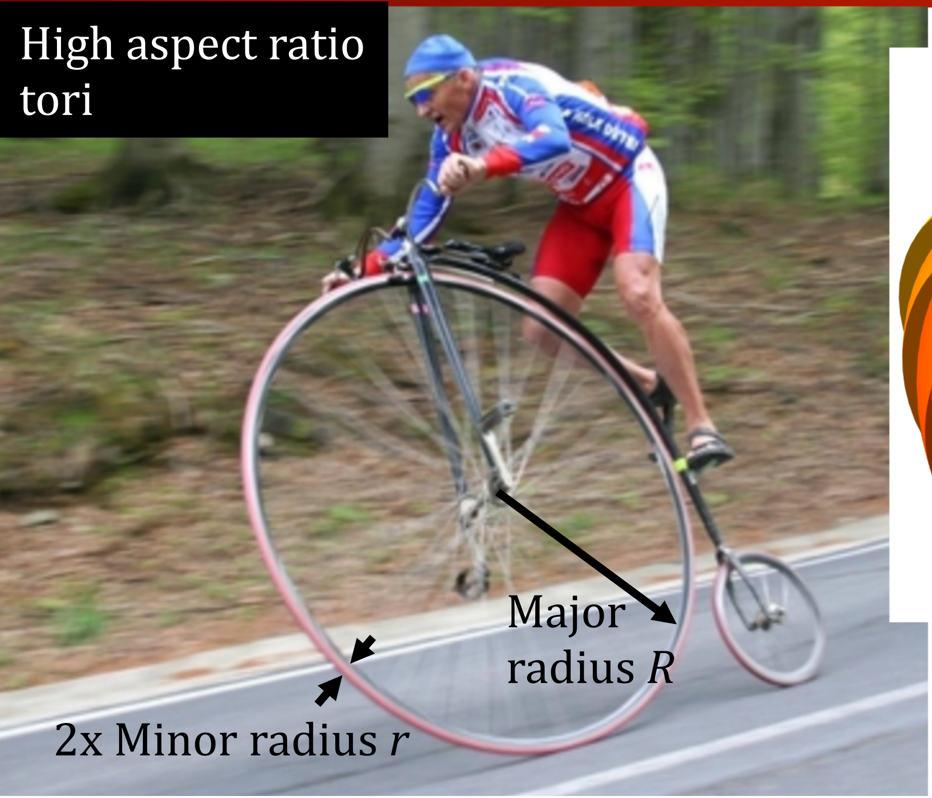
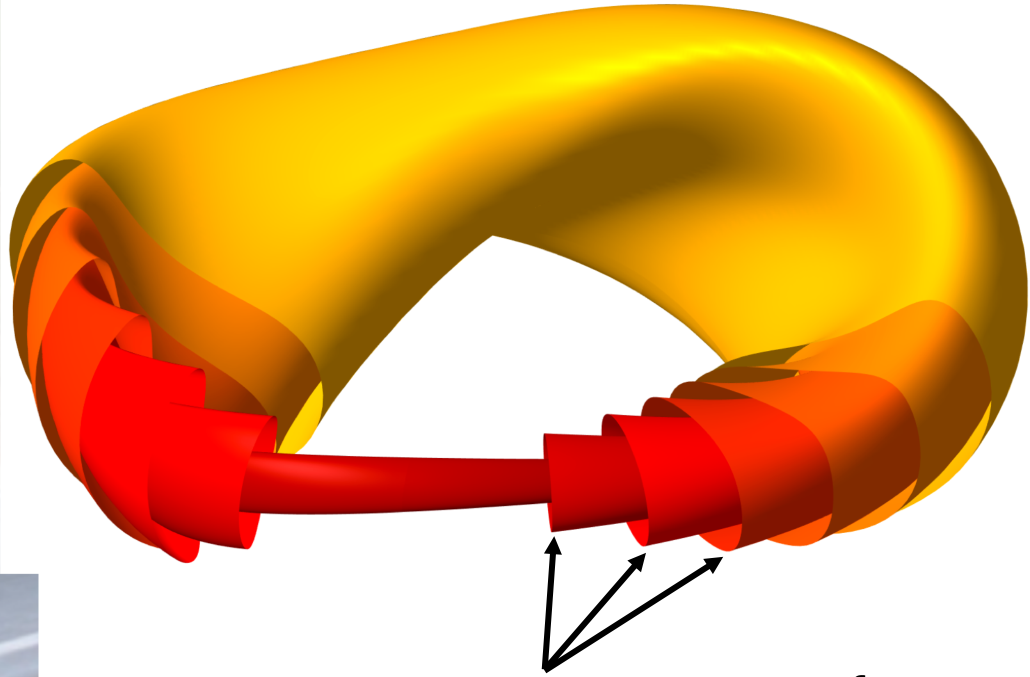
We will expand in the skinniness of the inner flux surfaces



$$\frac{r}{\text{radius of curvature of axis}} \ll 1 \qquad\qquad \text{Aspect ratio} = \frac{R}{r}$$

Even for a low aspect ratio stellarator, this expansion describes the core





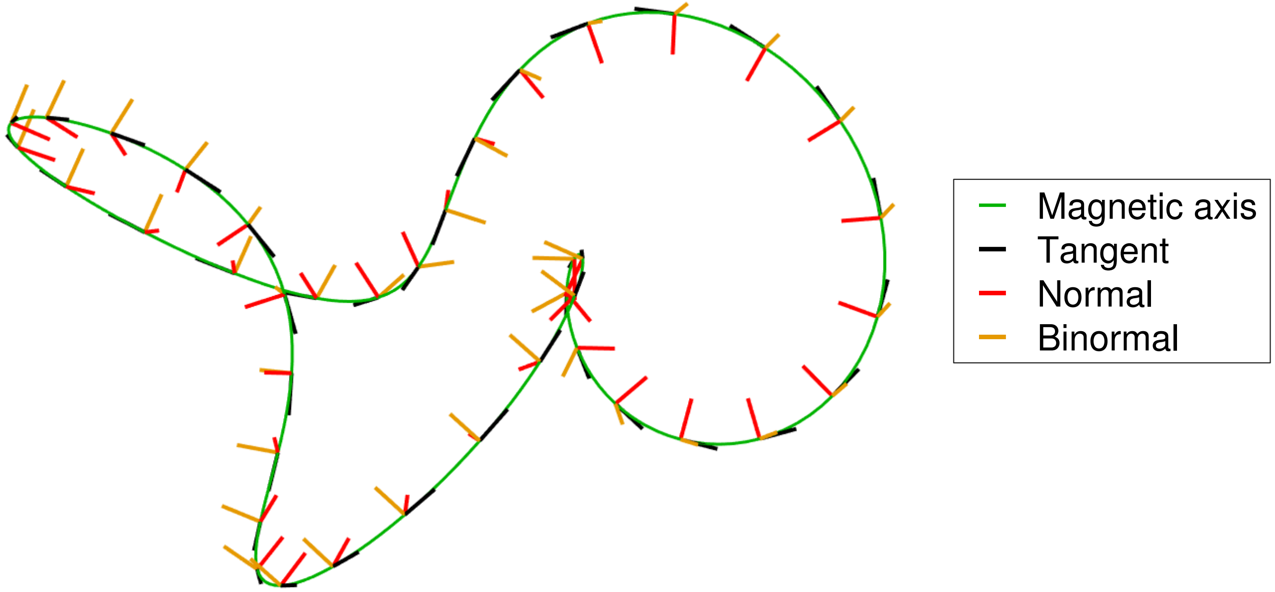
$$\frac{r}{\text{radius of curvature of axis}} \ll 1$$

$$\text{Aspect ratio} = \frac{R}{r}$$

A key ingredient of the theory is the Frenet frame of the magnetic axis

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}): \quad \frac{d\mathbf{x}_0}{d\ell}=\mathbf{t}, \quad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n}, \quad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b}, \quad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

$\mathbf{x}_0 =$ magnetic axis, $\kappa =$ curvature, $\tau =$ torsion, $\mathbf{t} =$ tangent, $\mathbf{n} =$ normal, $\mathbf{b} =$ binormal



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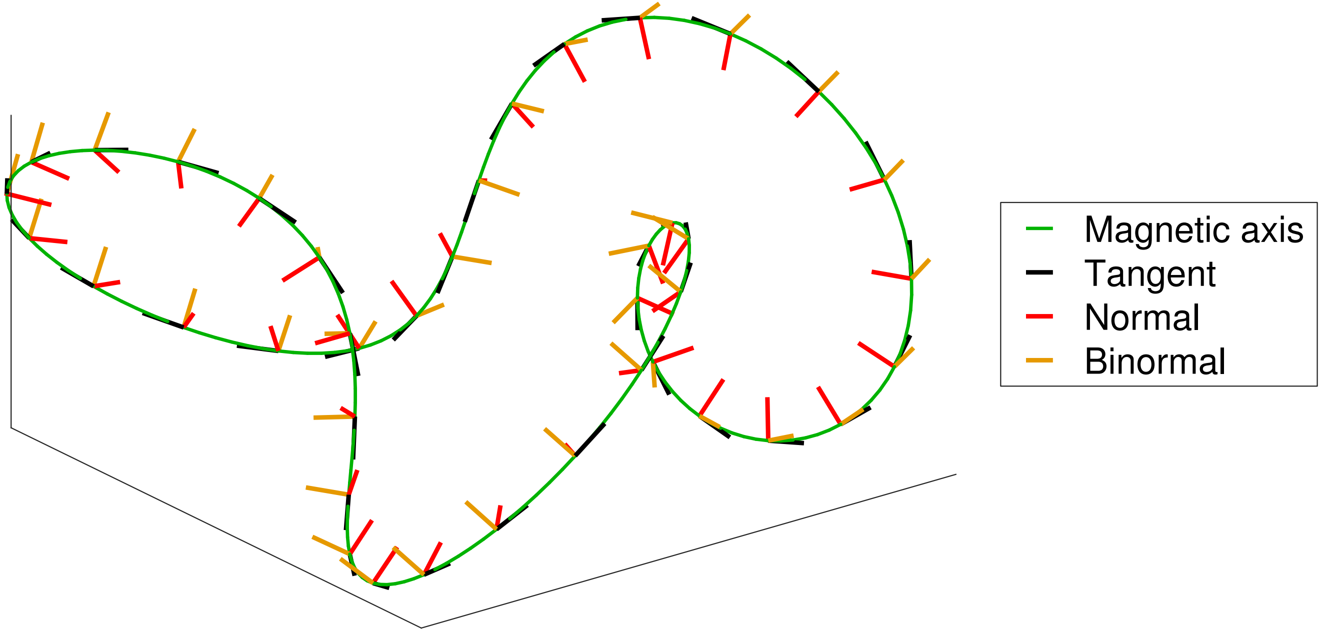
$\mathbf{x}_0$ = magnetic axis, κ = curvature, τ = torsion, $\mathbf{t}$ = tangent, $\mathbf{n}$ = normal, $\mathbf{b}$ = binormal



Theory: Write position vector **x** using axis's Frenet frame, expand in small *r*

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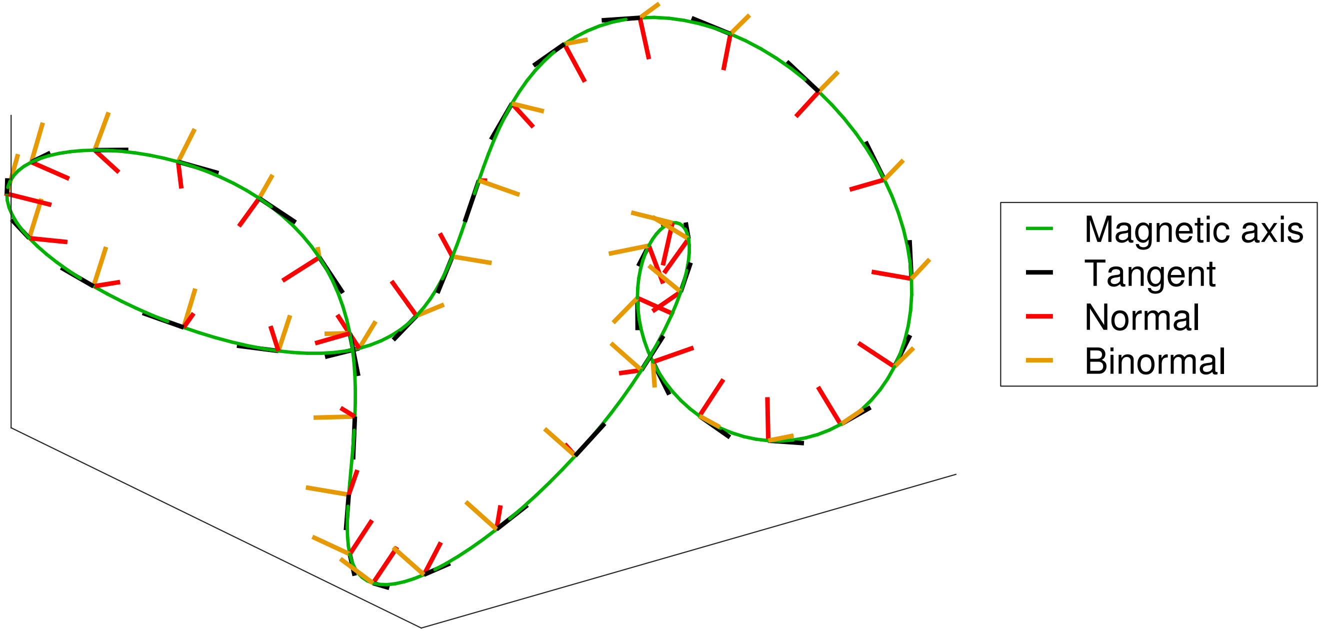
$\mathbf{x}_0=$ magnetic axis, $\kappa=$ curvature, $\tau=$ torsion, $\mathbf{t}=$ tangent, $\mathbf{n}=$ normal, $\mathbf{b}=$ binormal

$$\mathbf{J}\times\mathbf{B}=\nabla p,\qquad B=B(r,\ \theta-N\zeta),\qquad \mathbf{x}=\mathbf{x}_0+O(r)$$



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$$\mathbf{x}(r,\theta,\zeta)=\mathbf{x}_0(\zeta)+r\frac{\bar{\eta}}{\kappa(\zeta)}\cos\vartheta\,\mathbf{n}(\zeta)+r\left[\frac{\kappa(\zeta)}{\bar{\eta}}\sin\vartheta+\frac{\sigma(\zeta)\kappa(\zeta)}{\bar{\eta}}\cos\vartheta\right]\mathbf{b}(\zeta)+O(r^2)$$

Toroidal angle $\zeta\propto$ axis arclength ℓ, $\bar{\eta}=$ constant: $B=B_0\left[1+r\bar{\eta}\cos\vartheta+O(r^2)\right]$

$$\boxed{\frac{d\sigma}{d\zeta}+\iota\left[\frac{\bar{\eta}^4}{\kappa^4}+1+\sigma^2\right]-2\frac{\bar{\eta}^2}{\kappa^2}\left[I_2-\tau\right]=0}$$

$\vartheta=\theta-N\zeta$,

$\iota=$ rotational transform on axis,

$I_2=$ current density on axis

The size of the space of fields that are quasisymmetric to O(r) can be precisely understood.

Given $P(\zeta)>0$, $Q(\zeta)$, and $\sigma(0)$, with $P(\zeta)$ and $Q(\zeta)$

2π-periodic, bounded, and integrable, a solution to

$$\boxed{\frac{d\sigma}{d\zeta}+\iota\left(P+\sigma^2\right)+Q=0} \qquad (1)$$

is a pair $\{\iota,\ \sigma(\zeta)\}$ solving (1) where $\sigma(\zeta)$ is 2π-periodic.

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<u>**Theorem:**</u> A solution exists and it is unique.

ML, Sengupta, and Plunk (2019). Probably an earlier reference?

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Theorem: A solution exists and it is unique.

$\Rightarrow$ Numerical solution is very robust.

The theorem informs an algorithm for constructing quasisymmetric fields

Inputs:

- Shape of the magnetic axis.
- 3 real numbers:
  - I_2: Current density on the axis. (Usually 0).
  - $\sigma(0)$: Rotation of the elliptical flux surfaces at toroidal angle=0.
  - $\bar{\eta}$, which controls elongation and field strength: $B = B_0\left[1 + r\bar{\eta}\cos\vartheta + O\left(r^2\right)\right]$
- (Pressure doesn't matter to this order.)

Outputs:

- Shape of the surfaces around the axis. (Elongation & rotation of ellipses.)
- Rotational transform on axis.

Quasi-axisymmetry vs quasi-helical symmetry is determined purely by the axis normal vector

$$\mathbf{J}\times\mathbf{B}=\cancel{\nabla p} \quad \Rightarrow \quad \boxed{\nabla_{\perp} B = B\kappa\mathbf{n}}$$

So B contours rotate about axis with the same topology as $\mathbf{n}$.

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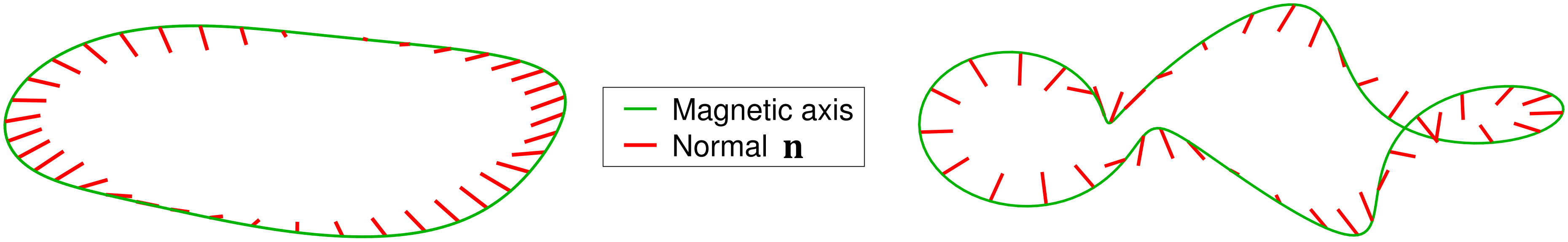

n does not rotate about the axis as you follow the axis around.

$\Rightarrow$ Quasi-axisymmetry

$$B=B\left(r,\theta\right)$$

n rotates about the axis 4 times as you follow the axis around.

$\Rightarrow$ Quasi-helical symmetry

$$B=B\left(r,\theta-4\zeta\right)$$

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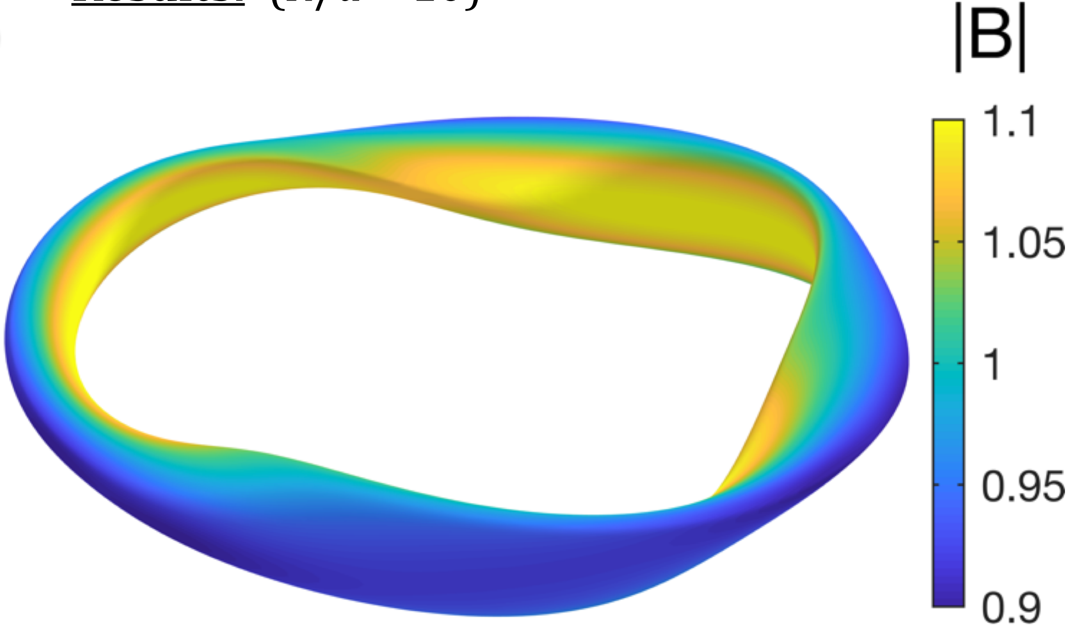
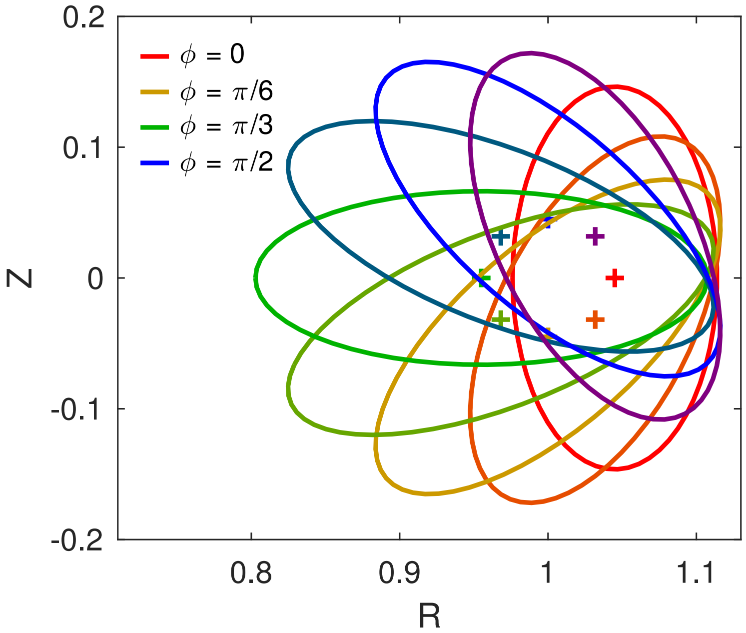
Example construction: quasi-axisymmetry

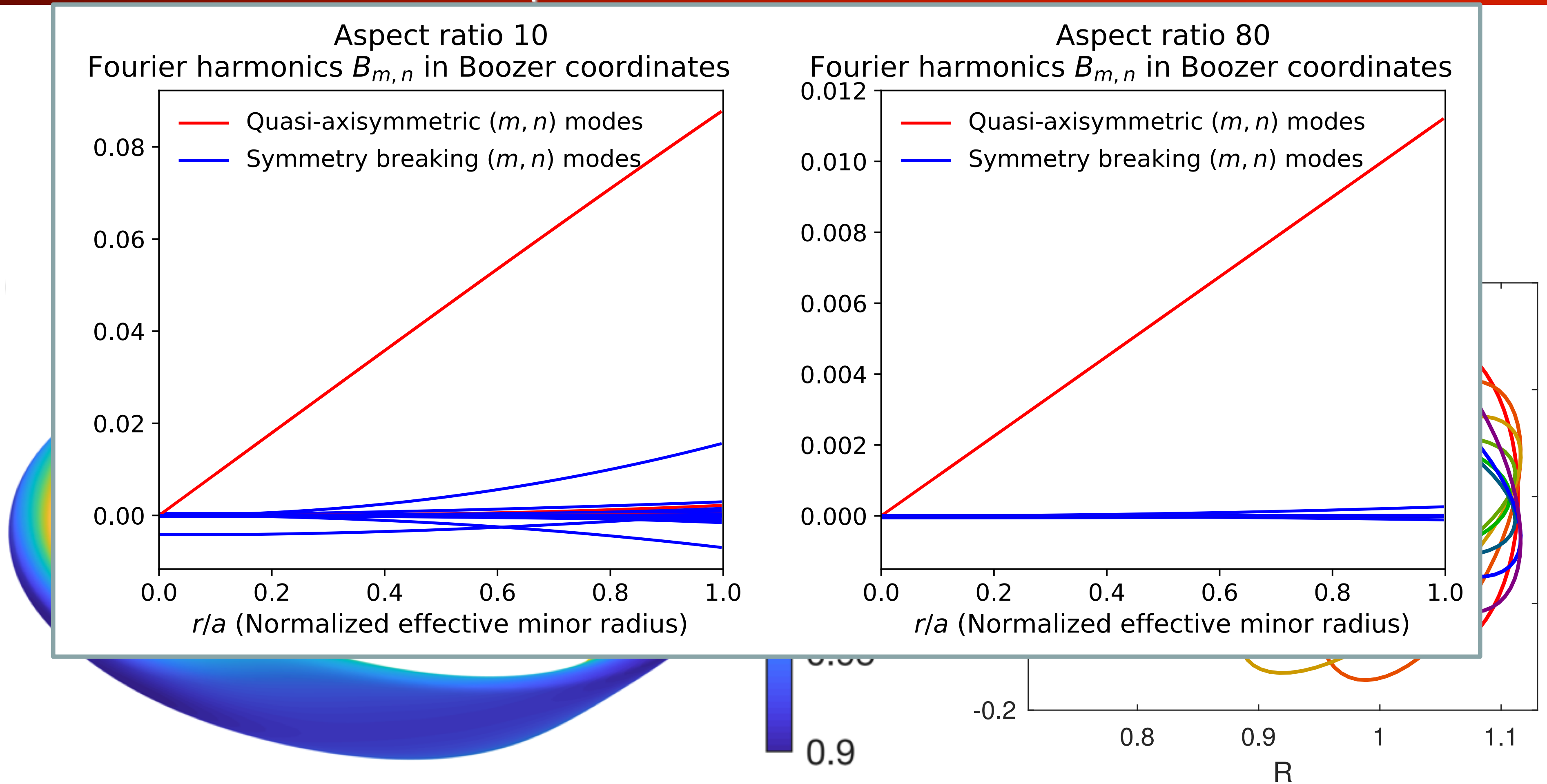
Inputs: axis shape $R_0(\phi) = 1 + 0.045\cos(3\phi)$, $Z_0(\phi) = -0.045\sin(3\phi)$, $I_2 = 0$, $\sigma(0) = 0$, $\bar{\eta} = -0.9$.

Results: $(R/a = 10)$



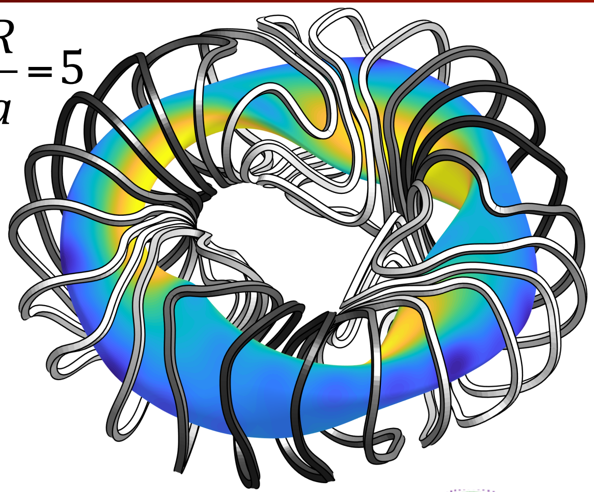


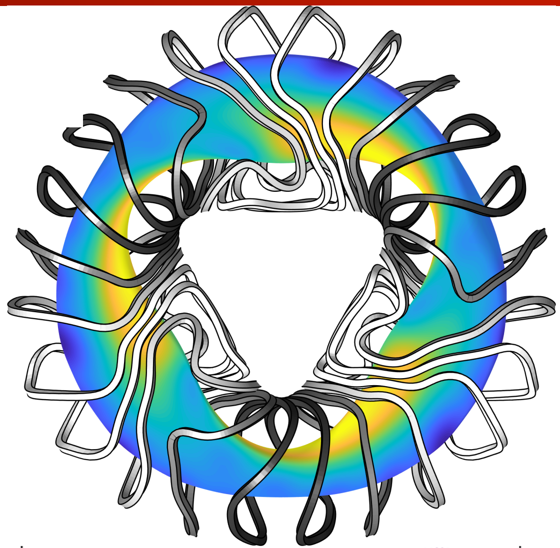
The construction can be verified by comparing to an MHD equilibrium calculation that does not make the *r* expansion.

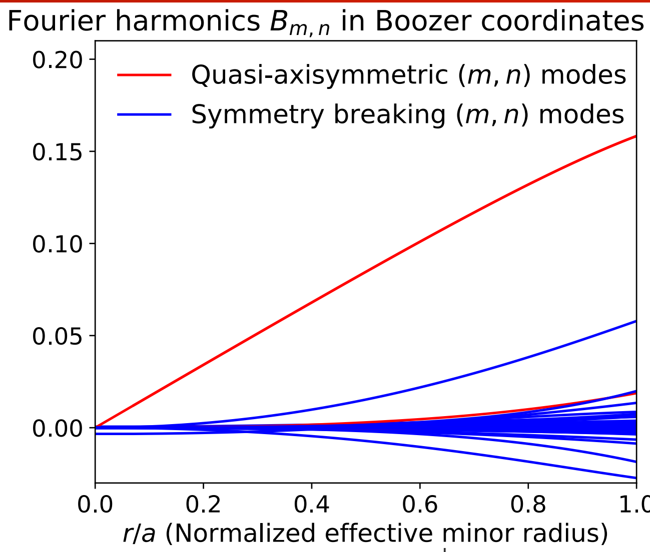


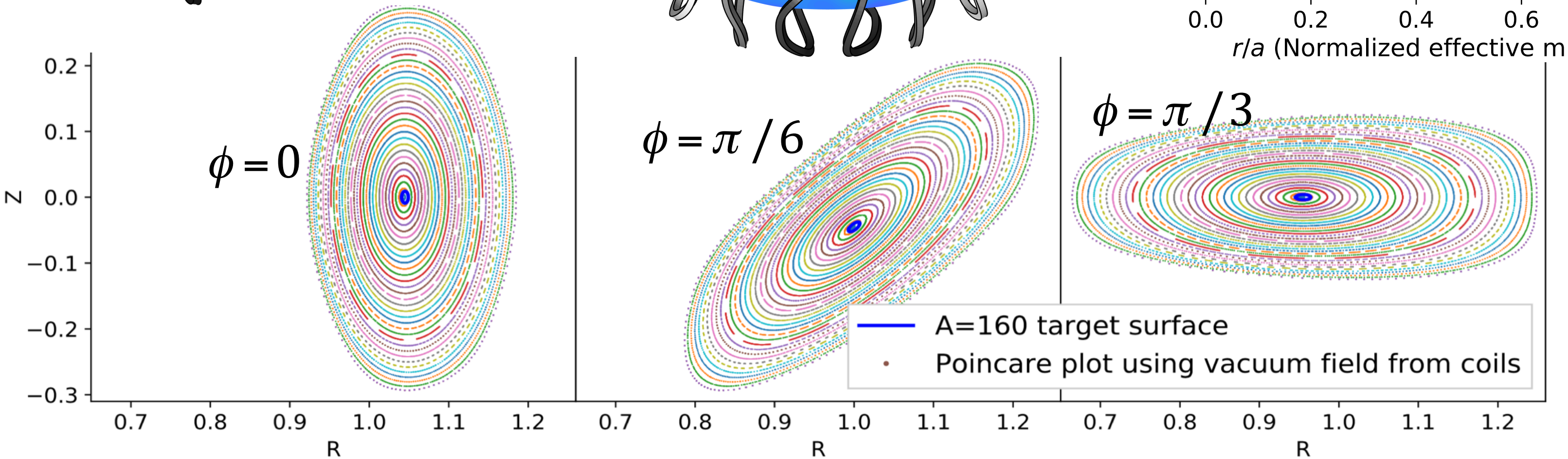
Alternative method to generate a finite-thickness boundary: find coils to make a skinny surface, then see what you get outside.

$$\frac{R}{a} = 5$$





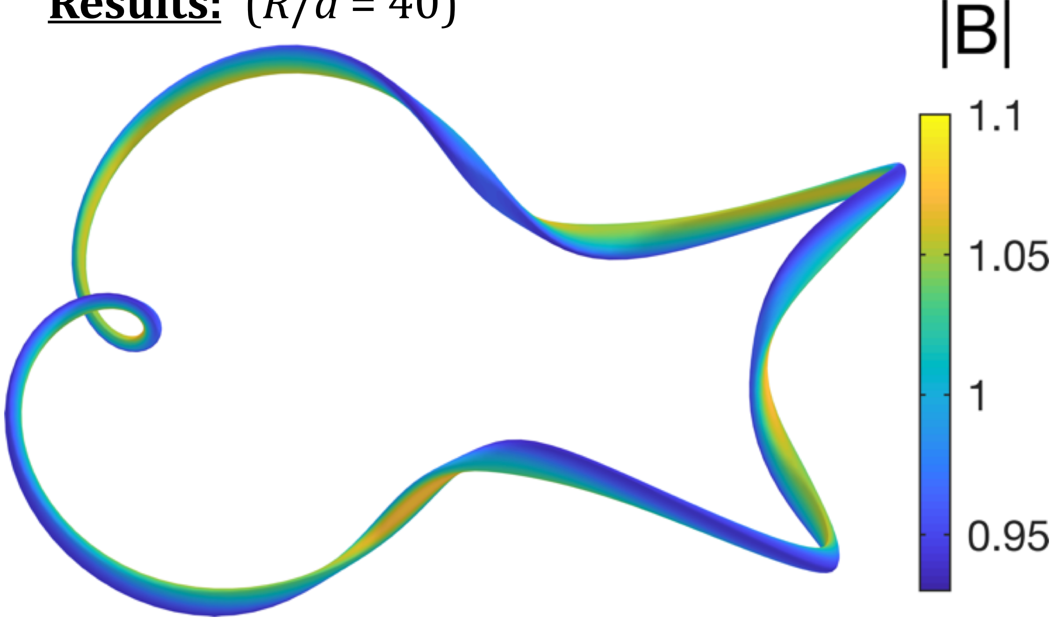


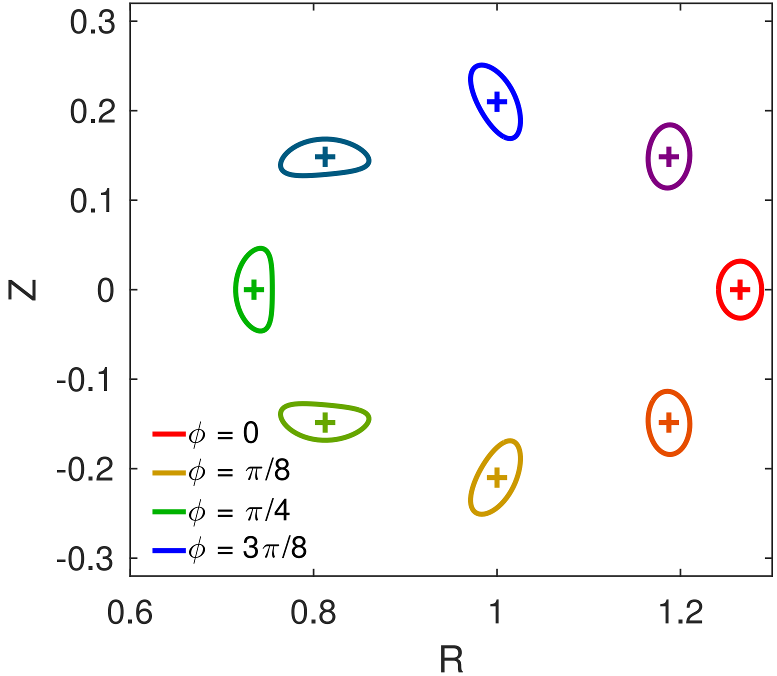


Example construction: Quasi-helical symmetry

Inputs: axis shape $R_0(\phi) = 1 + 0.265\cos(4\phi)$, $Z_0(\phi) = -0.21\sin(4\phi)$, $I_2 = 0$, $\sigma(0) = 0$, $\bar{\eta} = -2.25.$

Results: $(R/a = 40)$





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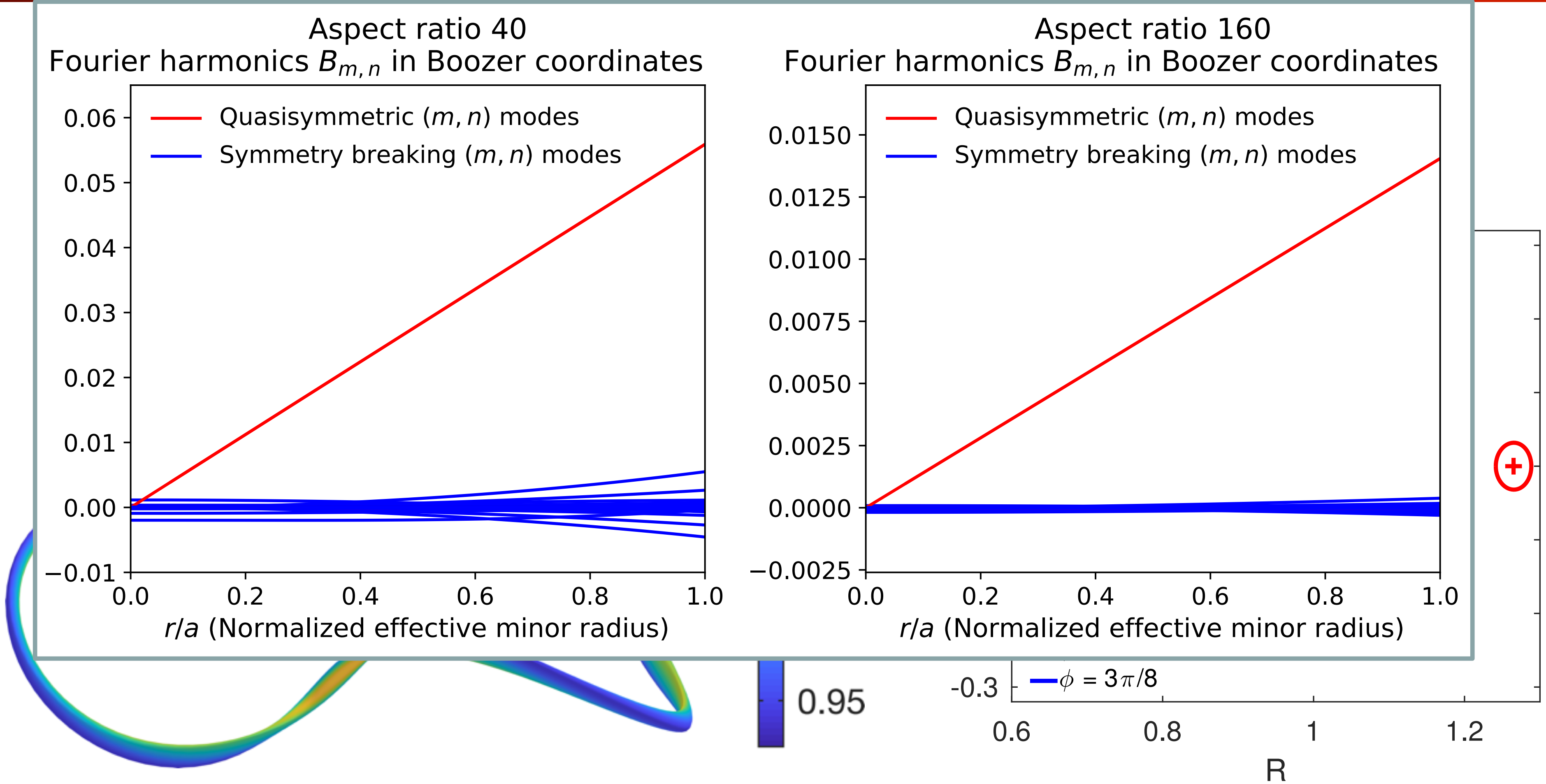
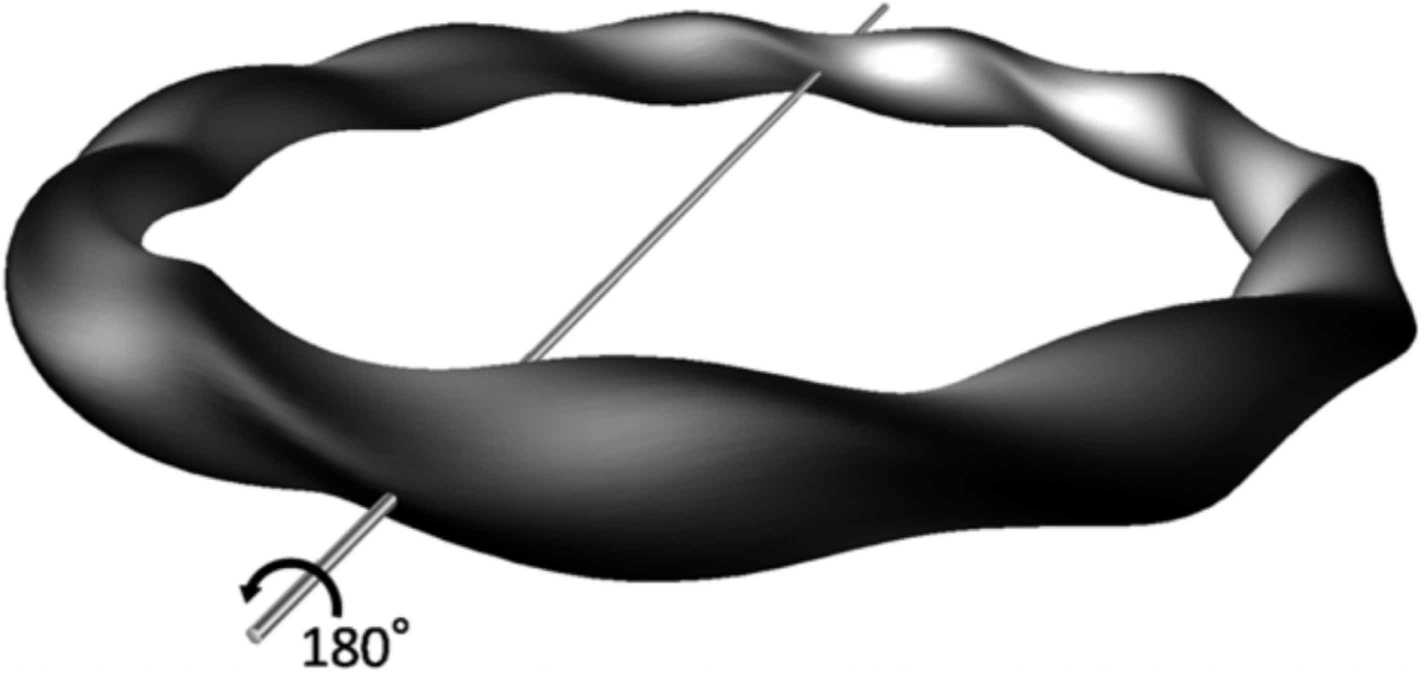


All stellarators built to date have 'stellarator symmetry', which is unrelated to quasisymmetry



Sugama et al (2011)

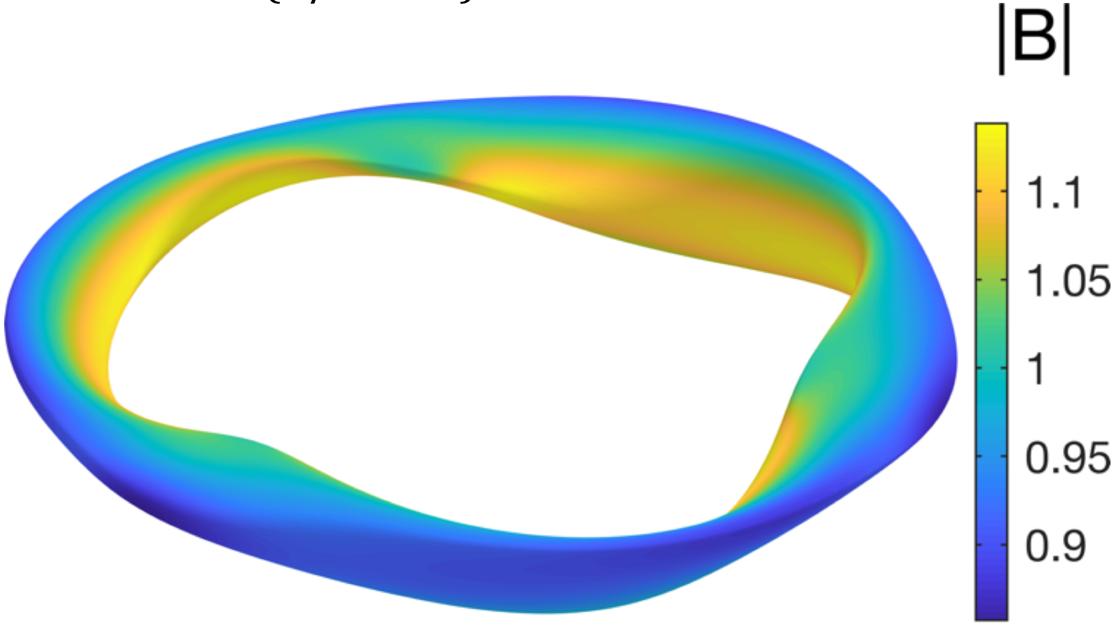
You can make a quasi-axisymmetric stellarator without stellarator symmetry

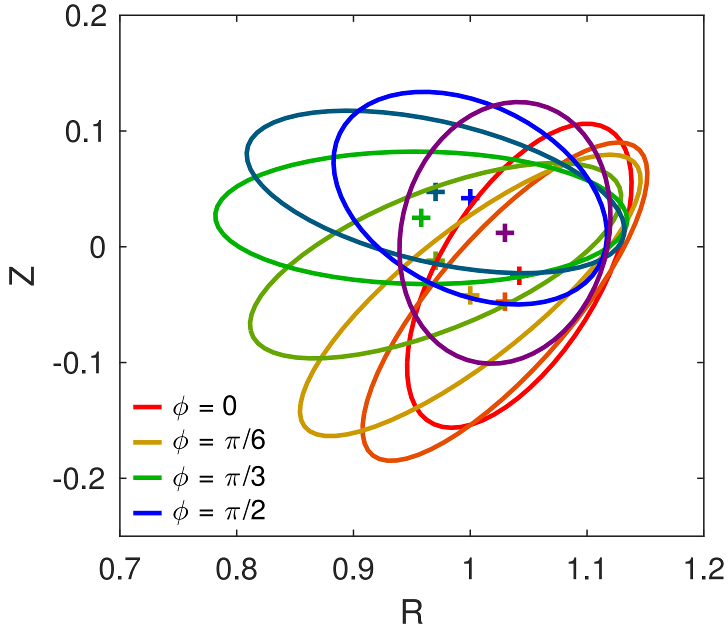
Inputs: axis shape $R_0(\phi) = 1 + 0.042\cos(3\phi)$, $I_2 = 0$, $\bar{\eta} = -1.1.$

$$Z_0(\phi) = -0.042\sin(3\phi) - 0.025\cos(3\phi),$$

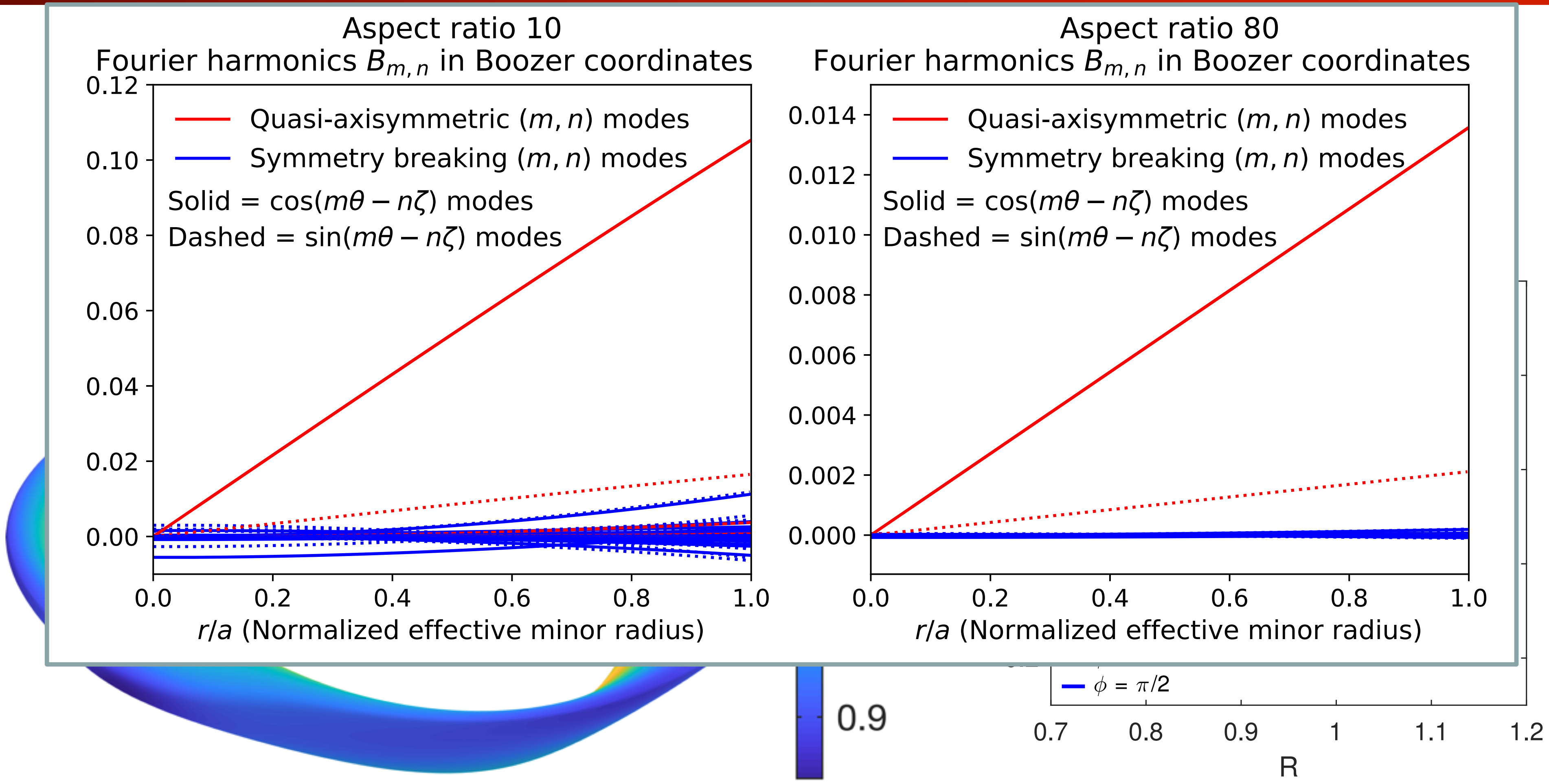
$$\sigma(0) = -0.6,$$

Results: $(R/a = 10)$

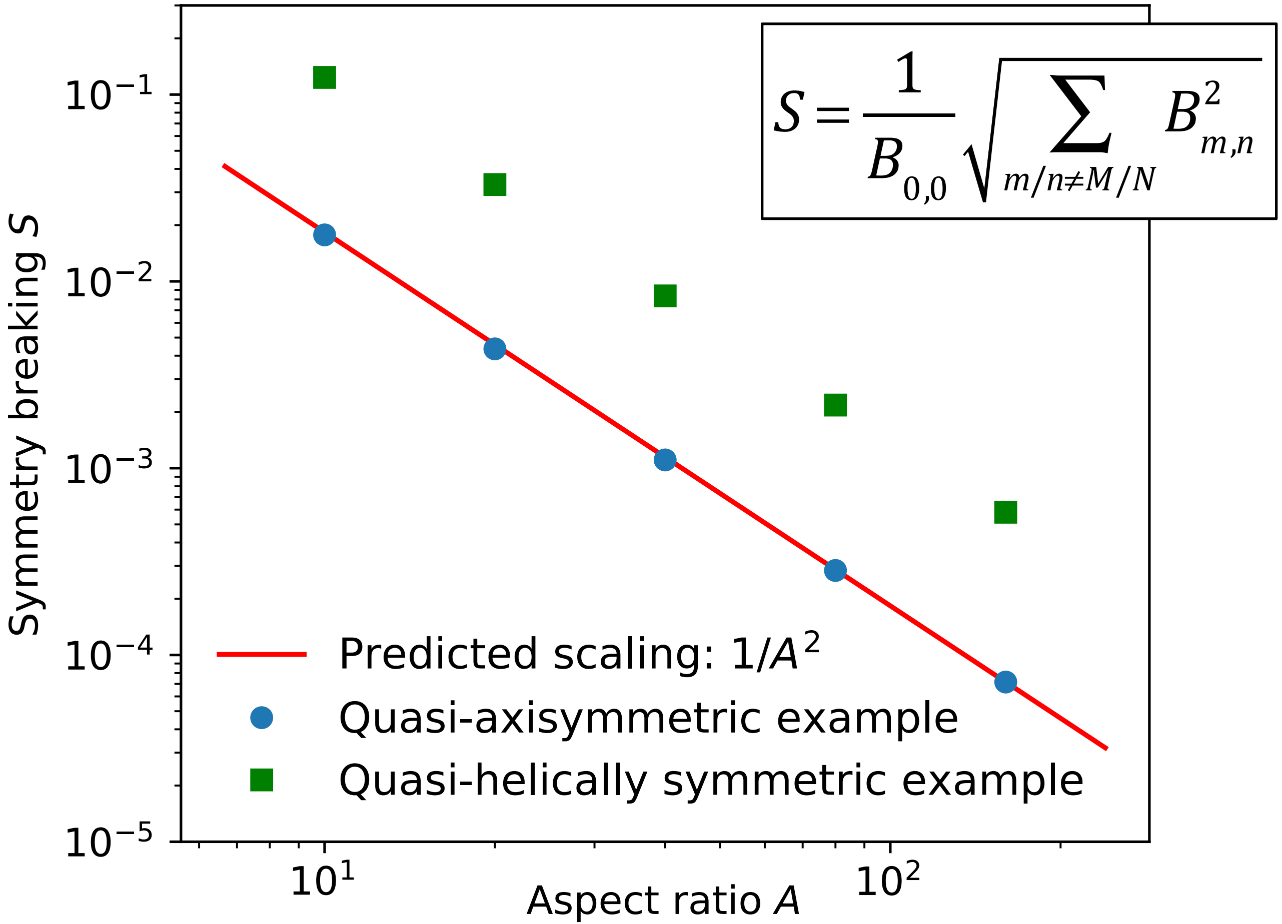




You can make a quasi-axisymmetric stellarator without stellarator symmetry



The symmetry-breaking Fourier amplitudes scale as predicted.



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  - $O(r^2)$ quasisymmetry

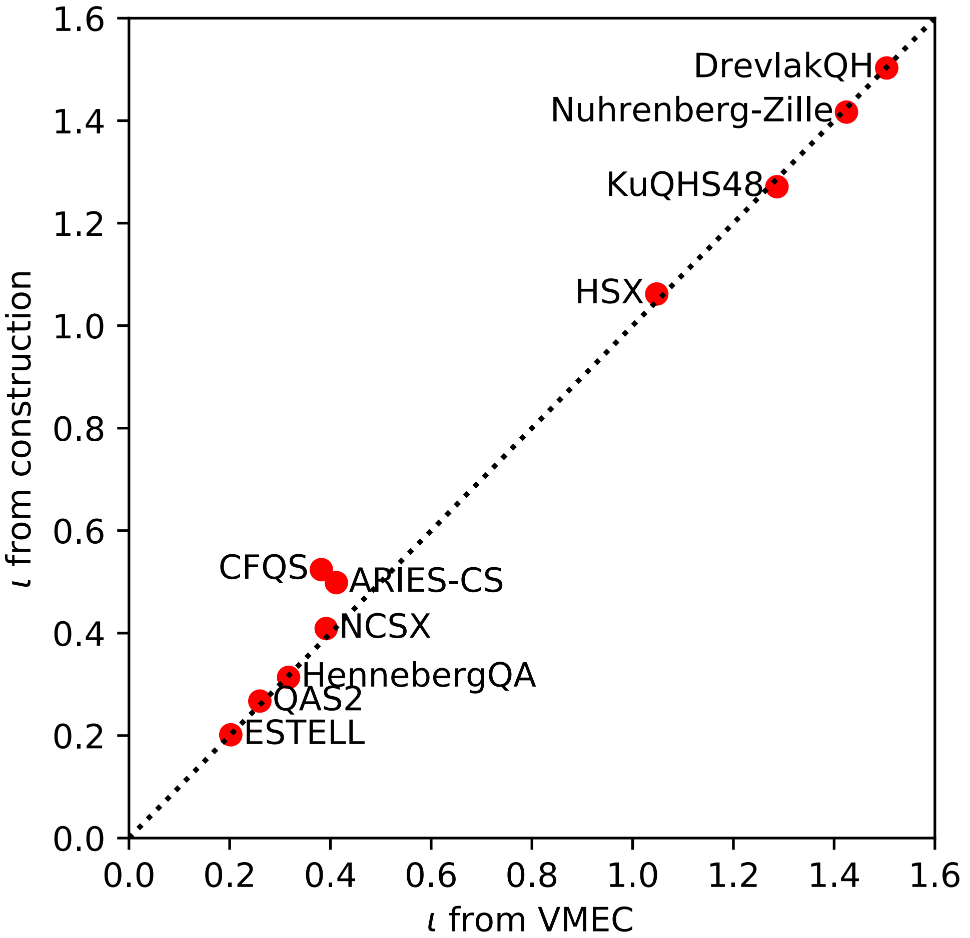
The construction can be fit to quasisymmetric stellarators designed by optimization

- Adopt the same axis shape.

- Fit $\bar{\eta}$ to minimize difference in the shapes of a near-axis surface.

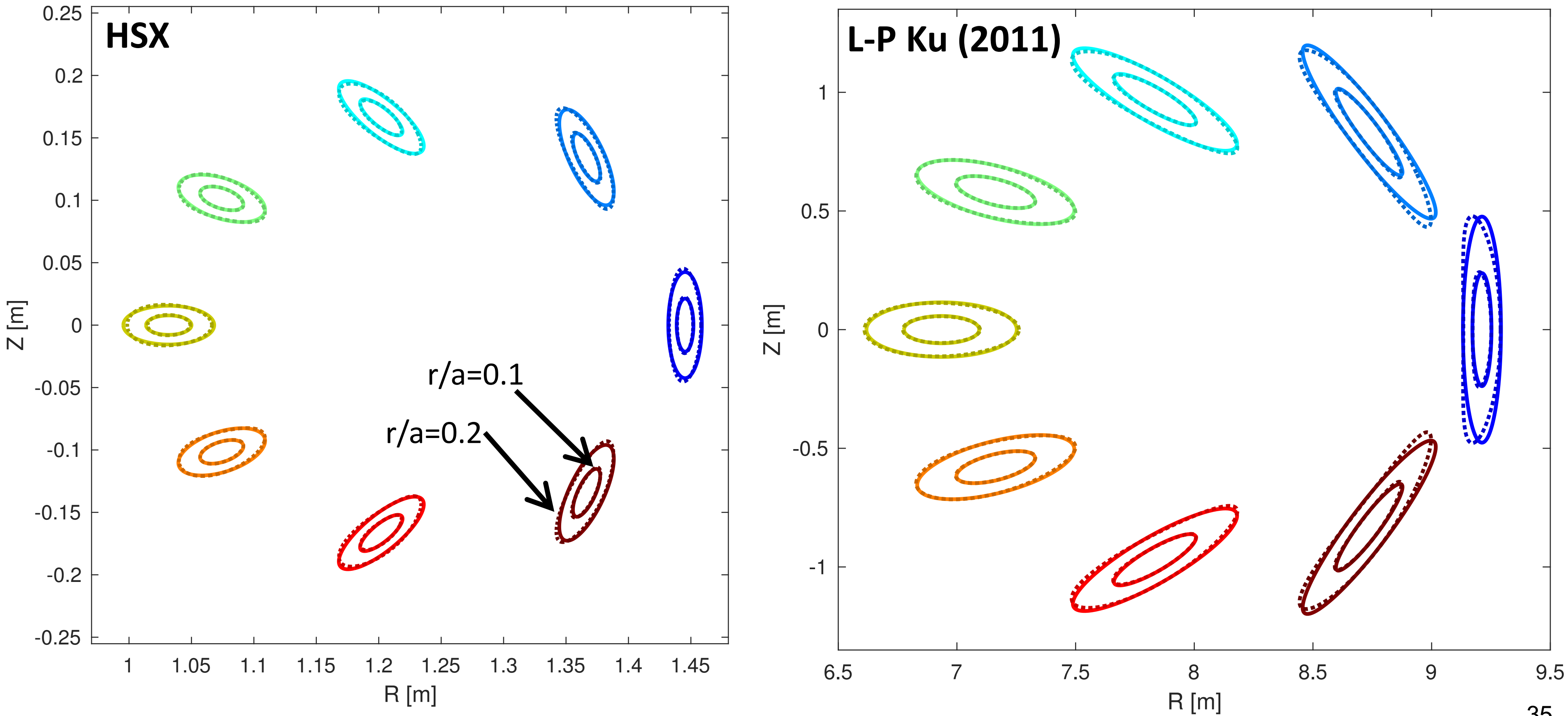
The direct construction gives an accurate match to the on-axis rotational transform in quasisymmetric stellarators designed by optimization

- Adopt the same axis shape.
- Fit $\overline{\eta}$ to minimize difference in the shapes of a near-axis surface.



Quasi-helically symmetric configurations

Dotted: VMEC equilibrium
Solid: Garren-Boozer construction



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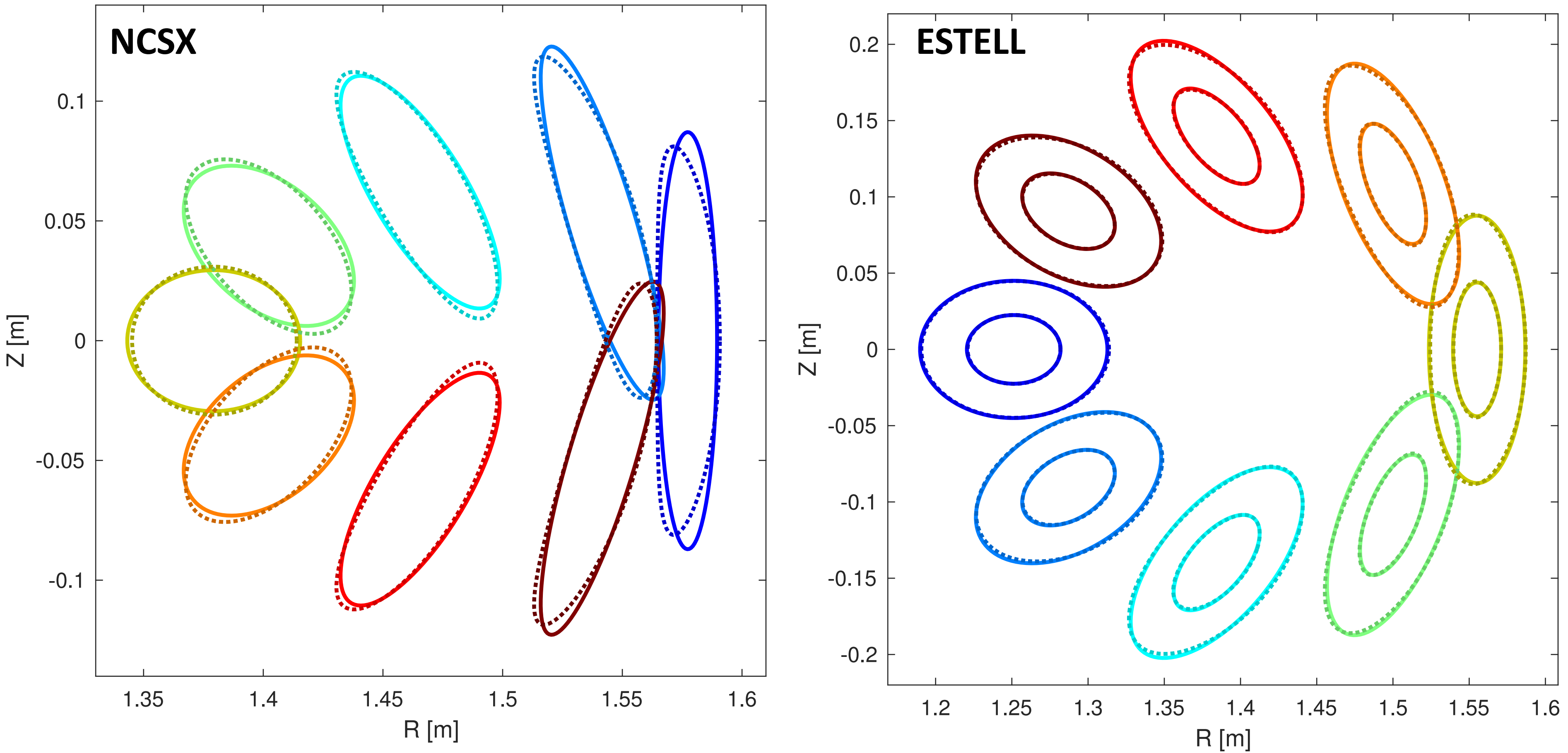

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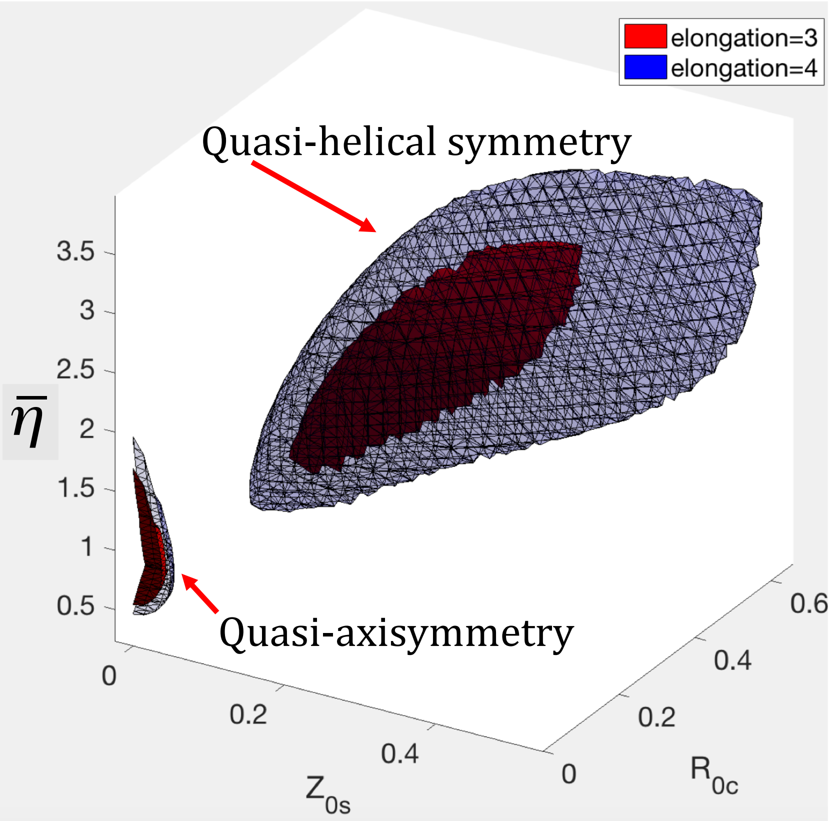
The construction enables fast scans over parameter space.

E.g. Scan over $\left(R_{0c}, Z_{0s}, \bar{\eta}\right)$ where magnetic axis shape is

$$R_0\left(\phi\right) = 1 + R_{0c}\cos\left(4\phi\right)$$

$$Z_0\left(\phi\right) = Z_{0s}\sin\left(4\phi\right)$$

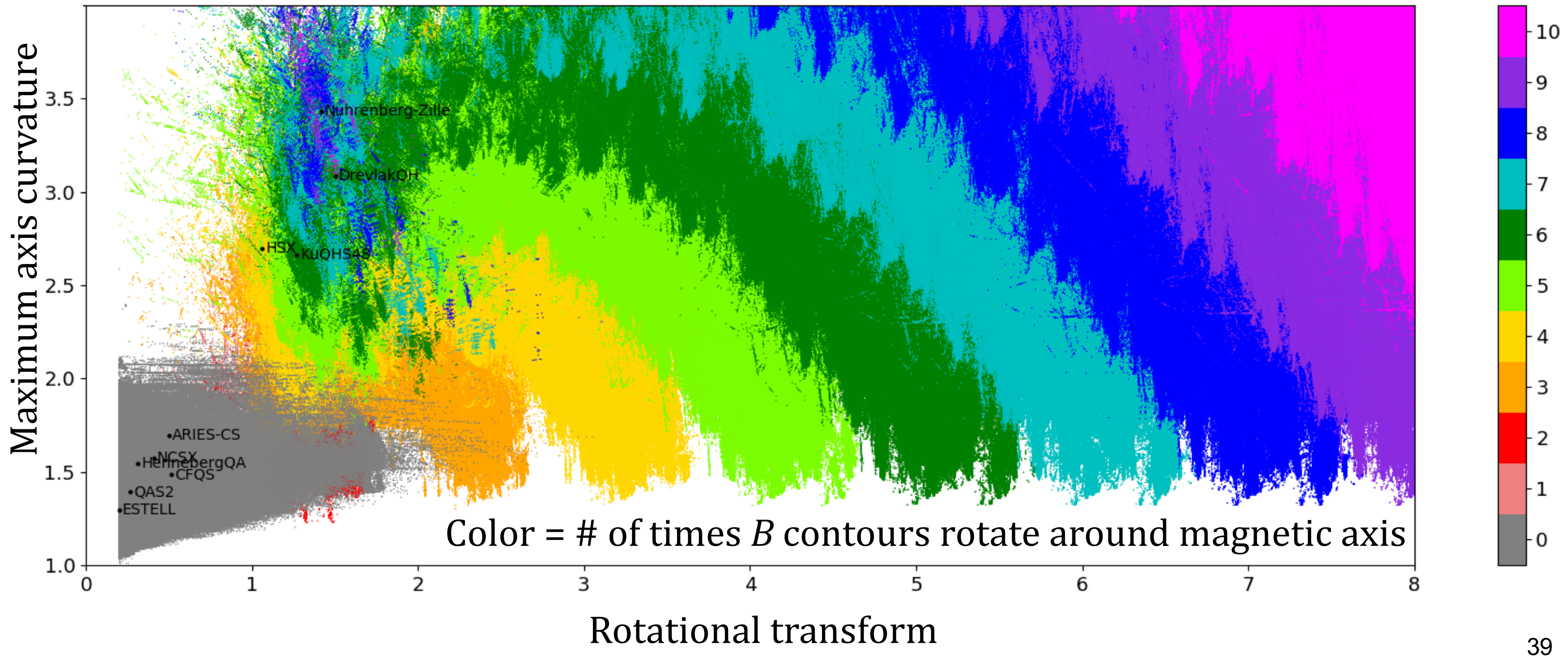
274,560 solutions generated in <30s on a laptop.



The fast construction enables brute-force surveys of "all" quasisymmetric fields

Axis shape: $R_0(\phi) = 1 + \sum_{j=1}^{3} R_j \cos(j n_{fp} \phi), \quad Z_0(\phi) = 1 + \sum_{j=1}^{3} Z_j \sin(j n_{fp} \phi)$

2.4×10^8 configurations



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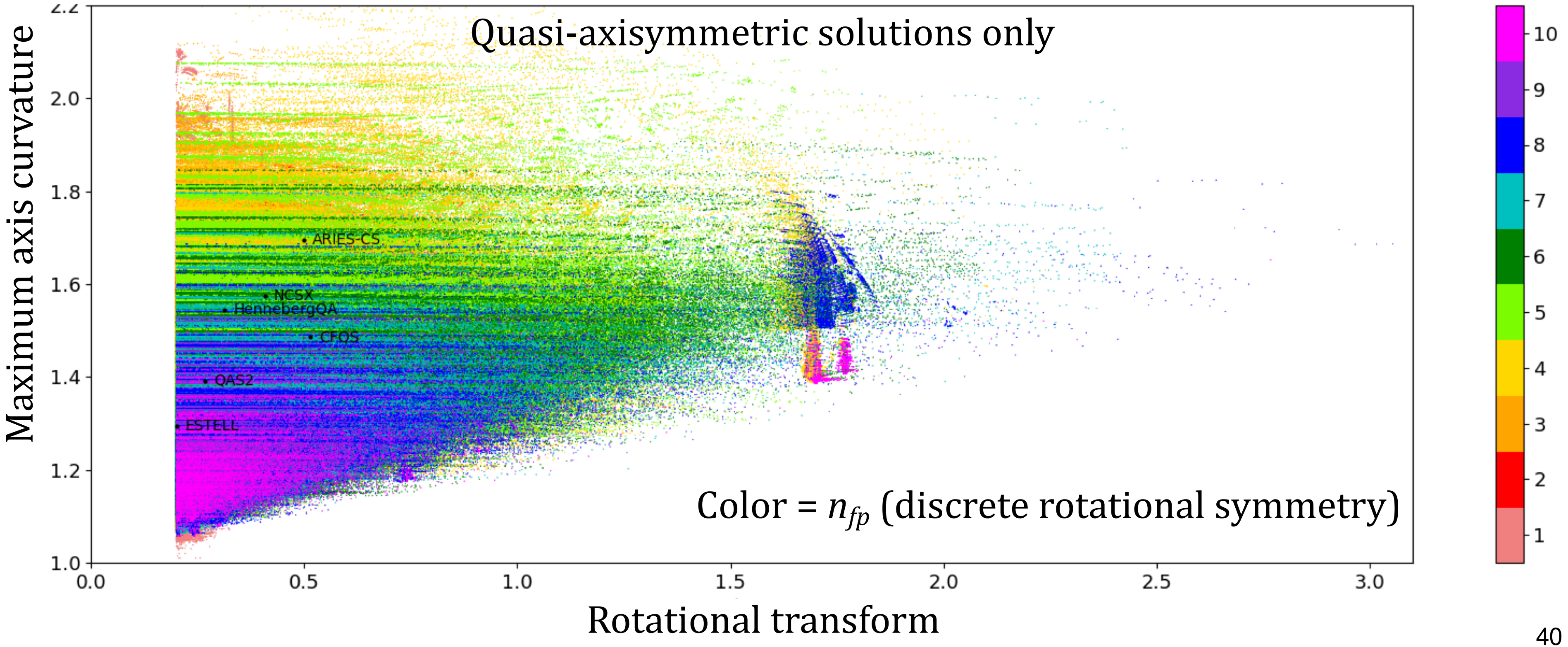
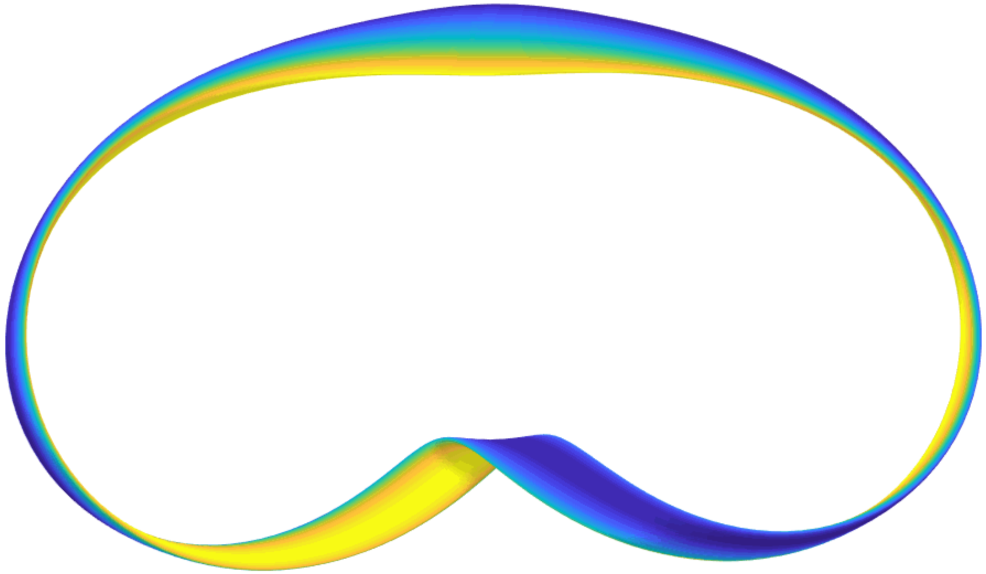
$4\text{x}10^6$ configurations



Brute-force searching is already yielding some new configurations

Quasi-helical symmetry with

1 field period

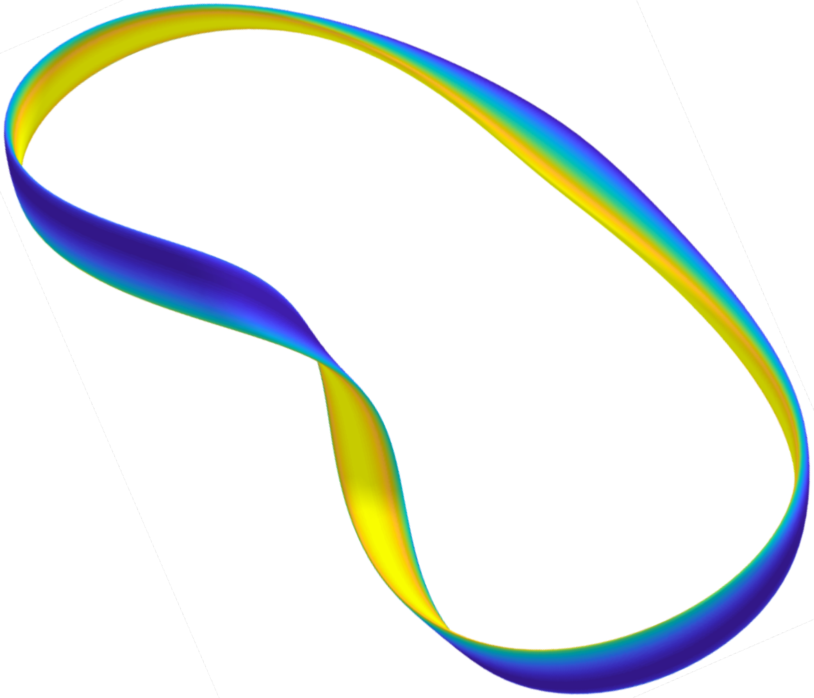


2 field periods

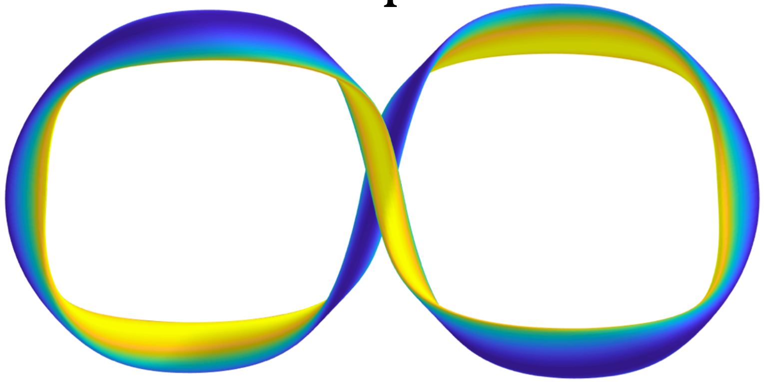
Brute-force searching is already yielding some new configurations

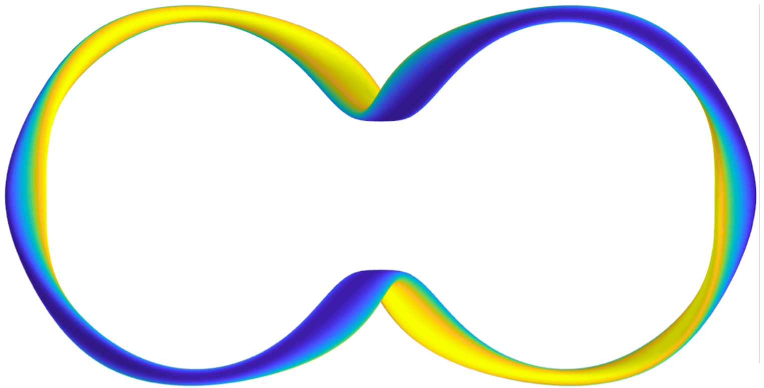
Quasi-helical symmetry with

1 field period



2 field periods





Outline

- Construction for $O(r)$ quasisymmetry
  - Theory, & the number of solutions
  - Numerical results
  - Comparison to “real experiments”
  - The landscape of solutions
- Extensions
  - Omnigenity
  - $O(r^2)$ quasisymmetry

Omnigenity is a weaker confinement condition than quasisymmetry.

Definition of omnigenity: The radial drift has a time average of 0 for all particles.

$$\oint \left(\mathbf{v}_d \cdot \nabla r\right) dt = 0 \quad \forall \text{ magnetic moments \& energies.}$$

- J Cary & S Shasharina, *Physics of Plasmas* **4**, 3323 (1997).
- J Cary & S Shasharina, *Physical Review Letters* **78**, 674 (1997).
- P Helander & J Nührenberg, *Plasma Physics and Controlled Fusion* 51, 055004 (2009).
- M Landreman & P J Catto, *Physics of Plasmas* **19**, 056103 (2012).

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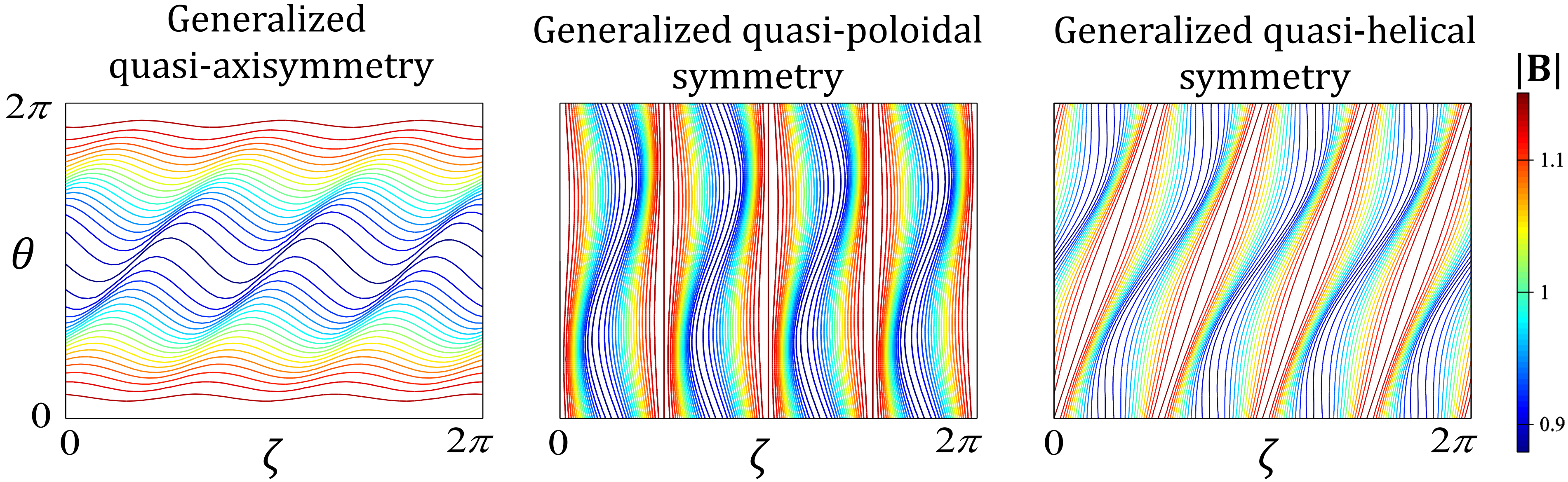
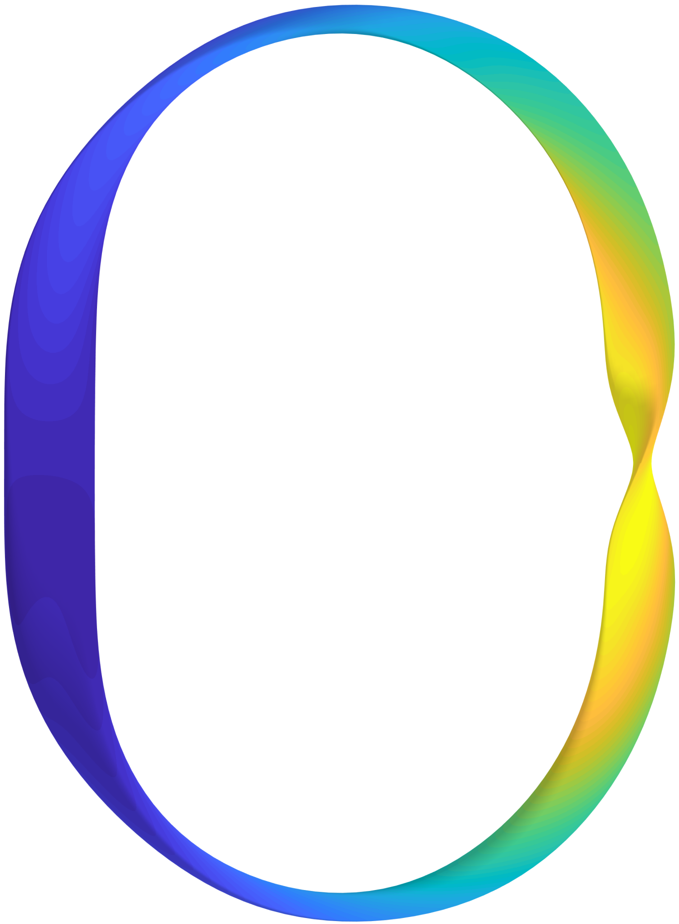
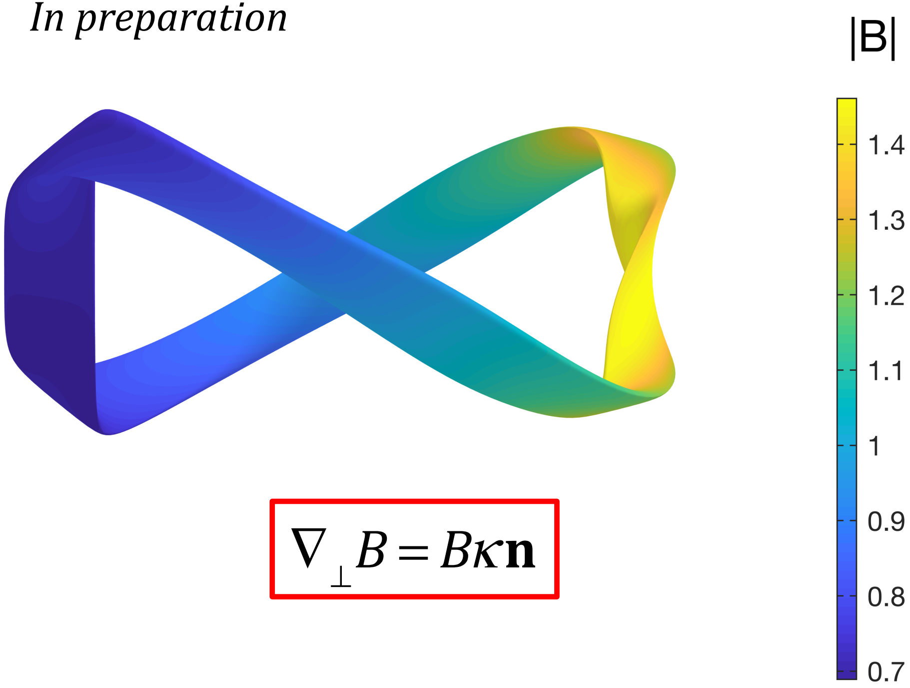


The near-axis analysis can be generalized to construct omnigenous configurations



G G Plunk, ML, and P Helander,
In preparation



$$\boxed{\nabla_{\perp} B = B\kappa \mathbf{n}}$$

Quasi-poloidal symmetry is not possible near the axis, but omnigenity is.

Outline

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Extending the construction to higher order is tricky

- We can only "half-specify" the axis shape:
  - A curve like the axis is given by 2 functions, e.g. {curvature, torsion} or $\{R(\boldsymbol{\phi}), Z(\boldsymbol{\phi})\}$.
  - At $O(r)$, (# unknowns)-(# equations)=2 so we can specify (almost) any axis. But at $\mathrm{O}(r^2)$, (# unknowns)-(# equations)=1 so we cannot.

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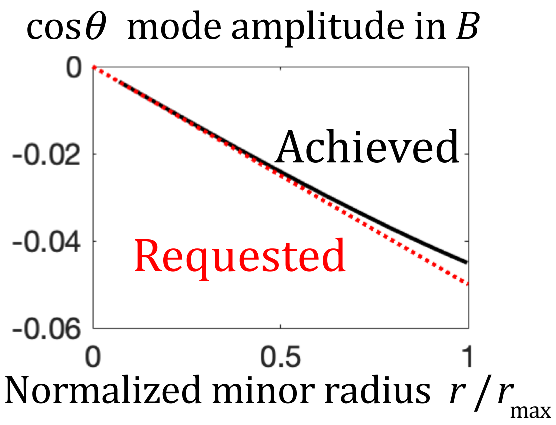
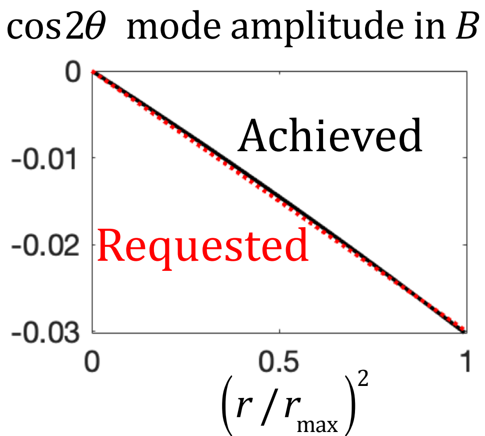
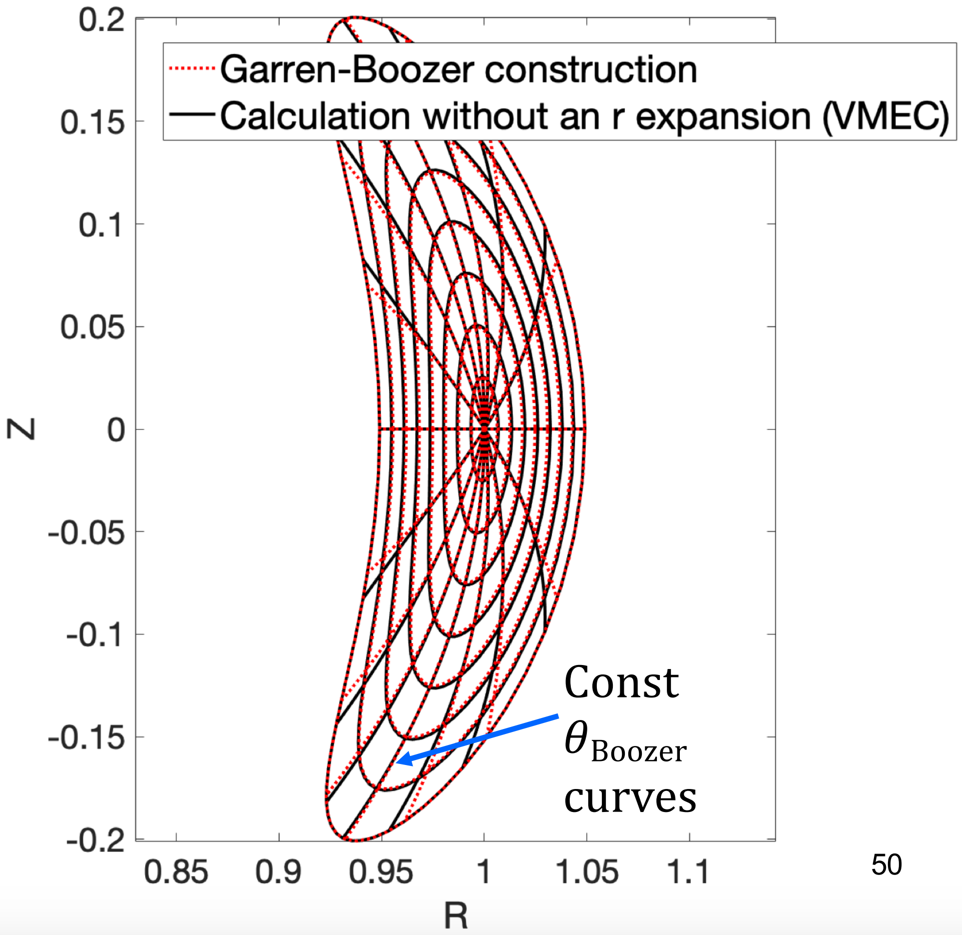
  - A curve like the axis is given by 2 functions, e.g. {curvature, torsion} or $\{R(\boldsymbol{\phi}), Z(\boldsymbol{\phi})\}$.
  - At $O(r)$, (# unknowns)-(# equations)=2 so we can specify (almost) any axis. But at $\mathrm{O}(r^2)$, (# unknowns)-(# equations)=1 so we cannot.
- No existence & uniqueness theorem for solutions (yet).
- Magnetic shear (variation of rotational transform) does not appear until $O(r^3)$.

We are working to extend the construction to O(r^2), enabling greater shaping

Axisymmetric example:







Questions – your input is welcome!

- How can coils be connected to this model?
  - Since we know **B** and **∇B** along the axis, can we say anything about how close coils must be?
- Are there other ways to extrapolate outward from the axis? E.g. Laplace's equation as initial value problem in r, with regularization.
- It is effective to optimize in the space of axis shapes? (H Mynick).
- For $O(r^2)$, how do you best half-specify the axis shape? Can anything be proved about the number or character of $O(r^2)$ solutions?
- Is there an analogous construction to give quasisymmetry at an off-axis surface?

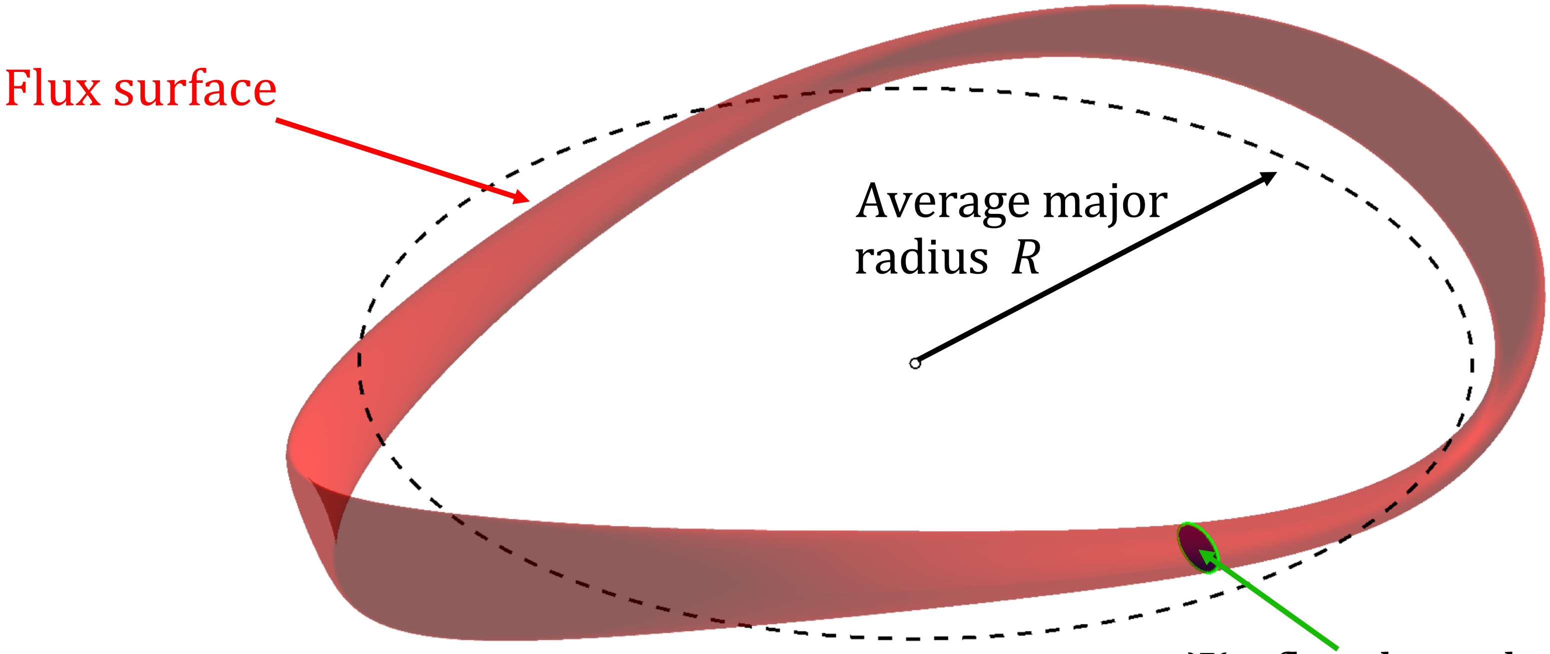
Conclusions

- The equations for quasisymmetric magnetic fields can be solved directly (without optimization) if you expand about the magnetic axis.
- The resulting construction can be useful for generating new initial conditions for optimization.
- We precisely understand the size of the space of magnetic fields that are quasisymmetric near the axis (i.e. to $O(r)$).
- The construction is consistent with configurations obtained by optimization.
- There is hope of definitively identifying all classes of practical quasisymmetric fields (near the axis).

J Plasma Phys (2019)   arXiv:1809.10246   https://github.com/landreman/quasisymmetry

Extra slides

We will expand in the skinniness of the inner flux surfaces



$\Psi =$ flux through patch $= \int \mathbf{B} \cdot d\mathbf{a}$

Define effective radius r by $\Psi = \pi r^2 B_{axis}$.

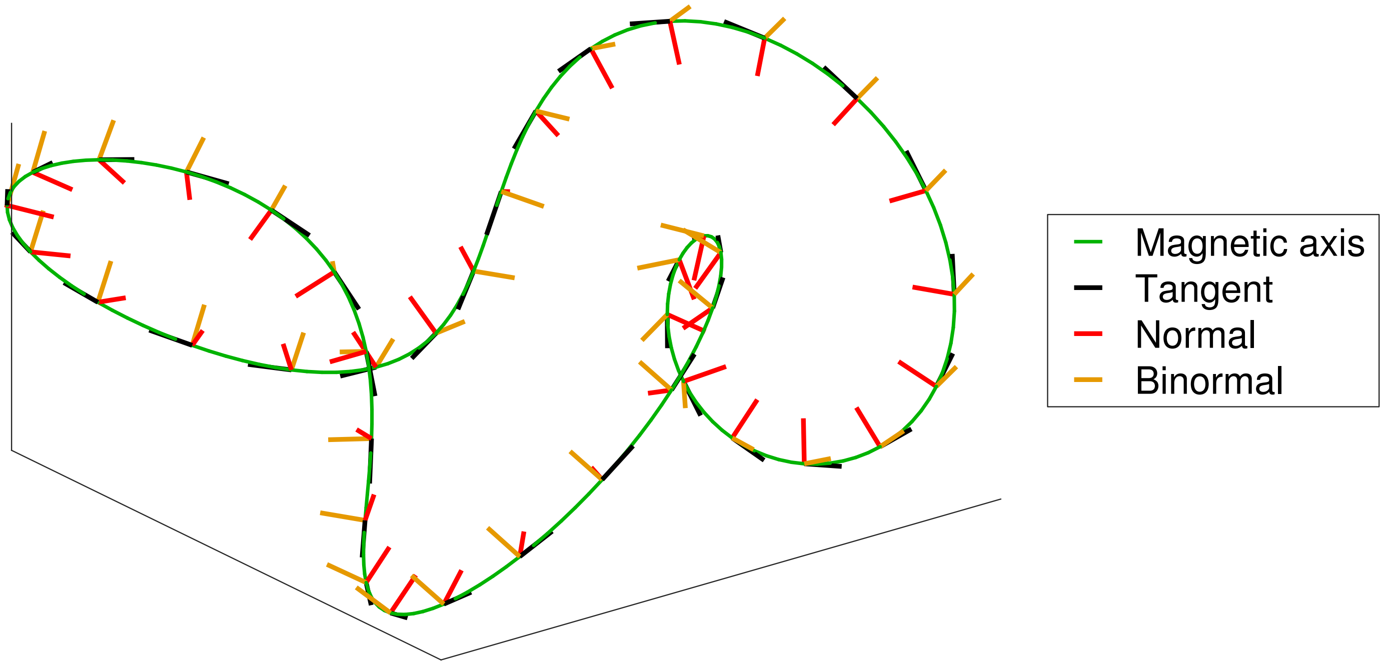
"Aspect ratio" $= \dfrac{R}{r}$ is $\gg 1$

Theory: Expand position vector using Frenet frame, equate 2 representations of B.

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}):\quad \frac{\partial \mathbf{r}_0}{\partial \ell} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{d\ell} = \kappa \mathbf{n}, \qquad \frac{d\mathbf{n}}{d\ell} = -\kappa \mathbf{t} + \tau \mathbf{b}, \qquad \frac{d\mathbf{b}}{d\ell} = -\tau \mathbf{n}$$

$$\mathbf{r}_0 = \text{magnetic axis}, \qquad \kappa = \text{curvature}, \qquad \tau = \text{torsion}$$

$$\mathbf{t} = \text{tangent}, \qquad \mathbf{n} = \text{normal}, \qquad \mathbf{b} = \text{binormal}$$

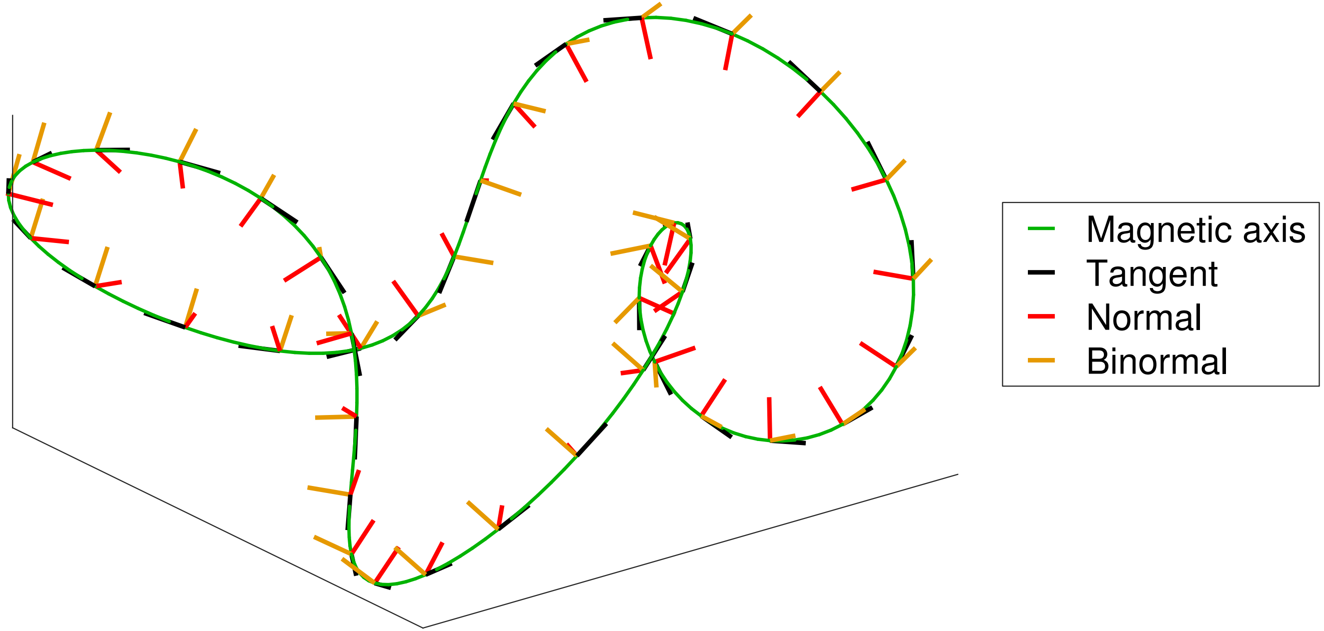


Theory: Write position vector using Frenet frame

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}):\quad \frac{d\mathbf{r}_0}{d\ell}=\mathbf{t},\qquad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n},\qquad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b},\qquad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

$\mathbf{r}_0 =$ magnetic axis, $\kappa =$ curvature, $\tau =$ torsion, $\mathbf{t} =$ tangent, $\mathbf{n} =$ normal, $\mathbf{b} =$ binormal

$$\mathbf{r}(r,\theta,\zeta)=\mathbf{r}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta)$$



Theory: Write position vector using Frenet frame, expand in small r = (flux)$^{1/2}$

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$$\mathbf{r}_0=\text{magnetic axis},\quad \kappa=\text{curvature},\quad \tau=\text{torsion},\quad \mathbf{t}=\text{tangent},\quad \mathbf{n}=\text{normal},\quad \mathbf{b}=\text{binormal}$$

$$\begin{aligned}\mathbf{r}(r,\theta,\zeta)&=\mathbf{r}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta)\\&=\mathbf{r}_0(\zeta)+rX_{1c}(\zeta)\cos\theta\,\mathbf{n}(\zeta)+r\left[Y_{1s}(\zeta)\sin\theta+Y_{1c}(\zeta)\cos\theta\right]\mathbf{b}(\zeta)+O(r^2)\end{aligned}$$

Using magnetohydrodynamic equilibrium $(\mathbf{J}\times\mathbf{B}=\nabla p)$

Theory: Write position vector using Frenet frame, expand in small r = (flux)$^{1/2}$

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$$X_{1c}(\zeta)=\frac{\bar{\eta}}{\kappa(\zeta)},\qquad Y_{1s}(\zeta)=\frac{\kappa(\zeta)}{\bar{\eta}},\qquad Y_{1c}(\zeta)=\frac{\sigma(\zeta)\kappa(\zeta)}{\bar{\eta}}$$

$$\text{Toroidal angle } \zeta\propto\text{arclength},\qquad \bar{\eta}=\text{constant:}\ \ B=B_0\left[1+r\bar{\eta}\cos(\theta-N\varphi)+O(r^2)\right]$$

$$\boxed{\frac{d\sigma}{d\zeta}+\iota\left[\frac{\bar{\eta}^4}{\kappa^4}+1+\sigma^2\right]-2\frac{\bar{\eta}^2}{\kappa^2}\left[I_2-\tau\right]=0}$$

$$I_2=\text{current density}$$

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$$X(r,\theta,\zeta) = r\left[X_{1s}(\zeta)\sin\theta + X_{1c}(\zeta)\cos\theta\right] + O(r^2). \quad \text{Same for } Y, Z.$$

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$$\mathbf{r}\left(r,\theta,\zeta\right) = \mathbf{r}_0\left(\zeta\right) + X\left(r,\theta,\zeta\right)\mathbf{n}\left(\zeta\right) + Y\left(r,\theta,\zeta\right)\mathbf{b}\left(\zeta\right) + Z\left(r,\theta,\zeta\right)\mathbf{t}\left(\zeta\right)$$

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$$\mathbf{B} = B_r \nabla r + B_\theta \nabla\theta + B_\zeta \nabla\zeta, \qquad \mathbf{B} = \nabla\psi \times \nabla\theta + \iota \nabla\zeta \times \nabla\psi$$

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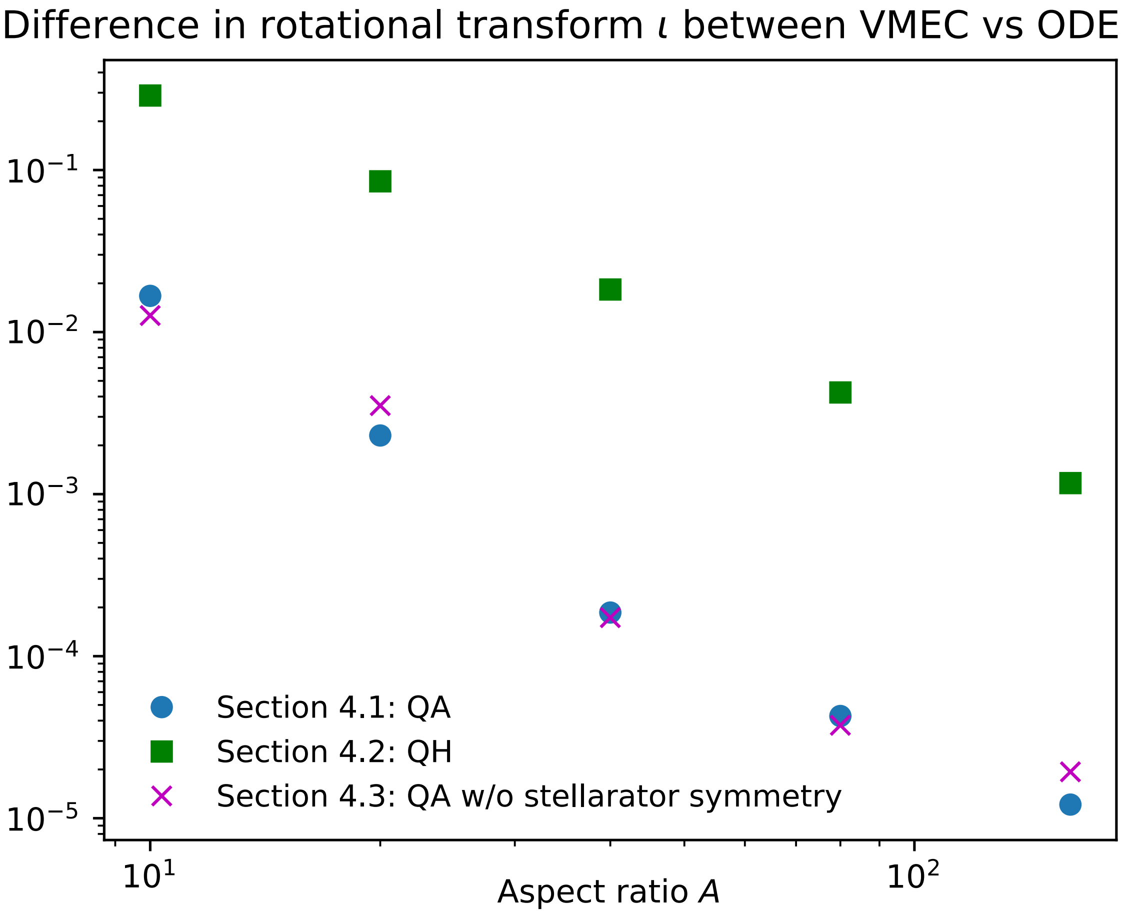
$$\text{Dual relations:} \quad \nabla r = \left[\frac{\partial \mathbf{r}}{\partial r} \cdot \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \zeta}\right]^{-1} \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \zeta}, \quad \text{cyclic permutations.}$$

The rotational transform computed by VMEC converges to the value computed by the Garren-Boozer approach.



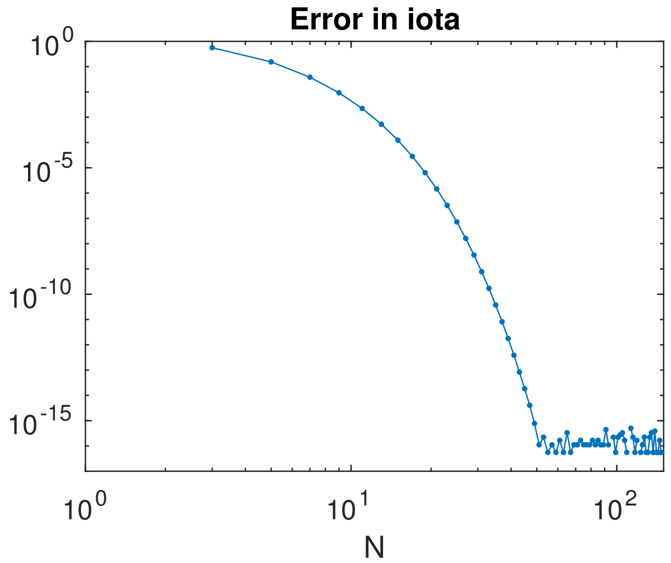
The ODE is solved with spectral accuracy using pseudospectral discretization + Newton iteration

Uniform grid in ϕ with N points: $\phi_1 = 0,\ \phi_2 = 2\pi / \left(N n_{fp}\right),\ \ldots,\ \phi_N = 2\pi\left(N-1\right)/\left(N n_{fp}\right)$.

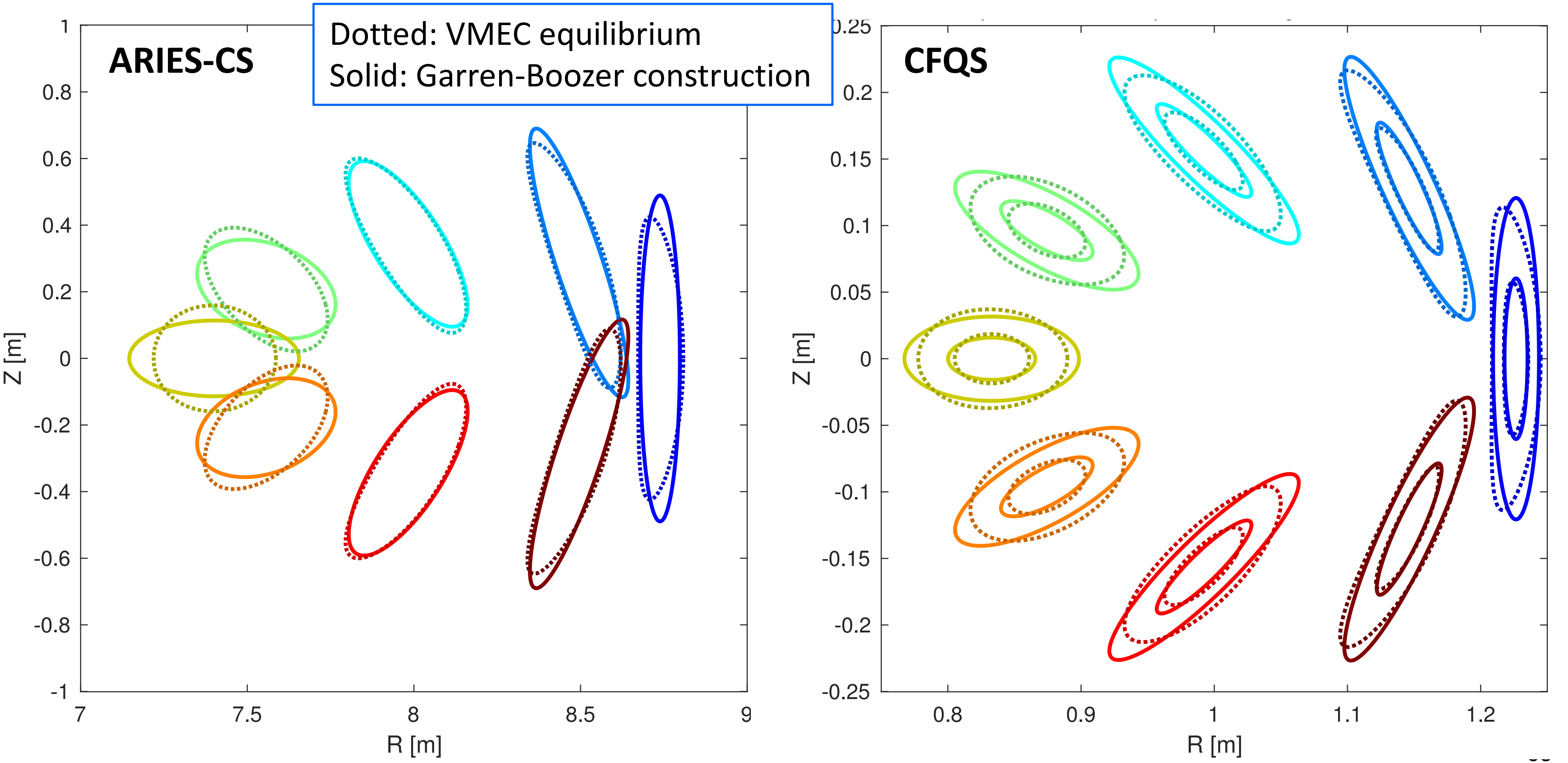
Vector of N unknowns: $\left(\iota,\ \sigma\left(\phi_2\right),\ \sigma\left(\phi_3\right),\ \ldots,\ \sigma\left(\phi_N\right)\right)^T$

N equations: impose ODE at $\phi_1,\ \ldots,\ \phi_N$.

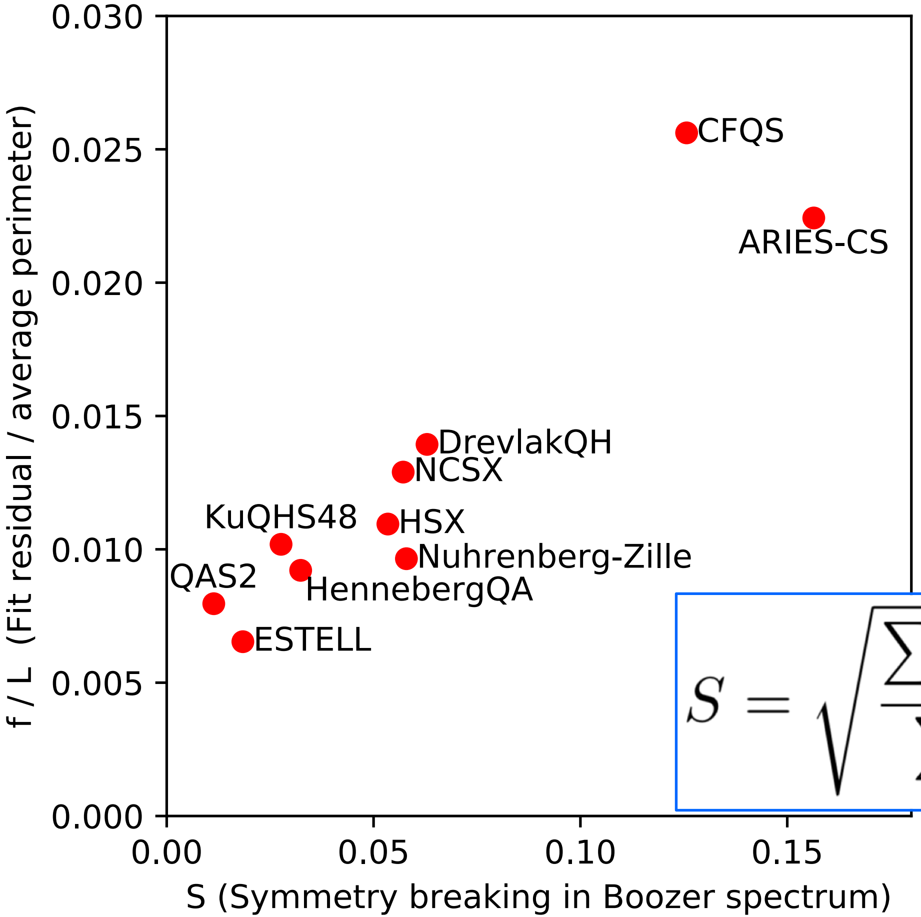
$\frac{d\sigma}{d\phi} \rightarrow D\sigma$ where D is a pseudospectral differentiation matrix.



Of 10 configurations examined, the fit is less good for 2



The configurations with relatively poor fits can be explained by their larger symmetry-breaking



$$S = \sqrt{\frac{\sum_{m\neq 0, n\neq mN} B_{m,n}^2}{\sum_{m,n=mN} B_{m,n}^2}}$$