Omnigenity: A generalization of quasisymmetry

Perhaps quasisymmetry is unnecessary?

Review of particle trajectories in magnetic fields

Magnetic field line
Charged particle trajectory
Guiding center drift $v_d$

Flux surface

Particles with small $|v_\parallel|/v$ are “trapped” in regions $B < B_{\text{max}}$.

Velocity along $\mathbf{B}$ slows, then reverses at “bounce points.”
Omnigenity is a condition of good confinement: the bounce-averaged radial drift vanishes for all velocities.

\[ \int (v_d \cdot \nabla \psi) \, dt = 0 \]

\( \forall \) magnetic moments & energies.

\( \psi \) = Flux surface label

Integral along a field line between bounce points, i.e. along the leading-order motion, neglecting \( v_d \).

\[ v_d \cdot \nabla \psi = \frac{m v^2}{q B^3} \left( 1 - \frac{\mu}{v^2} B \right) B \times \nabla B \cdot \nabla \psi \]
Omnigenity is a condition of good confinement: the bounce-averaged radial drift vanishes for all velocities.

Definition of omnigenity:
\[ \oint (v_d \cdot \nabla \psi) \, dt = 0 \]
\[ \forall \text{ magnetic moments \& energies.} \]

\( \psi = \) Flux surface label

Integral along a field line between bounce points, i.e. along the leading-order motion, neglecting \( v_d \).

\[ v_d \cdot \nabla \psi = \frac{mv^2}{qB^3} \left( 1 - \frac{\mu}{v^2} B \right) \mathbf{B} \times \nabla B \cdot \nabla \psi \]

Equivalent condition: For each magnetic moment, energy, and flux surface, the quantity \( J \) is uniform on the flux surface:

\[ J = \oint v_{||} \, d\ell, \quad \ell = \text{distance along field line} \]
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\[ \oint (v_d \cdot \nabla \psi) \, dt = 0 \]

for all magnetic moments and energies.  \[ \psi \] = Flux surface label

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Equivalent condition: For each magnetic moment, energy, and flux surface, the quantity \( J \) is uniform on the flux surface:

\[ J = \oint \vec{v}_\parallel \, d\ell, \quad \ell = \text{distance along field line} \]

Quasisymmetry is sufficient but not necessary!
Omnigenity places strong constraints on $B$.

If $\int (v_d \cdot \nabla \psi) dt = 0$, then $B$ contours can never be $\parallel$ to $B$.

**Contours of $B$ on a flux surface:**

Deeply trapped particles at $T$ would see a nonzero $v_d \cdot \nabla \psi \propto B \times \nabla B \cdot \nabla \psi$.
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**Contours of $B$ on a flux surface:**

Deeply trapped particles at $T$ would see a nonzero $v_d \cdot \nabla \psi \propto B \times \nabla B \cdot \nabla \psi$.

Similar argument for maxima of $B$.

$\Rightarrow$ All $B$ contours must link the torus toroidally, poloidally, or both.
Omnigenity places strong constraints on $B$.

The width of a “well” in $B$ must be the same everywhere on a flux surface. (Both in real distance and in Boozer coordinates.)

$\Rightarrow$ The $B_{\text{max}}$ contour must be straight in Boozer coordinates.
Omnigenous fields come in toroidal/poloidal/helical variants, just like quasisymmetric fields.
Omnigenity does not imply quasisymmetry in practice.

Cary & Shasharina (1997):

• If $B(\theta,\zeta)$ is omnigenous but not quasisymmetric, $B$ will be non-analytic at the “seam” along the $B_{\text{max}}$ contour.
• Thus, omnigenity + analyticity seems to imply quasisymmetry.
• But, there are analytic $B(\theta,\zeta)$ patterns that are nearly omnigenous but not close to quasisymmetry.

Omnigenous, non-analytic:

Almost omnigenous, analytic:

Fourier truncation
Omnigenity does not imply quasisymmetry in practice.

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• Thus, omnigenity + analyticity seems to imply quasisymmetry.
• But, there are analytic $B(\theta, \zeta)$ patterns that are nearly omnigenous but not close to quasisymmetry.

There are no experiments far from quasisymmetry that approach these omnigenity conditions (especially for $B_{\text{max}}$).

• Quasisymmetry is more restrictive than omnigenity, and is not obviously better for confinement.
Omnigenity is easier, so why might we want the more restrictive condition of quasisymmetry?

• Larger flows are permitted in quasisymmetry. These flows may improve stability and/or reduce turbulent transport?

• In quasisymmetric plasmas, the radial electric field and plasma flow are determined by turbulence, whereas they are determined by collisional processes in an omnigenous stellarator. Perhaps the former might turn out to be preferable?

• An outward collisional flux of impurities (“temperature screening”) is predicted for quasisymmetry.
Extra slides
Omnigenity can also be understood in terms of $B$ variation along field lines.

Tokamak or quasisymmetric stellarator:
$$\Delta \psi = 0$$

Omnigenous stellarator:
$$\Delta \psi = 0$$

Non-optimized stellarator:
$$\Delta \psi \neq 0$$
The quasisymmetry helicity \((M, N)\) can be generalized to omnigenity.

- Recall: all \(B\) contours encircle the torus poloidally, toroidally, or both.

- Define \(M\) and \(N\): contours of \(B\) close after linking the torus \(M\) times toroidally and \(N\) times poloidally.

**New geometric consequence of omnigenity:**

Apply Ampère’s Law to a \(B\) contour on a flux surface:

\[
\int \mathbf{B} \cdot d\mathbf{r} = \frac{4\pi}{c} \times (\text{enclosed current}) \frac{MG + NI}{MG + NI}
\]

\[
\Rightarrow \quad \frac{\mathbf{B} \times \nabla \psi \cdot \nabla \mathbf{B}}{\mathbf{B} \cdot \nabla \mathbf{B}} = \frac{2q}{c} \left( \frac{MG + NI + H}{M - qN} \right) \quad \text{where } \langle H \rangle = 0.
\]
Non-quasisymmetric stellarators:
• Neoclassical radial current depends on $E_r$.
• $\langle j_{\text{neoclassical}} \cdot \nabla \psi \rangle \gg \langle j_{\text{turbulence}} \cdot \nabla \psi \rangle$.
  

$\Rightarrow$ You can solve for $E_r$ using $\langle j_{\text{neoclassical}} \cdot \nabla \psi \rangle = 0$.

Tokamaks & quasisymmetric stellarators:
• Neoclassical radial fluxes of ions and electrons are always equal, regardless of $E_r$ (“intrinsic ambipolarity”)
  
  (Helander & Simakov, PRL 2008)

$\Rightarrow$ You cannot solve for $E_r$. 
$E_r$ is determined by ambipolarity.

Non-quasisymmetric stellarators:
• Neoclassical radial current depends on $E_r$.
• $\langle \mathbf{j}_{\text{neoclassical}} \cdot \nabla \psi \rangle \gg \langle \mathbf{j}_{\text{turbulence}} \cdot \nabla \psi \rangle$.


$\Rightarrow$ You can solve for $E_r$ using $\langle \mathbf{j}_{\text{neoclassical}} \cdot \nabla \psi \rangle = 0$.

Omnigenous stellarators:

$$\langle \mathbf{j} \cdot \nabla \psi \rangle = \left( \text{Zen}_i \frac{d\Phi}{d\psi} + T_i \frac{dn_i}{d\psi} - 0.17n_i \frac{dT_i}{d\psi} \right) \left( \text{departure from quasisymmetry} \right)^2$$

Universal result:

$$\frac{d\Phi}{d\psi} = \frac{1}{Ze} \left( -\frac{T_i}{n_i} \frac{dn_i}{d\psi} + 0.17 \frac{dT_i}{d\psi} \right)$$

Totally independent of the details of $\mathbf{B}$. 
**Single-particle confinement**

*Tokamak*

\[ B \times \nabla B \]

**Distance along field line**

- **Passing particle**
- **Trapped particle**

**Trapped particles**

**Flux surface**

*Tamm’s theorem: “In the absence of turbulence and collisions, all particles in a tokamak are confined.”*

*Non-optimized stellarator*

\[ B \times \nabla B \]

**Distance along field line**

- **Passing particle**
- **Trapped particles**

**Flux surface**
**Definition:** A field is “omnigenous” if the radial guiding-center drift averages over a bounce to zero for all trapped particles:

\[
\Delta \psi \text{ per bounce} = \int_{\text{bounce}} (v_d \cdot \nabla \psi) \, dt = 0.
\]

Many properties can be proven about omnigenous fields. E.g., all \(B\) contours link the torus toroidally, poloidally, or helically.
Omnigenity is more general than quasisymmetry.

Omnigenity: $B$ contours may be curved

Quasisymmetry: $B = B(M\theta - N\zeta)$

Quasisymmetry is more restrictive than omnigenity and does no more to reduce $\alpha$-particle loss or neoclassical transport.
Omnigenity places strong constraints on $B$.

If $\oint (v_d \cdot \nabla \psi) \, dt = 0$, then $B$ contours can never be $\parallel$ to $B$.

Contours of $B$ on a flux surface:

$J = \oint v_\parallel \, d\ell$ has different values above vs below $P$.

Also, for particles that bounce at $P$, there is a nonzero radial $v_d$ there, and bounce time diverges, so the radial excursion is very large.
Remaining slides:

Patterns of field strength, from
Beidler et al, Nuclear Fusion 51 (2011) 076001
LHD standard configuration

LHD inward-shifted

\( B/B_0 \)

(a) 

(b)