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Derivation of the drift-kinetic equation

Introduction

The "drift-kinetic equation" is the basis for all calculations of neoclassical transport and flows, as well as the bootstrap current. There are several variants of the equation; one standard form is

$$\upsilon_{\|}\mathbf{b}\cdot\nabla\bar{f}_1 + \mathbf{v}_d\cdot\nabla f_0 - \frac{Ze}{T}E_{\|}\upsilon_{\|}f_0 = C\left\{\bar{f}_1\right\} \tag{1}$$

where f_0 is the leading-order Maxwellian distribution function, $\bar{f}_1$ is the gyroaveraged perturbed distribution function, $\mathbf{b}=\mathbf{B}/B$, $B=\left|\mathbf{B}\right|$,

$$\mathbf{v}_d = \frac{c}{B}\mathbf{E}\times\mathbf{b} + \frac{\upsilon_{\|}^2}{\Omega}\mathbf{b}\times\boldsymbol{\kappa} + \frac{\upsilon_{\perp}^2}{2\Omega B}\mathbf{b}\times\nabla B + \frac{\upsilon_{\perp}^2}{2\Omega}\mathbf{b}\mathbf{b}\cdot\nabla\times\mathbf{b} \tag{2}$$

is the sum of magnetic, $\mathbf{E}\times\mathbf{B}$, and parallel drifts, $\Omega = ZeB/\left(mc\right)$ is the gyrofrequency, and $\boldsymbol{\kappa}=\mathbf{b}\cdot\nabla\mathbf{b}$ is the field curvature. The independent variables (which are held fixed in the gradients in (1)) are the magnetic moment $\mu=\upsilon_{\perp}^2/\left(2B\right)$ and leading-order total energy $W=\upsilon^2/2+Ze\Phi_0$, where Φ_0 is the leading-order electrostatic potential.

Several variations of the equation are possible. Often the parallel drift in (2) is dropped. Sometimes the $E_{\|}$ term in (1) is written $+\left(Ze/m\right)E_{\|}\upsilon_{\|}\partial f_0/\partial W$.

Orderings:

The drift-kinetic equation is derived from the Fokker-Planck equation by expanding in the small parameter $\rho_* = \rho/L = \upsilon_{th}/\left(\Omega L\right)$ where ρ is the thermal gyroradius, and L is the scale length for variation in all quantities: $\mathbf{B}$, f_0, f_1, and Φ. This is in contrast to gyrokinetics, in which f_1 and Φ_1 are permitted to vary on a scale length comparable to ρ. The collision frequency ν is ordered as $\nu \sim \rho_*\Omega$. The electric field is taken to be electrostatic to leading order: $\mathbf{E}=-\nabla\Phi_0+\mathbf{E}_*$ where $\mathbf{E}^* \sim \rho_*\mathbf{E}$, and the leading-order electric field $-\nabla\Phi_0$ is ordered using $\upsilon_{\mathbf{E}\times\mathbf{B}} \sim \rho_*\upsilon_{th}$. Time derivatives are taken to be small: $\partial/\partial t \sim \rho_*^2\Omega$.

Derivation

Begin with the Fokker-Planck equation $Df = C\left\{f\right\}$ where

$$D = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{v}} + \mathbf{v}\cdot\left(\nabla f\right)_{\mathbf{v}} + \frac{Ze}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\upsilon} f. \tag{3}$$

Subscripts on partial derivatives indicate quantities that are held fixed in differentiation.

We introduce cylindrical velocity-space coordinates $\left(\upsilon_{\perp},\ \varphi,\ \upsilon_{\|}\right)$ so that $\mathbf{v}=\upsilon_{\|}\mathbf{b}+\mathbf{v}_{\perp}$ where

$$\mathbf{v}_{\perp} = \upsilon_{\perp}\left(\mathbf{e}_1\cos\varphi + \mathbf{e}_2\sin\varphi\right), \tag{4}$$

$\mathbf{e}_1$ and $\mathbf{e}_2$ are position-dependent unit vectors orthogonal to $\mathbf{B}$, and φ is the gyrophase. The system $\left(\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{b}\right)$ is right handed. A brief calculation gives

$$\nabla_\upsilon Q = \mathbf{b}\left(\frac{\partial Q}{\partial \upsilon_\parallel}\right)_{\upsilon_\perp,\varphi} + \frac{\mathbf{v}_\perp}{\upsilon_\perp}\left(\frac{\partial Q}{\partial \upsilon_\perp}\right)_{\upsilon_\parallel,\varphi} + \frac{1}{\upsilon_\perp^2}\mathbf{b}\times\mathbf{v}_\perp\left(\frac{\partial Q}{\partial \varphi}\right)_{\upsilon_\perp,\upsilon_\parallel} \tag{5}$$

for any quantity Q. We next introduce

$$\mu = \upsilon_\perp^2/(2B) \quad \text{and} \quad W = \upsilon^2/2 + Ze\Phi_0/m \tag{6}$$

where $\upsilon^2 = \upsilon_\perp^2 + \upsilon_\parallel^2$. A bit of algebra gives

$$DW = (Ze/m)\mathbf{E}^* \cdot \mathbf{v} \tag{7}$$

where $\mathbf{E}^* = \mathbf{E} + \nabla\Phi_0$,

$$D\mu = -\frac{\mu}{B}\mathbf{v}\cdot\nabla B - \frac{\upsilon_\parallel \mathbf{vv}:(\nabla\mathbf{b})}{B} + \frac{Ze}{mB}\mathbf{E}\cdot\mathbf{v}_\perp, \tag{8}$$

and

$$D\varphi = -\Omega + G \tag{9}$$

where G is an ugly bunch of terms of order $\rho_*\Omega$ (arising from $(\nabla\varphi)_\mathbf{v}$). Thus, the Fokker-Planck equation can be written

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + (DW)\frac{\partial f}{\partial W} + (D\mu)\frac{\partial f}{\partial \mu} + (D\varphi)\frac{\partial f}{\partial \varphi} = C\{f\}. \tag{10}$$

Here, and for the rest of the calculation, partial derivatives hold μ, W, and φ fixed.

We now introduce the gyroaveraging operation $\bar{Q} = (2\pi)^{-1}\int_0^{2\pi} Q\, d\varphi$ where position, W, and μ are held fixed in the integration. Notice

$$\overline{DW} = (Ze/m)E_\parallel^* \upsilon_\parallel. \tag{11}$$

To compute $\overline{D\mu}$, we use

$$\overline{\mathbf{vv}} = \upsilon_\parallel^2 \mathbf{bb} + \frac{\upsilon_\perp^2}{2}\left(\ddot{\mathbf{I}} - \mathbf{bb}\right) \tag{12}$$

and $\mathbf{bb}:\nabla\mathbf{b} = 0$ to obtain $\overline{D\mu} = 0$. Introducing $\tilde{f} = f - \bar{f}$, it will turn out to be convenient to write the Fokker-Planck equation as

$$\frac{\partial \bar{f}}{\partial t} + \mathbf{v}\cdot\nabla\bar{f} + (DW)\frac{\partial \bar{f}}{\partial W} + (D\mu)\frac{\partial \bar{f}}{\partial \mu} + D\tilde{f} = C\{\bar{f} + \tilde{f}\}. \tag{13}$$

Applying a gyroaverage,

$$\frac{\partial \bar{f}}{\partial t} + \upsilon_\parallel \mathbf{b}\cdot\nabla\bar{f} + \frac{Ze}{m}E_\parallel^*\upsilon_\parallel\frac{\partial \bar{f}}{\partial W} + \overline{D\tilde{f}} = \overline{C\{\bar{f} + \tilde{f}\}}. \tag{14}$$

Subtracting this result from (13) gives

$$\mathbf{v}_\perp\cdot\nabla\bar{f} + \frac{Ze}{m}\mathbf{E}_\perp^*\cdot\mathbf{v}_\perp\frac{\partial \bar{f}}{\partial W} + (D\mu)\frac{\partial \bar{f}}{\partial \mu} + D\tilde{f} - \overline{D\tilde{f}} = C\{\bar{f} + \tilde{f}\} - \overline{C\{\bar{f} + \tilde{f}\}}. \tag{15}$$

Let us now begin to apply the ordering assumptions given above. The leading term in (10) is $-\Omega\, \partial f_0/\partial\varphi = 0$ from the $D\varphi$ term, so $\tilde{f}_0 = 0$, and $\tilde{f} \sim \rho_* \bar{f}$. We henceforth drop the overbar on f_0. Next, the leading terms in (14) are the $O(\rho_*\Omega\bar{f})$ terms

$$\upsilon_\parallel \mathbf{b}\cdot\nabla f_0 = \overline{C\{f_0\}}. \tag{16}$$

At this point, a rigorous derivation can be given to show f_0 must be a Maxwellian. For simplicity we will not give this derivation here. If f_0 is Maxwellian, then $C\{f_0\}=0$, so (16) becomes $\upsilon_{\|}\mathbf{b}\cdot\nabla f_0=0$. Also, we may linearize the collision operator and use $\overline{C_\ell\{g\}}=C_\ell\{\overline{g}\}$ to simplify the right-hand side of (14) to $C\{\overline{f}\}$.

Now consider the $O\left(\rho_*\Omega f_0\right)$ terms in (15):

$$\mathbf{v}_\perp\cdot\nabla f_0-\Omega\frac{\partial\tilde{f}_1}{\partial\varphi}=0\,. \tag{17}$$

Using

$$\mathbf{v}_\perp=\frac{\partial}{\partial\varphi}\left(\mathbf{v}\times\mathbf{b}\right) \tag{18}$$

then (17) may be integrated to obtain

$$\tilde{f}_1=-\boldsymbol{\rho}\cdot\nabla f_0 \tag{19}$$

where

$$\boldsymbol{\rho}=\Omega^{-1}\mathbf{b}\times\mathbf{v}\,. \tag{20}$$

We now form the drift-kinetic equation from the $O\left(\rho_*^2\Omega f_0\right)$ terms in (14):

$$\upsilon_{\|}\mathbf{b}\cdot\nabla\overline{f}_1-\frac{Ze}{T}E_{\|}^{*}\upsilon_{\|}f_0+\overline{D\tilde{f}_1}=C\left\{\overline{f}_1\right\}. \tag{21}$$

We must evaluate

$$\overline{D\tilde{f}_1}=-\overline{D\left[\boldsymbol{\rho}\cdot\nabla f_0\right]}=-\underbrace{\overline{\left(D\boldsymbol{\rho}\right)}}_{X}\cdot\nabla f_0-\underbrace{\overline{\boldsymbol{\rho}\cdot D\left(\nabla f_0\right)}}_{Y} \tag{22}$$

We can drop the time derivative in D since it is high order. First consider the term Y, writing

$$D\left(\nabla f_0\right)=\left[\mathbf{v}\cdot\nabla+\left(DW\right)\frac{\partial}{\partial W}\right]\nabla f_0=\mathbf{v}\cdot\nabla\nabla f_0+\frac{Ze}{m}\mathbf{E}^*\cdot\mathbf{v}\frac{\partial}{\partial W}\nabla f_0\,. \tag{23}$$

The $\mathbf{E}^*$ term is higher order than the others in (22), so it can be neglected. Then $Y=\overline{\boldsymbol{\rho}\mathbf{v}}\cdot\nabla\nabla f_0$. We find

$$\overline{\boldsymbol{\rho}\mathbf{v}}=\frac{\upsilon_\perp^2}{2\Omega}\left(\mathbf{e}_2\mathbf{e}_1-\mathbf{e}_1\mathbf{e}_2\right) \tag{24}$$

to be antisymmetric, so since $\nabla\nabla f_0$ is symmetric, $Y=0$. We can evaluate X using (3), finding

$$D\boldsymbol{\rho}=\mathbf{v}\cdot\nabla\left(\frac{1}{\Omega}\mathbf{b}\right)\times\mathbf{v}-\frac{c}{B}\mathbf{E}\times\mathbf{b}\,. \tag{25}$$

Gyroaveraging,

$$\begin{aligned}\overline{D\boldsymbol{\rho}}&=\left(\upsilon_{\|}^2-\frac{\upsilon_\perp^2}{2}\right)\mathbf{b}\cdot\nabla\left(\frac{1}{\Omega}\mathbf{b}\right)\times\mathbf{b}+\frac{\upsilon_\perp^2}{2}\sum_{i=1}^{3}\mathbf{e}_i\cdot\nabla\left(\frac{1}{\Omega}\mathbf{b}\right)\times\mathbf{e}_i-\frac{c}{B}\mathbf{E}\times\mathbf{b}\\&=\left(\upsilon_{\|}^2-\frac{\upsilon_\perp^2}{2}\right)\frac{1}{\Omega}\boldsymbol{\kappa}\times\mathbf{b}-\frac{\upsilon_\perp^2}{2\Omega}\nabla\times\mathbf{b}-\frac{\upsilon_\perp^2}{2\Omega B}\mathbf{b}\times\nabla B-\frac{c}{B}\mathbf{E}\times\mathbf{b}\end{aligned} \tag{26}$$

where $\boldsymbol{\kappa}=\mathbf{b}\cdot\nabla\mathbf{b}$. Then applying

$$\nabla\times\mathbf{b}=\mathbf{b}\mathbf{b}\cdot\nabla\times\mathbf{b}-\boldsymbol{\kappa}\times\mathbf{b}\,, \tag{27}$$

we obtain $\overline{D\boldsymbol{\rho}}=-\mathbf{v}_d$ where $\mathbf{v}_d$ is given in (2). Thus, (21) becomes

$$\upsilon_{\|} \mathbf{b} \cdot \nabla \overline{f}_1 - \frac{Ze}{T} E_{\|}^{*} \upsilon_{\|} f_0 + \mathbf{v}_d \cdot \nabla f_0 = C\left\{ \overline{f}_1 \right\} . \tag{28}$$

Taking $\mathbf{b} \cdot \nabla \Phi_0 = 0$ so $E_{\|}^{*} = E_{\|}$, we obtain the desired result (1), concluding the proof.