A Two-Species H Theorem and Its Small-Mass-Ratio Limit

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Abstract

This note gives a two-species version of Boltzmann's H theorem for the Landau collision operator. The species are electrons e and singly charged ions i in a fully ionized hydrogen plasma. For the exact two-species Landau operator with any fixed positive masses, vanishing entropy production implies that both species are Maxwellians with a common bulk flow velocity and a common temperature. Thus, if m_e/m_i is treated as order unity in an asymptotic expansion, the collisional local equilibrium has $\boldsymbol{U}_e = \boldsymbol{U}_i$ and $T_e = T_i$.

The last part of the note explains the small-mass-ratio limit. If $\mu = m_e/m_i \ll 1$ is taken before the interspecies equilibration constraint is imposed, the leading electron-ion operator is the Lorentz pitch-angle-scattering operator off an infinitely massive ion background. The leading nullspace of the combined collision operator is then

$$f_e = M_e(n_e, \boldsymbol{U}, T_e), \qquad f_i = M_i(n_i, \boldsymbol{U}, T_i),$$

with a common flow but with independent electron and ion temperatures. The exact finite-μ operator still drives T_e and T_i together, but the corresponding entropy production and temperature-equilibration rate are smaller by $O(m_e/m_i)$ relative to the leading electron collision rate. Hence, separate T_e and T_i are consistent in a mass-ratio expansion on time scales shorter than the electron-ion energy-equilibration time.

Contents

1 Conventions, assumptions, and what is being proved

We suppress the spatial coordinate throughout most of the proof. The Landau collision operator is local in position, so all statements below are pointwise in $\boldsymbol{x}$; spatial transport and force terms do not enter the collisional entropy production at a fixed point. The velocity variable for species s is denoted by $\boldsymbol{v} \in \mathbb{R}^3$. The two species are

$$s \in \{e, i\}, \qquad q_e = -e, \qquad q_i = +e,$$

with positive masses m_e, m_i. For hydrogen, quasineutrality gives $n_e = n_i$ in the usual plasma ordering, but quasineutrality is not needed for the H theorem itself. Collisions conserve each species' particle number separately and conserve the total momentum and total kinetic energy of the two species.

The distribution functions $f_s(\boldsymbol{v})$ are assumed to satisfy the following regularity hypotheses:

(A1) $f_s(\boldsymbol{v}) > 0$ and f_s is C^2 in velocity.

(A2) f_s, $\nabla_{\boldsymbol{v}} f_s$, and the velocity moments used below decay rapidly enough as $|\boldsymbol{v}| \to \infty$ for all integrations by parts to have zero boundary term.

(A3) All collision integrals are finite. The singular point $\boldsymbol{v} = \boldsymbol{v}'$ in the Landau kernel is harmless for the present argument because it has zero measure; equivalently, one may use a standard Landau cut-off and then remove it after the identities are obtained.

These assumptions can be weakened, but they keep the proof focused on the kinetic content rather than on functional analysis.

The entropy density in velocity space is

$$S[f_e, f_i] = -\sum_{s=e,i} \int_{\mathbb{R}^3} f_s(\boldsymbol{v}) \ln f_s(\boldsymbol{v})\, \mathrm{d}^3 v. \tag{1}$$

The collisional entropy production is

$$\dot{S}_{\mathrm{coll}} = -\sum_{s=e,i} \int_{\mathbb{R}^3} \ln f_s(\boldsymbol{v})\, C_s[f_e, f_i](\boldsymbol{v})\, \mathrm{d}^3 v, \tag{2}$$

where $C_s = \sum_r C_{sr}$ is the total Landau collision operator acting on species s.

The main finite-mass-ratio theorem proved below is:

Theorem 1 (Exact two-species H theorem)**.** *For the exact two-species Landau operator with $0 < m_e, m_i < \infty$,*

$$\dot{S}_{\mathrm{coll}} \geq 0.$$

Moreover, $\dot{S}_{\mathrm{coll}} = 0$ *if and only if*

$$f_e(\boldsymbol{v}) = M_e(n_e, \boldsymbol{U}, T)(\boldsymbol{v}), \qquad f_i(\boldsymbol{v}) = M_i(n_i, \boldsymbol{U}, T)(\boldsymbol{v}), \tag{3}$$

where

$$M_s(n_s, \boldsymbol{U}, T)(\boldsymbol{v}) = n_s \left(\frac{m_s}{2\pi T}\right)^{3/2} \exp\left[-\frac{m_s|\boldsymbol{v} - \boldsymbol{U}|^2}{2T}\right]. \tag{4}$$

The densities n_e, n_i *are arbitrary positive constants from the collision theorem alone; in a quasineutral hydrogen plasma one additionally has* $n_e = n_i$ *at leading order.*

The small-mass-ratio result is different because a different limiting operator is being used. If $\mu = m_e/m_i \ll 1$ is expanded first, the leading electron-ion operator only pitch-angle scatters electrons off an infinitely massive ion background and does not exchange electron and ion thermal energies. The leading local equilibria are then two Maxwellians with common flow but independent temperatures; see Theorem 2 below.

2 The multispecies Landau operator

For two species $s, r \in \{e, i\}$ define

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}', \qquad u = |\boldsymbol{u}|, \qquad \mathsf{U}(\boldsymbol{u}) = \frac{u^2\mathsf{I} - \boldsymbol{u}\boldsymbol{u}}{u^3} = \frac{1}{u}\left(\mathsf{I} - \hat{\boldsymbol{u}}\hat{\boldsymbol{u}}\right), \tag{5}$$

where $\hat{\boldsymbol{u}} = \boldsymbol{u}/u$ for $u \neq 0$ and I is the identity tensor. The dyadic $\boldsymbol{u}\boldsymbol{u}$ means $(\boldsymbol{u}\boldsymbol{u})_{ab} = u_a u_b$.

The exact numerical prefactor in the Landau operator is irrelevant for the H theorem, so we write it as a symmetric positive constant

$$\Gamma_{sr} = \Gamma_{rs} > 0. \tag{6}$$

For Coulomb collisions, Γ_{sr} is proportional to $q_s^2 q_r^2 \ln\Lambda$, so the sign of the electron charge does not affect entropy production. The multispecies Landau operator is written here in the symmetric mass-weighted form

$$\boxed{C_{sr}[f_s, f_r](\boldsymbol{v}) = \frac{\Gamma_{sr}}{m_s}\nabla_{\boldsymbol{v}} \cdot \int_{\mathbb{R}^3} \mathsf{U}(\boldsymbol{v} - \boldsymbol{v}') \cdot \left[\frac{f_r(\boldsymbol{v}')}{m_s}\nabla_{\boldsymbol{v}} f_s(\boldsymbol{v}) - \frac{f_s(\boldsymbol{v})}{m_r}\nabla_{\boldsymbol{v}'} f_r(\boldsymbol{v}')\right] \mathrm{d}^3v'.} \tag{7}$$

It is useful to set

$$f_s = f_s(\boldsymbol{v}), \qquad f_r' = f_r(\boldsymbol{v}'), \tag{8}$$

$$\boldsymbol{A}_{sr}(\boldsymbol{v}, \boldsymbol{v}') = \frac{1}{m_s}\nabla_{\boldsymbol{v}} \ln f_s(\boldsymbol{v}) - \frac{1}{m_r}\nabla_{\boldsymbol{v}'} \ln f_r(\boldsymbol{v}'), \tag{9}$$

so that

$$C_{sr}[f_s, f_r](\boldsymbol{v}) = \frac{\Gamma_{sr}}{m_s}\nabla_{\boldsymbol{v}} \cdot \int_{\mathbb{R}^3} f_s f_r' \,\mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{A}_{sr}(\boldsymbol{v}, \boldsymbol{v}')\,\mathrm{d}^3v'. \tag{10}$$

For $s = r$, this reduces to the single-species Landau form used in the original note, with the important proviso that in a multispecies proof the factors $1/m_s$ and $1/m_r$ in (9) must be retained. These mass factors are precisely what makes momentum and energy conservation work pairwise between unlike species.

2.1 Two elementary properties of the Landau tensor

For any vector $\boldsymbol{a} \in \mathbb{R}^3$,

$$\begin{aligned}\boldsymbol{a} \cdot \mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{a} &= \frac{u^2|\boldsymbol{a}|^2 - (\boldsymbol{u} \cdot \boldsymbol{a})^2}{u^3} \\ &= \frac{|(\mathsf{I} - \hat{\boldsymbol{u}}\hat{\boldsymbol{u}})\boldsymbol{a}|^2}{u} \geq 0. \end{aligned} \tag{11}$$

This is Cauchy's inequality in $\mathbb{R}^3$. It is slightly more precise than calling the argument the triangle inequality. Equation (11) shows that U is positive semidefinite and that

$$\mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{u} = \boldsymbol{0}, \qquad \mathrm{Null}\,\mathsf{U}(\boldsymbol{u}) = \mathrm{span}\{\boldsymbol{u}\} \quad (\boldsymbol{u} \neq \boldsymbol{0}). \tag{12}$$

The Landau tensor is also even:

$$\mathsf{U}(-\boldsymbol{u}) = \mathsf{U}(\boldsymbol{u}). \tag{13}$$

3 Pairwise conservation laws

The conservation laws are not needed for non-negativity of entropy production, but they are part of the consistency of the two-species operator and fix the mass factors in (7). Define the pair flux

$$\boldsymbol{G}_{sr}(\boldsymbol{v}, \boldsymbol{v}') = f_s(\boldsymbol{v}) f_r(\boldsymbol{v}')\, \mathsf{U}(\boldsymbol{v} - \boldsymbol{v}') \cdot \boldsymbol{A}_{sr}(\boldsymbol{v}, \boldsymbol{v}'). \tag{14}$$

Then

$$C_{sr}(\boldsymbol{v}) = \frac{\Gamma_{sr}}{m_s} \nabla_{\boldsymbol{v}} \cdot \int \boldsymbol{G}_{sr}(\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}^3 v'. \tag{15}$$

Using (9) and (13),

$$\boldsymbol{A}_{rs}(\boldsymbol{v}', \boldsymbol{v}) = -\boldsymbol{A}_{sr}(\boldsymbol{v}, \boldsymbol{v}'), \qquad \boldsymbol{G}_{rs}(\boldsymbol{v}', \boldsymbol{v}) = -\boldsymbol{G}_{sr}(\boldsymbol{v}, \boldsymbol{v}'). \tag{16}$$

3.1 Particle conservation

For each ordered pair (s, r),

$$\int C_{sr}(\boldsymbol{v})\, \mathrm{d}^3 v = \frac{\Gamma_{sr}}{m_s} \int \nabla_{\boldsymbol{v}} \cdot \left[\int \boldsymbol{G}_{sr}(\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}^3 v' \right] \mathrm{d}^3 v = 0, \tag{17}$$

where the boundary term at $|\boldsymbol{v}| = \infty$ vanishes by assumption. Thus collisions of s with r do not convert particles of one species into the other.

3.2 Momentum conservation

The momentum transfer to species s by species r is

$$\begin{aligned}\boldsymbol{R}_{sr} &= \int m_s \boldsymbol{v}\, C_{sr}(\boldsymbol{v})\, \mathrm{d}^3 v \\ &= \Gamma_{sr} \int \boldsymbol{v}\, \nabla_{\boldsymbol{v}} \cdot \left[\int \boldsymbol{G}_{sr}(\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}^3 v' \right] \mathrm{d}^3 v \\ &= -\Gamma_{sr} \iint \boldsymbol{G}_{sr}(\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}^3 v'\, \mathrm{d}^3 v. \end{aligned} \tag{18}$$

Here the identity $\int v_a \partial_{v_b} H_b \, \mathrm{d}^3v = -\int H_a \, \mathrm{d}^3v$ was used componentwise. Similarly,

$$\begin{aligned}
\boldsymbol{R}_{rs} &= \int m_r \boldsymbol{v}' \, C_{rs}(\boldsymbol{v}') \, \mathrm{d}^3v' \\
&= -\Gamma_{rs} \iint \boldsymbol{G}_{rs}(\boldsymbol{v}', \boldsymbol{v}) \, \mathrm{d}^3v \, \mathrm{d}^3v' \\
&= +\Gamma_{sr} \iint \boldsymbol{G}_{sr}(\boldsymbol{v}, \boldsymbol{v}') \, \mathrm{d}^3v' \, \mathrm{d}^3v = -\boldsymbol{R}_{sr}.
\end{aligned} \tag{19}$$

Therefore

$$\boldsymbol{R}_{sr} + \boldsymbol{R}_{rs} = \boldsymbol{0}. \tag{20}$$

Collisions transfer momentum between the two species but conserve the total momentum.

3.3 Kinetic-energy conservation

The kinetic-energy transfer to species s by species r is

$$\begin{aligned}
Q_{sr} &= \int \frac{m_s v^2}{2} \, C_{sr}(\boldsymbol{v}) \, \mathrm{d}^3v \\
&= \frac{\Gamma_{sr}}{2} \int v^2 \, \nabla_{\boldsymbol{v}} \cdot \left[\int \boldsymbol{G}_{sr}(\boldsymbol{v}, \boldsymbol{v}') \, \mathrm{d}^3v' \right] \mathrm{d}^3v \\
&= -\Gamma_{sr} \iint \boldsymbol{v} \cdot \boldsymbol{G}_{sr}(\boldsymbol{v}, \boldsymbol{v}') \, \mathrm{d}^3v' \, \mathrm{d}^3v.
\end{aligned} \tag{21}$$

For species r,

$$\begin{aligned}
Q_{rs} &= -\Gamma_{rs} \iint \boldsymbol{v}' \cdot \boldsymbol{G}_{rs}(\boldsymbol{v}', \boldsymbol{v}) \, \mathrm{d}^3v \, \mathrm{d}^3v' \\
&= +\Gamma_{sr} \iint \boldsymbol{v}' \cdot \boldsymbol{G}_{sr}(\boldsymbol{v}, \boldsymbol{v}') \, \mathrm{d}^3v' \, \mathrm{d}^3v.
\end{aligned} \tag{22}$$

Adding (21) and (22) gives

$$\begin{aligned}
Q_{sr} + Q_{rs} &= \Gamma_{sr} \iint (\boldsymbol{v}' - \boldsymbol{v}) \cdot \left[f_s f_r' \, \mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{A}_{sr} \right] \mathrm{d}^3v' \, \mathrm{d}^3v \\
&= -\Gamma_{sr} \iint f_s f_r' \, \boldsymbol{u} \cdot \mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{A}_{sr} \, \mathrm{d}^3v' \, \mathrm{d}^3v \\
&= 0,
\end{aligned} \tag{23}$$

using (12). Thus unlike-species collisions exchange kinetic energy but conserve total kinetic energy.

4 Entropy production for the exact two-species operator

The total collision operator acting on species s is

$$C_s = C_{ss} + C_{sr}, \qquad r \neq s. \tag{24}$$

The collisional entropy production (2) can be decomposed into a self-collision contribution for each species and one electron-ion contribution:

$$\dot{S}_{\mathrm{coll}} = \sigma_{ee} + \sigma_{ii} + \sigma_{ei}. \tag{25}$$

4.1 Self-collision entropy production

Fix one species s. From (15),

$$-\int \ln f_s(\boldsymbol{v})\, C_{ss}(\boldsymbol{v})\, \mathrm{d}^3v = \frac{\Gamma_{ss}}{m_s} \iint \nabla_{\boldsymbol{v}} \ln f_s(\boldsymbol{v}) \cdot \boldsymbol{G}_{ss}(\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}^3v'\, \mathrm{d}^3v. \tag{26}$$

Interchanging the dummy variables $\boldsymbol{v}$ and $\boldsymbol{v}'$ in the same integral and using (16) with $r = s$ gives

$$-\int \ln f_s C_{ss}\, \mathrm{d}^3v = -\frac{\Gamma_{ss}}{m_s} \iint \nabla_{\boldsymbol{v}'} \ln f_s(\boldsymbol{v}') \cdot \boldsymbol{G}_{ss}(\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}^3v'\, \mathrm{d}^3v. \tag{27}$$

Averaging (26) and (27), and using

$$\boldsymbol{A}_{ss}(\boldsymbol{v}, \boldsymbol{v}') = \frac{1}{m_s} \left[\nabla_{\boldsymbol{v}} \ln f_s(\boldsymbol{v}) - \nabla_{\boldsymbol{v}'} \ln f_s(\boldsymbol{v}')\right], \tag{28}$$

we obtain

$$\boxed{\sigma_{ss} = \frac{\Gamma_{ss}}{2} \iint f_s f_s'\, \boldsymbol{A}_{ss} \cdot \mathsf{U}(\boldsymbol{v} - \boldsymbol{v}') \cdot \boldsymbol{A}_{ss}\, \mathrm{d}^3v'\, \mathrm{d}^3v.} \tag{29}$$

By (11), $\sigma_{ss} \geq 0$.

4.2 Electron-ion entropy production

For the electron-ion pair, one must combine the entropy change of both species. Let $s = e$ and $r = i$ for a moment. The electron part is

$$-\int \ln f_e\, C_{ei}\, \mathrm{d}^3v = \frac{\Gamma_{ei}}{m_e} \iint \nabla_{\boldsymbol{v}} \ln f_e(\boldsymbol{v}) \cdot \boldsymbol{G}_{ei}(\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}^3v'\, \mathrm{d}^3v. \tag{30}$$

The ion part is

$$\begin{aligned} -\int \ln f_i(\boldsymbol{v}')\, C_{ie}(\boldsymbol{v}')\, \mathrm{d}^3v' &= \frac{\Gamma_{ie}}{m_i} \iint \nabla_{\boldsymbol{v}'} \ln f_i(\boldsymbol{v}') \cdot \boldsymbol{G}_{ie}(\boldsymbol{v}', \boldsymbol{v})\, \mathrm{d}^3v\, \mathrm{d}^3v' \\ &= -\frac{\Gamma_{ei}}{m_i} \iint \nabla_{\boldsymbol{v}'} \ln f_i(\boldsymbol{v}') \cdot \boldsymbol{G}_{ei}(\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}^3v'\, \mathrm{d}^3v. \end{aligned} \tag{31}$$

Adding (30) and (31) gives

$$\boxed{\sigma_{ei} = \Gamma_{ei} \iint f_e(\boldsymbol{v}) f_i(\boldsymbol{v}')\, \boldsymbol{A}_{ei}(\boldsymbol{v}, \boldsymbol{v}') \cdot \mathsf{U}(\boldsymbol{v} - \boldsymbol{v}') \cdot \boldsymbol{A}_{ei}(\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}^3v'\, \mathrm{d}^3v.} \tag{32}$$

Again $\sigma_{ei} \geq 0$ by (11).

Equations (25), (29), and (32) prove the non-negativity part of the H theorem:

$$\dot{S}_{\text{coll}} = \sigma_{ee} + \sigma_{ii} + \sigma_{ei} \geq 0. \tag{33}$$

5 Characterization of equality for finite mass ratio

It remains to determine when (33) is an equality. Since each σ is non-negative, $\dot{S}_{\text{coll}} = 0$ if and only if

$$\sigma_{ee} = 0, \qquad \sigma_{ii} = 0, \qquad \sigma_{ei} = 0. \tag{34}$$

For a non-negative continuous integrand, a zero integral implies that the integrand is zero almost everywhere; with the smoothness assumptions above, the identities below may be read pointwise for all $\boldsymbol{v}, \boldsymbol{v}'$ with $\boldsymbol{v} \neq \boldsymbol{v}'$.

5.1 Self-collisions force each species to be a Maxwellian

Lemma 1 (Self-collision nullspace). *Let $f > 0$ be C^2, rapidly decaying, and normalized with finite density, momentum, and energy. If*

$$[\nabla_{\boldsymbol{v}} \ln f(\boldsymbol{v}) - \nabla_{\boldsymbol{v}'} \ln f(\boldsymbol{v}')] \cdot \mathsf{U}(\boldsymbol{v} - \boldsymbol{v}') \cdot [\nabla_{\boldsymbol{v}} \ln f(\boldsymbol{v}) - \nabla_{\boldsymbol{v}'} \ln f(\boldsymbol{v}')] = 0 \tag{35}$$

for all $\boldsymbol{v} \neq \boldsymbol{v}'$, then f is a Maxwellian.

Proof. Set

$$\boldsymbol{g}(\boldsymbol{v}) = \nabla_{\boldsymbol{v}} \ln f(\boldsymbol{v}). \tag{36}$$

By (12), condition (35) is equivalent to

$$\boldsymbol{g}(\boldsymbol{v}) - \boldsymbol{g}(\boldsymbol{v}') \in \operatorname{span}\{\boldsymbol{v} - \boldsymbol{v}'\} \qquad \text{for all } \boldsymbol{v} \neq \boldsymbol{v}'. \tag{37}$$

Equivalently, if $\boldsymbol{h} \cdot (\boldsymbol{v} - \boldsymbol{v}') = 0$, then

$$\boldsymbol{h} \cdot [\boldsymbol{g}(\boldsymbol{v}) - \boldsymbol{g}(\boldsymbol{v}')] = 0. \tag{38}$$

Fix $\boldsymbol{v}$ and an arbitrary vector $\boldsymbol{a} \neq 0$. Put $\boldsymbol{v}' = \boldsymbol{v} + \epsilon \boldsymbol{a}$, where ϵ is nonzero and small. If $\boldsymbol{h} \cdot \boldsymbol{a} = 0$, then (38) gives

$$\boldsymbol{h} \cdot \frac{\boldsymbol{g}(\boldsymbol{v} + \epsilon \boldsymbol{a}) - \boldsymbol{g}(\boldsymbol{v})}{\epsilon} = 0. \tag{39}$$

Taking $\epsilon \to 0$ yields

$$\boldsymbol{h} \cdot [D\boldsymbol{g}(\boldsymbol{v})]\boldsymbol{a} = 0 \qquad \text{for every } \boldsymbol{h} \perp \boldsymbol{a}. \tag{40}$$

Thus $[D\boldsymbol{g}(\boldsymbol{v})]\boldsymbol{a}$ is parallel to $\boldsymbol{a}$ for every vector $\boldsymbol{a}$. A linear map with every vector as an eigenvector must be a scalar multiple of the identity. Therefore, for each $\boldsymbol{v}$ there is a scalar $\lambda(\boldsymbol{v})$ such that

$$D\boldsymbol{g}(\boldsymbol{v}) = \lambda(\boldsymbol{v})\mathsf{I}. \tag{41}$$

In components,

$$\partial_{v_j} g_k = 0 \quad (j \neq k), \qquad \partial_{v_1} g_1 = \partial_{v_2} g_2 = \partial_{v_3} g_3 = \lambda(\boldsymbol{v}). \tag{42}$$

The first relation says g_1 depends only on v_1, g_2 only on v_2, and g_3 only on v_3. Hence $\partial_{v_1} g_1$ is a function only of v_1, while $\partial_{v_2} g_2$ is a function only of v_2. Since they are equal for all (v_1, v_2, v_3), both must be a constant. The same comparison with the third coordinate shows that

$$\lambda(\boldsymbol{v}) = \lambda_0 \tag{43}$$

for a constant λ_0. Thus

$$\boldsymbol{g}(\boldsymbol{v}) = \lambda_0 \boldsymbol{v} + \boldsymbol{b} \tag{44}$$

for some constant vector $\boldsymbol{b}$. Since $\boldsymbol{g} = \nabla_{\boldsymbol{v}} \ln f$, integration in $\boldsymbol{v}$ gives

$$\ln f(\boldsymbol{v}) = \frac{\lambda_0}{2}|\boldsymbol{v}|^2 + \boldsymbol{b} \cdot \boldsymbol{v} + c_0. \tag{45}$$

The integrability of f over $\mathbb{R}^3$ requires $\lambda_0 < 0$. Define

$$T = -\frac{m}{\lambda_0} > 0, \qquad \boldsymbol{U} = -\frac{\boldsymbol{b}}{\lambda_0}. \tag{46}$$

Then (45) can be written as

$$f(\boldsymbol{v}) = A \exp\left[-\frac{m|\boldsymbol{v} - \boldsymbol{U}|^2}{2T}\right]. \tag{47}$$

Choosing $A = n(m/2\pi T)^{3/2}$ gives the Maxwellian (4). This proves the lemma. $\square$

Applying the lemma to $\sigma_{ee} = 0$ and $\sigma_{ii} = 0$ gives

$$f_e = M_e(n_e, \boldsymbol{U}_e, T_e), \qquad f_i = M_i(n_i, \boldsymbol{U}_i, T_i), \tag{48}$$

with, at this stage, unrelated $\boldsymbol{U}_e, \boldsymbol{U}_i, T_e, T_i$.

5.2 Electron-ion collisions force common temperature and common flow

We now impose $\sigma_{ei} = 0$ on the two Maxwellians (48). For a Maxwellian,

$$\nabla_{\boldsymbol{v}} \ln M_s(n_s, \boldsymbol{U}_s, T_s) = -\frac{m_s}{T_s}(\boldsymbol{v} - \boldsymbol{U}_s). \tag{49}$$

Therefore the mass-weighted difference (9) is

$$\boldsymbol{A}_{ei}(\boldsymbol{v}, \boldsymbol{v}') = -\frac{\boldsymbol{v} - \boldsymbol{U}_e}{T_e} + \frac{\boldsymbol{v}' - \boldsymbol{U}_i}{T_i}. \tag{50}$$

Since $\sigma_{ei} = 0$, (12) implies

$$\boldsymbol{A}_{ei}(\boldsymbol{v}, \boldsymbol{v}') \in \operatorname{span}\{\boldsymbol{v} - \boldsymbol{v}'\} \qquad \text{for all } \boldsymbol{v} \neq \boldsymbol{v}'. \tag{51}$$

Introduce the variables

$$\boldsymbol{V} = \frac{\boldsymbol{v} + \boldsymbol{v}'}{2}, \qquad \boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}'. \tag{52}$$

Then $\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{u}/2$ and $\boldsymbol{v}' = \boldsymbol{V} - \boldsymbol{u}/2$, so (50) becomes

$$\boldsymbol{A}_{ei} = \left(\frac{1}{T_i} - \frac{1}{T_e}\right)\boldsymbol{V} - \frac{1}{2}\left(\frac{1}{T_e} + \frac{1}{T_i}\right)\boldsymbol{u} + \frac{\boldsymbol{U}_e}{T_e} - \frac{\boldsymbol{U}_i}{T_i}. \tag{53}$$

Let

$$\mathsf{P}_{\boldsymbol{u}}^{\perp} = \mathsf{I} - \hat{\boldsymbol{u}}\hat{\boldsymbol{u}} \tag{54}$$

be the projection onto the plane perpendicular to $\boldsymbol{u}$. Condition (51) is equivalent to

$$\mathsf{P}_{\boldsymbol{u}}^{\perp}\boldsymbol{A}_{ei} = \boldsymbol{0}. \tag{55}$$

For any fixed nonzero $\boldsymbol{u}$, the vector $\boldsymbol{V}$ can be varied independently in all three directions. In particular, $\mathsf{P}_{\boldsymbol{u}}^{\perp}\boldsymbol{V}$ can be any vector in the two-dimensional plane perpendicular to $\boldsymbol{u}$. Projecting (53) perpendicular to $\boldsymbol{u}$ gives

$$\left(\frac{1}{T_i} - \frac{1}{T_e}\right)\mathsf{P}_{\boldsymbol{u}}^{\perp}\boldsymbol{V} + \mathsf{P}_{\boldsymbol{u}}^{\perp}\left(\frac{\boldsymbol{U}_e}{T_e} - \frac{\boldsymbol{U}_i}{T_i}\right) = \boldsymbol{0} \quad \text{for all } \boldsymbol{V}. \tag{56}$$

The coefficient of $\mathsf{P}_{\boldsymbol{u}}^{\perp}\boldsymbol{V}$ must vanish. Therefore

$$\frac{1}{T_i} - \frac{1}{T_e} = 0, \qquad \text{so} \qquad T_i = T_e \equiv T. \tag{57}$$

With equal temperatures, (56) reduces to

$$\mathsf{P}_{\boldsymbol{u}}^{\perp}(\boldsymbol{U}_e - \boldsymbol{U}_i) = \boldsymbol{0} \qquad \text{for every nonzero } \boldsymbol{u}. \tag{58}$$

The only vector parallel to every possible $\boldsymbol{u}$ is the zero vector. Hence

$$\boldsymbol{U}_e = \boldsymbol{U}_i \equiv \boldsymbol{U}. \tag{59}$$

Thus the cross-species entropy production can vanish only for common T and common $\boldsymbol{U}$.

Conversely, if (3) holds, then

$$\boldsymbol{A}_{ei} = -\frac{\boldsymbol{v}-\boldsymbol{U}}{T} + \frac{\boldsymbol{v}'-\boldsymbol{U}}{T} = -\frac{\boldsymbol{v}-\boldsymbol{v}'}{T} = -\frac{\boldsymbol{u}}{T}, \tag{60}$$

so $\mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{A}_{ei} = 0$. The self-collision terms also vanish for Maxwellians. This completes the proof of Theorem 1.

Remark 1 (What "mass ratio of order unity" means here). The finite-mass theorem did not use any special numerical value of m_e/m_i; it holds for every fixed positive mass ratio. In an asymptotic calculation, however, saying $m_e/m_i = O(1)$ means that the exact interspecies collision operator is retained at the same asymptotic order as the self-collision operators. Then the equality condition is the exact one: a common flow and a common temperature.

6 The small-mass-ratio limit

Now set

$$\mu = \frac{m_e}{m_i} \ll 1. \tag{61}$$

The exact theorem above remains true for every fixed $\mu > 0$. The point of the small-mass-ratio calculation is different: one expands the collision operator in μ and asks for the nullspace of the leading operator. That leading operator does not contain electron-ion thermal-energy exchange.

Throughout this section assume $T_e/T_i = O(1)$ and write the thermal speeds as

$$v_{te} = \sqrt{\frac{2T_e}{m_e}}, \qquad v_{ti} = \sqrt{\frac{2T_i}{m_i}}, \qquad \frac{v_{ti}}{v_{te}} = O(\sqrt{\mu}). \tag{62}$$

6.1 Leading electron-ion operator: Lorentz pitch-angle scattering

Let the ion distribution be a smooth distribution with bulk flow $\boldsymbol{U}_i$ and ion thermal spread $O(v_{ti})$. On the electron velocity scale, $v_{ti}/v_{te} = O(\sqrt{\mu})$, so to leading order the ions are an infinitely massive, monoenergetic scattering background located at $\boldsymbol{v}' = \boldsymbol{U}_i$. In (7), the term containing $m_i^{-1}\nabla_{\boldsymbol{v}'} f_i$ is smaller, and the leading electron-ion operator is

$$\boxed{C_{ei}^{(0)}[f_e](\boldsymbol{v}) = \frac{\Gamma_{ei} n_i}{m_e^2} \nabla_{\boldsymbol{c}} \cdot \left[\mathsf{U}(\boldsymbol{c}) \cdot \nabla_{\boldsymbol{c}} f_e(\boldsymbol{c}+\boldsymbol{U}_i)\right], \qquad \boldsymbol{c} = \boldsymbol{v} - \boldsymbol{U}_i.} \tag{63}$$

This is the Lorentz pitch-angle-scattering operator. It diffuses only in the angular direction of $\boldsymbol{c}$ because $\mathsf{U}(\boldsymbol{c}) \cdot \boldsymbol{c} = 0$.

The corresponding entropy production is

$$\begin{aligned}
\sigma_{ei}^{(0)} &= -\int \ln f_e \, C_{ei}^{(0)}[f_e] \, \mathrm{d}^3 v \\
&= \frac{\Gamma_{ei} n_i}{m_e^2} \int f_e \, \nabla_{\boldsymbol{c}} \ln f_e \cdot \mathsf{U}(\boldsymbol{c}) \cdot \nabla_{\boldsymbol{c}} \ln f_e \, \mathrm{d}^3 c \\
&= \frac{\Gamma_{ei} n_i}{m_e^2} \int f_e \, \frac{|(\mathsf{I} - \hat{\boldsymbol{c}}\hat{\boldsymbol{c}}) \nabla_{\boldsymbol{c}} \ln f_e|^2}{|\boldsymbol{c}|} \, \mathrm{d}^3 c \geq 0.
\end{aligned} \tag{64}$$

The equality condition is

$$(\mathsf{I} - \hat{\boldsymbol{c}}\hat{\boldsymbol{c}}) \nabla_{\boldsymbol{c}} \ln f_e = \boldsymbol{0} \qquad \text{for all } \boldsymbol{c} \neq \boldsymbol{0}. \tag{65}$$

This says that $\ln f_e$ has zero derivative tangent to each sphere $|\boldsymbol{c}| = \text{constant}$. Hence

$$f_e(\boldsymbol{v}) = F_e(|\boldsymbol{v} - \boldsymbol{U}_i|) \tag{66}$$

for some scalar function F_e. Thus the Lorentz operator alone preserves electron speed in the ion frame and cannot determine an electron temperature.

6.2 Leading nullspace with self-collisions included

The relevant local-equilibrium ordering for a two-temperature plasma is: each species' self-collisions are retained strongly enough to Maxwellianize that species, while the electron-ion energy-exchange part of the interspecies operator is ordered smaller by $O(\mu)$. If one measures time in electron collision times, ion self-collisions may themselves be slower by a mass-ratio factor; including σ_{ii} below means that we are characterizing the state after ion self-collisions have also relaxed the ion distribution, but before the still slower electron-ion temperature equilibration has acted. With this convention, the leading entropy production has the form

$$\dot{S}^{(0)}_{\text{coll}} = \sigma_{ee} + \sigma_{ii} + \sigma^{(0)}_{ei}. \tag{67}$$

Ion response to electron collisions and electron-ion energy exchange are higher order in μ in this ordering. The self terms imply, by Lemma 1,

$$f_e = M_e(n_e, \boldsymbol{U}_e, T_e), \qquad f_i = M_i(n_i, \boldsymbol{U}_i, T_i). \tag{68}$$

The Lorentz nullspace condition (66) requires the electron Maxwellian to be a function of $|\boldsymbol{v} - \boldsymbol{U}_i|$. This is true if and only if

$$\boldsymbol{U}_e = \boldsymbol{U}_i \equiv \boldsymbol{U}. \tag{69}$$

There is no condition on T_e from the Lorentz operator, and ion self-collisions impose no condition relating T_i to T_e. We have therefore proved:

Theorem 2 (Leading small-mass-ratio local equilibria)**.** *At leading order in $\mu = m_e/m_i \ll 1$, the collision operator*

$$C_{ee} + C_{ii} + C^{(0)}_{ei}$$

has zero entropy production if and only if

$$f_e = M_e(n_e, \boldsymbol{U}, T_e), \qquad f_i = M_i(n_i, \boldsymbol{U}, T_i), \tag{70}$$

with T_e and T_i independent. Thus, after the small-mass-ratio expansion has been taken, the leading H theorem permits different electron and ion temperatures.

6.3 Why the exact finite-μ theorem is not contradicted

The exact electron-ion entropy production for two Maxwellians with a common flow $\boldsymbol{U}$ but different temperatures is small but not zero. Let

$$\boldsymbol{c}_e = \boldsymbol{v} - \boldsymbol{U}, \qquad \boldsymbol{c}_i = \boldsymbol{v}' - \boldsymbol{U}, \qquad \boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}' = \boldsymbol{c}_e - \boldsymbol{c}_i. \tag{71}$$

For the two Maxwellians

$$f_e = M_e(n_e, \boldsymbol{U}, T_e), \qquad f_i = M_i(n_i, \boldsymbol{U}, T_i), \tag{72}$$

formula (50) gives

$$\begin{aligned}\boldsymbol{A}_{ei} &= -\frac{\boldsymbol{c}_e}{T_e} + \frac{\boldsymbol{c}_i}{T_i} \\ &= -\frac{\boldsymbol{c}_e - \boldsymbol{c}_i}{T_e} + \left(\frac{1}{T_i} - \frac{1}{T_e}\right)\boldsymbol{c}_i \\ &= -\frac{\boldsymbol{u}}{T_e} + \left(\frac{1}{T_i} - \frac{1}{T_e}\right)\boldsymbol{c}_i. \end{aligned} \tag{73}$$

Because $\mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{u} = 0$, only the second term contributes to $\boldsymbol{A}_{ei} \cdot \mathsf{U} \cdot \boldsymbol{A}_{ei}$:

$$\boldsymbol{A}_{ei} \cdot \mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{A}_{ei} = \left(\frac{1}{T_i} - \frac{1}{T_e}\right)^2 \boldsymbol{c}_i \cdot \mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{c}_i. \tag{74}$$

In the region that dominates the Maxwellian integral,

$$|\boldsymbol{c}_e| = O(v_{te}), \qquad |\boldsymbol{c}_i| = O(v_{ti}) = O(\sqrt{\mu}\, v_{te}), \qquad |\boldsymbol{u}| = O(v_{te}). \tag{75}$$

Therefore

$$\boldsymbol{c}_i \cdot \mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{c}_i = \frac{|(\mathsf{I} - \hat{\boldsymbol{u}}\hat{\boldsymbol{u}})\boldsymbol{c}_i|^2}{|\boldsymbol{u}|} = O\left(\frac{v_{ti}^2}{v_{te}}\right) = O(\mu) \times O(v_{te}). \tag{76}$$

Equivalently, the component of $\boldsymbol{A}_{ei}$ perpendicular to $\boldsymbol{u}$ is $O(\sqrt{\mu})$ compared with its natural electron-scale size, so the quadratic entropy production associated with $T_e - T_i = O(T_e)$ is $O(\mu)$.

Thus the exact finite-μ operator does eventually equilibrate the temperatures, but that equilibration is a higher-order effect in a small-m_e/m_i expansion. In the common convention for defining the electron-ion collision frequency, the Maxwellian energy exchange has the schematic form

$$\frac{3}{2} n_e \frac{\mathrm{d}T_e}{\mathrm{d}t}\bigg|_{ei} = 3\frac{m_e}{m_i}\, n_e \nu_{ei}\, (T_i - T_e) + O(\mu^2), \tag{77}$$

with the ion equation carrying the opposite energy change. Consequently

$$\dot{S}_{ei}^{(T)} = Q_{ei}\left(\frac{1}{T_e} - \frac{1}{T_i}\right) = 3\frac{m_e}{m_i} n_e \nu_{ei}\, \frac{(T_i - T_e)^2}{T_e T_i} + O(\mu^2) \geq 0, \tag{78}$$

where Q_{ei} is the energy per unit volume per unit time transferred to electrons. The coefficient depends on the precise normalization of ν_{ei}, but the m_e/m_i ordering and the sign are invariant.

6.4 A note about relative flows in the small-mass-ratio limit

The exact finite-mass H theorem forces $\boldsymbol{U}_e = \boldsymbol{U}_i$. The leading Lorentz operator (63) also has zero entropy production for a Maxwellian only when it is centered on the ion flow, so Theorem 2 has a common leading flow.

There is one asymptotic nuance. If a current-carrying relative drift is ordered as

$$|\boldsymbol{U}_e - \boldsymbol{U}_i| = O(v_{ti}) = O(\sqrt{\mu}\, v_{te}), \tag{79}$$

then its contribution to the electron-ion entropy production is also $O(\mu)$ on the electron thermal scale. Indeed, for equal temperatures $T_e = T_i = T$ but different flows,

$$\boldsymbol{A}_{ei} = -\frac{\boldsymbol{u}}{T} + \frac{\boldsymbol{U}_e - \boldsymbol{U}_i}{T}, \tag{80}$$

so

$$\boldsymbol{A}_{ei} \cdot \mathsf{U}(\boldsymbol{u}) \cdot \boldsymbol{A}_{ei} = \frac{1}{T^2}(\boldsymbol{U}_e - \boldsymbol{U}_i) \cdot \mathsf{U}(\boldsymbol{u}) \cdot (\boldsymbol{U}_e - \boldsymbol{U}_i). \tag{81}$$

If the drift is $O(v_{te})$, this is a leading-order entropy production and the drift is not in the leading nullspace. If the drift is ordered as (79), it enters one order later, just as temperature equilibration does. This is a statement about the chosen asymptotic ordering, not a change in the exact equality condition for any finite μ.

7 Summary

For the exact Landau collision operator in a two-species fully ionized hydrogen plasma,

$$\dot{S}_{\text{coll}} = \sigma_{ee} + \sigma_{ii} + \sigma_{ei} \geq 0, \tag{82}$$

with the three terms given explicitly by (29) and (32). The two self-collision terms force electrons and ions separately to be Maxwellians. The electron-ion term then forces the two Maxwellians to have the same temperature and the same mean flow. Hence, if m_e/m_i is treated as $O(1)$, the collisional equilibrium is

$$f_e = M_e(n_e, \boldsymbol{U}, T), \qquad f_i = M_i(n_i, \boldsymbol{U}, T). \tag{83}$$

If instead $m_e/m_i \ll 1$ is used as an expansion parameter, the leading electron-ion operator is Lorentz pitch-angle scattering. Its nullspace, combined with the self-collision nullspaces, gives

$$f_e = M_e(n_e, \boldsymbol{U}, T_e), \qquad f_i = M_i(n_i, \boldsymbol{U}, T_i), \tag{84}$$

with T_e and T_i independent at leading order. For every finite nonzero m_e/m_i, the exact next-order terms still produce entropy and relax $T_e - T_i$ on the longer electron-ion energy-equilibration time.

References

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