Derivation of the magnetohydrodynamic equilibrium equation

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1 Overview

In this note we give a somewhat unconventional derivation of the magnetohydrodynamic equilibrium equation

$$\mathbf{J}\times\mathbf{B} = \nabla p \tag{1}$$

where p is the pressure, $\mathbf{B}$ is the magnetic field, and $\mathbf{J} = \mu_0^{-1}\nabla\times\mathbf{B}$ is the current density, for general 3D or 2D geometry, assuming nested flux surfaces. We get most of the way there by considering the total momentum conservation equation, obtained by summing the momentum moment of the kinetic equation over species. The main step that requires additional work is showing that the stress tensor is approximately an isotropic scalar pressure. To show this, we apply a small-ρ_* expansion to the kinetic equation, using a drift-kinetic approach. *If we assume that good flux surfaces exist*, it can be shown that the leading-order distribution function is a stationary Maxwellian, which implies the desired form of the stress tensor.

2 Total momentum equation

Start with the Fokker-Planck kinetic equation:

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s + \frac{q_s}{m_s}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f_s = C_s + S_s, \tag{2}$$

where S_s denotes any source or sink. Multiply by $m_s\mathbf{v}$ and integrate over velocity:

$$\frac{\partial}{\partial t} m_s \int d^3v\,\mathbf{v} f_s + \nabla\cdot m_s\int d^3v\,\mathbf{v}\mathbf{v} f_s + q_s\int d^3v\,\mathbf{v}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f_s = m_s\int d^3v\,\mathbf{v}C_s + m_s\int d^3v\,\mathbf{v}S_s. \tag{3}$$

The first term is related to the macroscopic fluid flow

$$\mathbf{V}_s = \frac{1}{n_s}\int d^3v\,\mathbf{v} f_s \tag{4}$$

where

$$n_s = \int d^3v\, f_s \tag{5}$$

is the density. The momentum flux tensor is defined as

$$\Pi_s = m_s\int d^3v\,\mathbf{v}\mathbf{v} f_s. \tag{6}$$

The acceleration term is integrated by parts:

$$\int d^3v\,\mathbf{v}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f_s = \int d^3v\,\nabla_v\cdot\left[\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) f_s\mathbf{v}\right] - \int d^3v\, f_s\nabla_v\cdot\left[\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\mathbf{v}\right]. \tag{7}$$

On the right-hand side, using the divergence theorem in velocity space, the first term becomes an area integral over a large surface that goes to infinity:

$$\int d^3v\,\nabla_v\cdot\left[\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) f_s\mathbf{v}\right] = \int d^2\mathbf{a}\cdot\left[\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) f_s\mathbf{v}\right]. \tag{8}$$

Since $f_s \to 0$ exponentially at large $|\mathbf{v}|$, the integral goes to 0 as the integration volume gets larger. So we are left with only the last term in (7):

$$\int d^3v\,\mathbf{v}\,(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_s = -\int d^3v\, f_s \nabla_v\cdot[(\mathbf{E}+\mathbf{v}\times\mathbf{B})\,\mathbf{v}]\,. \tag{9}$$

The electric field term on the right is evaluated using

$$\nabla_v\cdot[\mathbf{E}\mathbf{v}] = \mathbf{E}\cdot\nabla_v\mathbf{v} = \mathbf{E}\cdot\mathbf{I} = \mathbf{E}. \tag{10}$$

The magnetic field term is evaluated as

$$\nabla_v\cdot[(\mathbf{v}\times\mathbf{B})\,\mathbf{v}] = \varepsilon_{ijk}\partial_{v_i}(v_j B_k v_\ell \mathbf{e}_\ell) = \underbrace{\varepsilon_{ijk}\delta_{ij}}_{=0} B_k v_\ell \mathbf{e}_\ell + \varepsilon_{ijk} v_j B_k \delta_{i\ell}\mathbf{e}_\ell \tag{11}$$

$$= \varepsilon_{ijk} v_j B_k \mathbf{e}_i = \mathbf{v}\times\mathbf{B}. \tag{12}$$

So, we end up with

$$\int d^3v\,\mathbf{v}\,(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_s = -\int d^3v\, f_s\,(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = -n_s\,(\mathbf{E}+\mathbf{V}_s\times\mathbf{B})\,. \tag{13}$$

Our moment equation then becomes

$$\frac{\partial}{\partial t}(m_s n_s \mathbf{V}_s) = q_s n_s\,(\mathbf{E}+\mathbf{V}_s\times\mathbf{B}) - \nabla\cdot\Pi_s + m_s\int d^3v\,\mathbf{v}C_s + m_s\int d^3v\,\mathbf{v}S_s. \tag{14}$$

Now sum over species. We employ quasineutrality,

$$\sum_s q_s n_s = 0. \tag{15}$$

We can also define the current

$$\mathbf{J} = \sum_s q_s n_s \mathbf{V}_s \tag{16}$$

and total momentum flux tensor

$$\Pi = \sum_s \Pi_s. \tag{17}$$

We also have momentum conservation in collisions:

$$\sum_s m_s \int d^3v\,\mathbf{v}C_s = 0. \tag{18}$$

So, we are left with

$$\frac{\partial}{\partial t}\left(\sum_s m_s n_s \mathbf{V}_s\right) = \mathbf{J}\times\mathbf{B} - \nabla\cdot\Pi + \sum_s m_s\int d^3v\,\mathbf{v}S_s. \tag{19}$$

Often, Π is separated into terms associated with the mean flow and a pressure tensor associated with the deviations from that mean, but for the present calculation this step would make no difference in the end.

To get the result (19) we have not had to introduce any orderings or make significant approximations besides quasineutrality. It remains to show that the time derivative and source are small and that the momentum flux tensor is nearly isotropic.

3 Orderings

We will expand in the small parameter $\rho_* \ll 1$ where $\rho_* = \rho_i/L$, $\rho_i = v_i/\Omega_i$ is the ion gyroradius, $v_i = \sqrt{2T_i/m_i}$ is the ion thermal speed, T_i is the ion temperature, $\Omega_i = q_i B/m_i$ is the ion gyrofrequency, and L is the scale length of all leading-order quantities. If the macroscopic flow is ordered as large as the thermal speed, $V \sim v$ so the Mach number is

~ 1, a contradiction is reached in a general stellarator, as shown by Helander (2007). Therefore we will use a standard low-flow ordering, $V \sim \rho_* v$.

To order time derivatives, we expect there to be drift wave fluctuations with frequencies $\sim v/L = \rho_* \Omega$, suggesting $\partial/\partial t \sim v/L$. However we do not expect leading-order quantities to vary on this timescale.

Let us use the ordering to compare the sizes of the terms in (19). First compare $\mathbf{J} \times \mathbf{B}$ and $\nabla \cdot \Pi$:

$$\frac{\mathbf{J} \times \mathbf{B}}{\nabla \cdot \Pi} \sim \frac{qnVB}{nT/L} \sim \frac{qB}{m} \frac{mVL}{T} \sim \frac{\Omega L}{v} \frac{V}{v} \sim \frac{1}{\rho_*} \rho_* \sim 1. \tag{20}$$

So, those terms are comparable. Now consider the size of the time derivative term:

$$\frac{\frac{\partial}{\partial t}\left(\sum_s m_s n_s \mathbf{V}_s\right)}{\nabla \cdot \Pi} \sim \frac{\frac{v}{L} mnV}{nT/L} \sim \frac{vmV}{T} \sim \frac{V}{v} \sim \rho_*. \tag{21}$$

So, the time derivative term in (19) is small compared to the $\mathbf{J} \times \mathbf{B} \sim \nabla \cdot \Pi$ terms but could matter at the next order.

We expect the $\mathbf{E} \times \mathbf{B}$ velocity to be comparable to the total velocity, $\sim \rho_* v$. This implies the electrostatic potential Φ has a magnitude $e\Phi/T \sim 1$:

$$\frac{e\Phi}{T} \sim \frac{eBL}{T} \frac{\Phi/L}{B} \sim \frac{eBL}{m} \frac{m}{T} v_{E\times B} \sim \frac{\Omega L}{v} \frac{v_E}{v} \sim \frac{L}{\rho} \rho_* \sim 1. \tag{22}$$

The collision frequency is ordered as $\nu \sim \rho_* \Omega$, as is usual in gyrokinetics or drift-kinetics. We will not make any subsidiary ordering to differentiate banana vs Pfirsch-Schluter regimes.

For the source term, it will be sufficient to assume

$$S_s \ll \rho_* \Omega f_s. \tag{23}$$

4 Asymptotic solution of the kinetic equation

These next steps are essentially identical to the initial steps in the derivation of the drift-kinetic equation.

4.1 Velocity variables

We introduce cylindrical velocity-space coordinates $\left(v_\perp, \vartheta, v_{||}\right)$ so that $\mathbf{v} = v_{||}\mathbf{b} + \mathbf{v}_\perp$ where

$$\mathbf{v}_\perp = v_\perp \left(\mathbf{e}_1 \cos\vartheta + \mathbf{e}_2 \sin\vartheta\right), \tag{24}$$

$\mathbf{b} = B^{-1}\mathbf{B}$, $B = |\mathbf{B}|$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors orthogonal to $\mathbf{B}$ such that $(\mathbf{b}, \mathbf{e}_1, \mathbf{e}_2)$ form a right-handed orthonormal system. A brief calculation gives

$$\nabla_v Q = \mathbf{b}\left(\frac{\partial Q}{\partial v_{||}}\right)_{v_\perp,\vartheta} + \frac{\mathbf{v}_\perp}{v_\perp}\left(\frac{\partial Q}{\partial v_\perp}\right)_{v_{||},\vartheta} + \frac{1}{v_\perp^2}\mathbf{b} \times \mathbf{v}_\perp \left(\frac{\partial Q}{\partial \vartheta}\right)_{v_{||},v_\perp} \tag{25}$$

for any quantity Q.

Next we introduce the magnetic moment

$$\mu = \frac{v_\perp^2}{2B} \tag{26}$$

and total specific energy

$$W_s = \frac{v^2}{2} + \frac{q_s \Phi_0}{m_s} \tag{27}$$

where Φ_0 is the leading-order electrostatic potential. Then

$$v_{||} = \sigma\sqrt{v^2 - v_\perp^2} = \sigma\sqrt{2\left(W_s - \frac{q_s\Phi_0}{m_s} - \mu B\right)}. \tag{28}$$

Later we will need to integrate over all velocity. To do this we first observe

$$\int d^3v = \int_0^{2\pi} d\vartheta \int_{-\infty}^{\infty} dv_{||} \int_0^{\infty} dv_\perp v_\perp. \tag{29}$$

Using (26),

$$\int d^3v = \int_0^{2\pi} d\vartheta \int_{-\infty}^{\infty} dv_{||} \int_0^{\infty} d\mu\, B. \tag{30}$$

We now write the energy as

$$W_s = \frac{v_{||}^2}{2} + \mu B + \frac{q_s \Phi_0}{m_s} \tag{31}$$

so

$$dW_s = v_{||} dv_{||}. \tag{32}$$

Hence

$$\begin{aligned} \int d^3v &= \sum_\sigma \int_0^{2\pi} d\vartheta \int_0^{\infty} d\mu \int_{\mu B + q_s\Phi_0/m_s}^{\infty} dW_s \frac{B}{|v_{||}|} \\ &= \sum_\sigma \int_0^{2\pi} d\vartheta \int_{q_s\Phi_0/m_s}^{\infty} dW_s \int_0^{(W_s - q_s\Phi_0/m_s)/B} d\mu \frac{B}{|v_{||}|}. \end{aligned} \tag{33}$$

In contrast to gyrokinetics, for this calculation we do not need to change variables from the original position vector $\mathbf{r}$ to the guiding-center position.

4.2 Change of variables in the kinetic equation

We now return to (2), writing it as

$$D_s f_s = C_s + S_s \tag{34}$$

where

$$D_s = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v \tag{35}$$

is the Vlasov operator. To change to the new variables in the kinetic equation, we must evaluate D_s applied to each of the new variables. A bit of algebra gives

$$D_s W_s = \frac{q_s}{m_s}\left(\mathbf{E}^*\cdot\mathbf{v} + \frac{\partial \Phi_0}{\partial t}\right) \tag{36}$$

where

$$\mathbf{E}^* = \mathbf{E} + \nabla\Phi_0. \tag{37}$$

We also find

$$D_s \mu = -\frac{\mu}{B}\mathbf{v}\cdot\nabla B - \frac{v_{||}\mathbf{v}\mathbf{v} : \nabla\mathbf{b}}{B} + \frac{q_s}{m_s B}\mathbf{E}\cdot\mathbf{v}_\perp \tag{38}$$

and

$$D_s \vartheta = -\Omega_s + G_s \tag{39}$$

where G_s is an ugly bunch of terms of order $\rho_* \Omega_s$ arising from $(\nabla\vartheta)_\mathbf{v}$. Thus, the Fokker-Planck equation can be written

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla f_s + (D_s W_s)\frac{\partial f_s}{\partial W_s} + (D_s\mu)\frac{\partial f_s}{\partial \mu} + (D_s\vartheta)\frac{\partial f_s}{\partial \vartheta} = C_s + S_s. \tag{40}$$

Here and for the rest of the document, partial derivatives are taken at fixed (μ, W_s, ϑ).

We now introduce the gyro-averaging operation, denoted with an over-bar:

$$\bar{Q}=\frac{1}{2\pi}\int_0^{2\pi}d\vartheta\,Q \tag{41}$$

for any quantity Q, where the position $\mathbf{r}$, W_s, and μ are held fixed in the integral. Notice

$$\overline{D_sW_s}=\frac{q_s}{m_s}\left(E_{||}^{*}v_{||}+\frac{\partial\Phi_0}{\partial t}\right). \tag{42}$$

To compute $\overline{D_s\mu}$, we use

$$\overline{\mathbf{vv}}=v_{||}^2\mathbf{bb}+\frac{v_\perp^2}{2}\left(\mathbf{I}-\mathbf{bb}\right) \tag{43}$$

and $\mathbf{bb}:\nabla\mathbf{b}=0$ to obtain $\overline{D_s\mu}=0$. Introducing $\tilde{f}=f-\bar{f}$, it will turn out to be convenient to write the kinetic equation as

$$\frac{\partial\bar{f}_s}{\partial t}+\mathbf{v}\cdot\nabla\bar{f}_s+\left(D_sW_s\right)\frac{\partial\bar{f}_s}{\partial W_s}+\left(D_s\mu\right)\frac{\partial\bar{f}_s}{\partial\mu}+D_s\tilde{f}_s=C_s+S_s. \tag{44}$$

The gyro-average of (40) is

$$\frac{\partial\bar{f}_s}{\partial t}+v_{||}\mathbf{b}\cdot\nabla\bar{f}_s+\frac{q_s}{m_s}\left(E_{||}^{*}v_{||}+\frac{\partial\Phi_0}{\partial t}\right)\frac{\partial\bar{f}_s}{\partial W_s}+\overline{D\tilde{f}_s}=\bar{C}_s+\bar{S}_s. \tag{45}$$

4.3 Leading order

We now expand the distribution in ρ_*:

$$f_s=f_{s0}+f_{s1}+\ldots \tag{46}$$

The largest term in (40) comes from the $-\Omega_s$ in $D_s\vartheta$, giving a term of size $\Omega_s f_{s0}$:

$$-\Omega_s\frac{\partial f_{s0}}{\partial\vartheta}=0, \tag{47}$$

so $\partial f_{s0}/\partial\vartheta=0$. Therefore $\tilde{f}_{s0}=0$ and $\tilde{f}_s\sim\rho_*\bar{f}_s$. We therefore will not write an overbar on f_{s0}.

4.4 Next order

Next, we proceed to the leading terms in (45). These are of size $\rho_*\Omega_s f_{s0}$ and are

$$v_{||}\mathbf{b}\cdot\nabla f_{s0}=\overline{C_s\left\{f_{s0}\right\}}, \tag{48}$$

where we have used $\overline{f_{s0}}=f_{s0}$. It turns out that for this equation to be satisfied, the right and left hand sides must both separately be 0. We show this rigorously now.

Multiplying by $(1+\ln f_{s0})$, and using $d\left(f\ln f\right)=df+\ln f\,df=\left(1+\ln f\right)df$, we have

$$v_{||}\mathbf{b}\cdot\nabla\left(f_{s0}\ln f_{s0}\right)=\left(1+\ln f_{s0}\right)\overline{C_s\left\{f_{s0}\right\}}. \tag{49}$$

Now integrate over velocity space by applying (33):

$$\sum_\sigma\sigma\int_0^{2\pi}d\vartheta\int_0^{\infty}d\mu\int_{\mu B+q_s\Phi_0/m_s}^{\infty}dW_s\,\mathbf{B}\cdot\nabla\left(f_{s0}\ln f_{s0}\right)=\int d^3v\left(1+\ln f_{s0}\right)C_s\left\{f_{s0}\right\}. \tag{50}$$

On the right-hand side we have removed the gyroaverage because it is superfluous with the ϑ integral in $\int d^3v$. Due to density conservation by collisions, $\int d^3v\,C_s=0$, so the "1" term on the right vanishes. On the left-hand side, it turns out that we can pull the $\mathbf{B}\cdot\nabla$ outside the integrals. This requires a bit of care because the endpoint of the W_s integral depends on position through B and Φ. The rigorous argument is as follows.

The Leibniz rule for differentiation of an integral with moving endpoints is

$$\frac{d}{dx}\int_{a(x)}^{b(x)} dy\, g(x,y) = \int_{a(x)}^{b(x)} dy \frac{\partial g}{\partial x} + \frac{db}{dx} g(x, b(x)) - \frac{da}{dx} g(x, a(x)). \tag{51}$$

In our case, let the integrand be a function $G_{s\sigma}(\mathbf{r}, W_s, \mu)$ which depends on phase space. The Leibniz rule applied to the W_s integral gives

$$\begin{aligned}\mathbf{B}\cdot\nabla \int_{\mu B + q_s\Phi_0/m_s}^{\infty} dW_s\, G_{s\sigma} &= \int_{\mu B + q_s\Phi_0/m_s}^{\infty} dW_s\, \mathbf{B}\cdot\nabla G_{s\sigma} \\ &\quad - \left(\mathbf{B}\cdot\nabla\left[\mu B + \frac{q_s\Phi_0}{m_s}\right]\right) G_{s\sigma}\left(\mathbf{r}, \mu B + \frac{q_s\Phi_0}{m_s}, \mu\right).\end{aligned} \tag{52}$$

The term on the last line is not generally 0. However in this term we evaluate $G_{s\sigma}$ at $v_{||} = 0$ (because $W_s = \mu B + q_s\Phi_0/m_s$), and $G_{s\sigma}$ must be independent of σ at this point in phase space. Therefore when we sum over $\sigma = \pm 1$ with a factor of σ, the last line vanishes:

$$\mathbf{B}\cdot\nabla\sum_\sigma \sigma \int_{\mu B + q_s\Phi_0/m_s}^{\infty} dW_s\, G_{s\sigma} = \sum_\sigma \sigma \int_{\mu B + q_s\Phi_0/m_s}^{\infty} dW_s\, \mathbf{B}\cdot\nabla G_{s\sigma}. \tag{53}$$

Thus, from (50) we arrive at

$$\mathbf{B}\cdot\nabla\sum_\sigma \sigma \int_0^{2\pi} d\vartheta \int_0^\infty d\mu \int_{\mu B + q_s\Phi/m_s}^{\infty} dW_s f_{s0} \ln f_{s0} = \int d^3v \ln f_{s0} C_s\{f_{s0}\}. \tag{54}$$

At this point we assume that good nested flux surfaces exist. With this assumption, we can apply a flux surface average to both sides, giving:

$$\left\langle \mathbf{B}\cdot\nabla\sum_\sigma \sigma \int_0^{2\pi} d\vartheta \int_0^\infty d\mu \int_{\mu B + q_s\Phi_0/m_s}^{\infty} dW_s\, f_{s0} \ln f_{s0} \right\rangle = \left\langle \int d^3v \ln f_{s0} C_s\{f_{s0}\} \right\rangle. \tag{55}$$

The left-hand side vanishes due to the general result $\langle \mathbf{B}\cdot\nabla w\rangle = 0$ for any single-valued quantity w. Thus,

$$\left\langle \int d^3v \ln f_{s0} C_s\{f_{s0}\} \right\rangle = 0. \tag{56}$$

4.5 H theorem

It can be shown that

$$\int d^3v \ln f_{s0} C_s\{f_{s0}\} \leq 0. \tag{57}$$

Due to this sign-definiteness, eq (56) implies that the quantity inside the average must be 0 even without the average:

$$\int d^3v \ln f_{s0} C_s\{f_{s0}\} = 0. \tag{58}$$

The only solution of this equation is a Maxwellian:

$$f_{s0} = n_s\left(\frac{m_s}{2\pi T_s}\right)^{3/2} \exp\left(-\frac{m_s}{2T_s}(\mathbf{v} - \mathbf{V}_{s0})^2\right) \tag{59}$$

where the density n_s, temperature T_s, and mean flow $\mathbf{V}_{s0}$ are allowed to depend on position. The results (57)-(59) are known as the H-theorem and are proved in notes on the group website. There are several ways to handle the H-theorem for multiple species, which are discussed in one of these notes, which either allow for different temperatures or force the temperatures to be equal; either approach will give the same result here in the end. While it could be reasonable to set $\mathbf{V}_{s0} = 0$ due to our ordering assumption that the Mach number is small, we can also allow for a nonzero $\mathbf{V}_{s0}$ for now and then rigorously derive that it must vanish.

Due to $\partial f_{s0}/\partial\vartheta = 0$, any $\mathbf{V}_{s0}$ must be parallel to $\mathbf{B}$, so we can write $\mathbf{V}_{s0} = V_{s0}\mathbf{b}$, and

$$f_{s0} = n_s \left(\frac{m_s}{2\pi T_s}\right)^{3/2} \exp\left(-\frac{m_s}{2T_s}\left(v_\perp^2 + v_{||}^2 - 2v_{||}V_{s0} + V_{s0}^2\right)\right). \tag{60}$$

Equivalently,

$$f_{s0} = N_s \left(\frac{m_s}{2\pi T_s}\right)^{3/2} \exp\left(-\frac{m_s W_s}{T_s} + \frac{m_s v_{||} V_{s0}}{T_s}\right) \tag{61}$$

where

$$N_s(\mathbf{r}) = n_s \exp\left(\frac{q_s \Phi_0}{T_s} - \frac{m_s V_{s0}^2}{2T_s}\right). \tag{62}$$

4.6 More information from the 1st order equation

We can now return to (48), which reduces to

$$v_{||}\mathbf{b}\cdot\nabla f_{s0} = 0. \tag{63}$$

Let us substitute our Maxwellian into this equation. Since we will then need to differentiate our Maxwellian, it is useful to first differentiate

$$\frac{v_{||}^2}{2} = W_s - \frac{q_s\Phi_0}{m_s} - \mu B \tag{64}$$

with the result

$$v_{||}\mathbf{b}\cdot\nabla v_{||} = -\frac{q_s}{m_s}\mathbf{b}\cdot\nabla\Phi_0 - \mu\mathbf{b}\cdot\nabla B. \tag{65}$$

Thus, when the Maxwellian (61) is substituted into (63), we obtain (after dividing through by f_{s0})

$$\frac{v_{||}}{N_s}\mathbf{b}\cdot\nabla N_s - \frac{3v_{||}}{2T_s}\mathbf{b}\cdot\nabla T_s + v_{||}\frac{m_s}{T_s^2}\left(W_s - v_{||}V_{s0}\right)\mathbf{b}\cdot\nabla T_s + \frac{m_s V_{s0}}{T_s}\left(-\frac{q_s}{m_s}\mathbf{b}\cdot\nabla\Phi_0 - \mu\mathbf{b}\cdot\nabla B\right) + \frac{m_s v_{||}^2}{T_s}\mathbf{b}\cdot\nabla V_{s0} = 0. \tag{66}$$

This equation must hold at all points in phase space, so each factor with a distinct velocity dependence must vanish separately. There is one term with velocity dependence $\propto \mu$, so this term must vanish:

$$-\frac{m_s V_{s0}}{T_s}\mu\mathbf{b}\cdot\nabla B = 0. \tag{67}$$

For a toroidal plasma, $\mathbf{b}\cdot\nabla B$ is nonzero everywhere except on a set of measure zero. Therefore the only way for this term to vanish everywhere in phase space with continuous V_{s0} is for $V_{s0} = 0$. So, the Maxwellian has no mean flow. Next, there is one term in (66) with velocity dependence $\propto v_{||}W_s$, and this term must vanish:

$$v_{||}\frac{m_s}{T_s^2}W_s\mathbf{b}\cdot\nabla T_s = 0. \tag{68}$$

The only way for this term to vanish everywhere in phase space is if

$$\mathbf{b}\cdot\nabla T_s = 0. \tag{69}$$

The only remaining term in (66) is

$$\frac{v_{||}}{N_s}\mathbf{b}\cdot\nabla N_s = 0 \tag{70}$$

so

$$\mathbf{b}\cdot\nabla N_s = 0. \tag{71}$$

If ψ labels the magnetic flux surfaces, we therefore have $T_s = T_s(\psi)$ and $N_s = N_s(\psi)$, with

$$n_s = N_s(\psi)\exp\left(-\frac{q_s\Phi_0}{T_s}\right). \tag{72}$$

4.7 Final momentum balance equation

From (59), now that we know $\mathbf{V}_{s0} = 0$, we have proven that the leading-order distribution function is a stationary and isotropic Maxwellian:

$$f_{s0} = n_s \left(\frac{m_s}{2\pi T_s}\right)^{3/2} \exp\left(-\frac{m_s v^2}{2T_s}\right). \tag{73}$$

Evaluating the momentum flux tensor (17) with (6) gives

$$\Pi = p\mathbf{I} \tag{74}$$

where

$$p = \sum_s n_s T_s \tag{75}$$

is the total scalar pressure and $\mathbf{I}$ is the identity tensor. Thus, the leading order (in ρ_*) terms in the momentum balance equation (19) are

$$0 = \mathbf{J} \times \mathbf{B} - \nabla p, \tag{76}$$

as desired.

4.8 Quasineutrality

While we have already derived the desired result $\mathbf{J} \times \mathbf{B} = \nabla p$, there is a bit of additional information that can be extracted from the results so far, by examining the quasineutrality condition. Consider for concreteness the case of a pure hydrogenic plasma, with ions ($s = i$, $q_i = e$ with e the elementary charge) and electrons ($s = e$, $q_e = -e$). Then the quasineutrality equation is

$$N_i(\psi) \exp\left(-\frac{e\Phi_0}{T_i}\right) = N_e(\psi) \exp\left(\frac{e\Phi_0}{T_e}\right). \tag{77}$$

Rearranging,

$$\frac{N_i(\psi)}{N_e(\psi)} = \exp\left(\frac{e\Phi_0}{T_e} + \frac{e\Phi_0}{T_i}\right). \tag{78}$$

The left-hand side is constant on a flux surface, so the right-hand side must be so as well. This is only possible if Φ_0 is constant on the surfaces:

$$\Phi_0 = \Phi_0(\psi). \tag{79}$$

It follows then from (72) that each density is also constant on surfaces:

$$n_i = n_i(\psi), \quad n_e = n_e(\psi) = n_i. \tag{80}$$