Width of magnetic islands

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In this note we will show that the width w of a magnetic island can be described by

$$w \propto \sqrt{\frac{B_{mn}}{m\iota'}} \tag{1}$$

where m is the mode number, B_{mn} is the amplitude of a radial magnetic field perturbation that resonates with the field line pitch (i.e. with the rotational transform), and $\iota' = d\iota/d\Psi$ is the magnetic shear, with Ψ the toroidal flux. (The precise definition of B_{mn} is given below.) Some important consequences of this result are

1. Increasing magnetic shear makes islands smaller.
2. Due to the power $1/2$ in the formula, even a small resonant field B_{mn} can result in a large island.
3. Higher m islands are smaller, both due to the explicit m factor and the fact that the Fourier coefficients B_{mn} will be smaller.

This note follows page 1078 of Boozer, "Physics of magnetically confined plasmas" (2004). A different approach to get the result using the field line Hamiltonian can be found in Helander "Theory of plasma confinement in non-axisymmetric magnetic fields" (2014).

We suppose the total magnetic field $\mathbf{B}$ is not too dissimilar from a field $\mathbf{B}_0$ that does have nested magnetic surfaces:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 \tag{2}$$

where

$$|\mathbf{B}_1| \ll |\mathbf{B}_0|. \tag{3}$$

We call $\mathbf{B}_0$ the "unperturbed field." Let p be some label (not necessarily pressure) for the surfaces of the perturbed field:

$$p = p_0 + p_1 \tag{4}$$

where

$$|p_1| \ll |p_0|. \tag{5}$$

The condition that p is constant along the field is

$$\mathbf{B} \cdot \nabla p = 0 \tag{6}$$

At leading order,

$$\mathbf{B}_0 \cdot \nabla p_0 = 0. \tag{7}$$

At next order,

$$\mathbf{B}_0 \cdot \nabla p_1 + \mathbf{B}_1 \cdot \nabla p_0 = 0. \tag{8}$$

This is a magnetic differential equation for p_1, so for solving it, it is helpful to introduce magnetic coordinates.

Now, let $\Psi(p_0)$ be the toroidal flux of the unperturbed field, and let (θ, φ) be some set of straight-field-line coordinates for this field, so

$$\mathbf{B}_0 = \frac{1}{2\pi}\left(\nabla\Psi \times \nabla\theta + \iota\nabla\varphi \times \nabla\Psi\right). \tag{9}$$

Then eq (8) becomes

$$\mathbf{B}_0 \cdot \nabla\theta \frac{\partial p_1}{\partial \theta} + \mathbf{B}_0 \cdot \nabla\varphi \frac{\partial p_1}{\partial \varphi} = -\mathbf{B}_1 \cdot \nabla p_0. \tag{10}$$

The first two terms can be grouped as

$$\mathbf{B}_0 \cdot \nabla\varphi \left(\iota \frac{\partial p_1}{\partial \theta} + \frac{\partial p_1}{\partial \varphi} \right) = -\mathbf{B}_1 \cdot \nabla p_0 \tag{11}$$

and we can then rearrange to get

$$\iota \frac{\partial p_1}{\partial \theta} + \frac{\partial p_1}{\partial \varphi} = -\frac{\mathbf{B}_1 \cdot \nabla p_0}{\mathbf{B}_0 \cdot \nabla\varphi} = -\frac{dp_0}{d\Psi} \frac{\mathbf{B}_1 \cdot \nabla\Psi}{\mathbf{B}_0 \cdot \nabla\varphi}. \tag{12}$$

The total (perturbed) field is not perfectly normal to Ψ surfaces, so $\mathbf{B} \cdot \nabla\Psi = \mathbf{B}_1 \cdot \nabla\Psi$ is nonzero. We are free to write this quantity as a Fourier series in the two angles (θ, φ). Actually because of the right-hand side of (12), it turns out to be more convenient to add a factor of

$$\mathbf{B}_0 \cdot \nabla\varphi = \frac{1}{2\pi} \nabla\Psi \cdot \nabla\theta \times \nabla\varphi \tag{13}$$

first before forming the Fourier series. We thus define the Fourier amplitudes B_{mn}:

$$\frac{\mathbf{B} \cdot \nabla\Psi}{\mathbf{B}_0 \cdot \nabla\varphi} = \sum_{m,n} \left[B^s_{mn} \sin(m\theta - n\varphi) + B^c_{mn} \cos(m\theta - n\varphi) \right]. \tag{14}$$

For simplicity, here we will analyze the case with a single sine term. The cosine terms could be analyzed in an analogous way. Dropping the superscript to simplify notation,

$$\frac{\mathbf{B} \cdot \nabla\Psi}{\mathbf{B}_0 \cdot \nabla\varphi} = B_{mn} \sin(m\theta - n\varphi). \tag{15}$$

(Boozer's definition differs by a minus sign.) You can think of this equation as the definition of B_{mn}.

Returning now to (12), we have

$$\iota \frac{\partial p_1}{\partial \theta} + \frac{\partial p_1}{\partial \varphi} = -\frac{dp_0}{d\Psi} B_{mn} \sin(m\theta - n\varphi). \tag{16}$$

Guess a solution of the form

$$p_1 = \hat{p} \cos(m\theta - n\varphi), \tag{17}$$

so

$$\frac{\partial p_1}{\partial \theta} = -m\hat{p} \sin(m\theta - n\varphi), \tag{18}$$

$$\frac{\partial p_1}{\partial \varphi} = n\hat{p} \sin(m\theta - n\varphi). \tag{19}$$

Then (16) becomes

$$\hat{p} = \frac{1}{\iota m - n} \frac{dp_0}{d\Psi} B_{mn} \tag{20}$$

so

$$p_1 = \frac{1}{\iota m - n} \frac{dp_0}{d\Psi} B_{mn} \cos(m\theta - n\varphi). \tag{21}$$

You can see that p_1 will diverge at the rational surface $\iota = n/m$ unless $dp_0/d\Psi \to 0$ there. We can linearize $dp_0/d\Psi$ in the neighborhood of the rational surface by writing

$$\frac{dp_0}{d\Psi} \approx (\iota m - n)\, c \tag{22}$$

for some constant c so this factor goes to 0 near the rational surface equally fast to the resonant denominator. Then

$$p_1 = cB_{mn} \cos(m\theta - n\varphi). \tag{23}$$

Near the rational surface,

$$\iota(\Psi) \approx \frac{n}{m} + \frac{d\iota}{d\Psi}(\Psi - \Psi_{mn}) \tag{24}$$

where Ψ_{mn} is the flux at the rational surface. Plugging this expansion into (22),

$$\frac{dp_0}{d\Psi} \approx m\frac{d\iota}{d\Psi}(\Psi - \Psi_{mn})\, c \tag{25}$$

which can be integrated with respect to Ψ to get

$$p_0(\Psi) \approx p_0(\Psi_{mn}) + \frac{cm}{2}\frac{d\iota}{d\Psi}(\Psi - \Psi_{mn})^2. \tag{26}$$

We can now form the total surface label $p = p_0 + p_1$:

$$p = p_0(\Psi_{mn}) + c\left[\frac{m}{2}\frac{d\iota}{d\Psi}(\Psi - \Psi_{mn})^2 + B_{mn}\cos(m\theta - n\varphi)\right]. \tag{27}$$

Now apply the trigonometric identity

$$\cos x = 1 - 2\sin^2(x/2) \tag{28}$$

to write

$$p = p_0(\Psi_{mn}) + cB_{mn} + c\left[\frac{m}{2}\frac{d\iota}{d\Psi}(\Psi - \Psi_{mn})^2 - 2B_{mn}\sin^2\left(\frac{m\theta - n\varphi}{2}\right)\right]. \tag{29}$$

Let us rearrange this expression to solve for $\Psi - \Psi_{mn}$:

$$\frac{p - p_0(\Psi_{mn}) - cB_{mn}}{c} = \left[\frac{m}{2}\frac{d\iota}{d\Psi}(\Psi - \Psi_{mn})^2 - 2B_{mn}\sin^2\left(\frac{m\theta - n\varphi}{2}\right)\right]. \tag{30}$$

$$(\Psi - \Psi_{mn})^2 = \frac{p - p_0(\Psi_{mn}) - cB_{mn}}{\frac{cm}{2}\frac{d\iota}{d\Psi}} + \frac{4B_{mn}}{m\frac{d\iota}{d\Psi}}\sin^2\left(\frac{m\theta - n\varphi}{2}\right). \tag{31}$$

The first terms on the right-hand side depend on position only through p, so let's introduce a new label of the total surfaces s to collect these terms, defined by

$$\frac{p - p_0(\Psi_{mn}) - cB_{mn}}{\frac{cm}{2}\frac{d\iota}{d\Psi}} = s\frac{4B_{mn}}{m\frac{d\iota}{d\Psi}}. \tag{32}$$

(Boozer writes s^2 instead of s but I believe it makes more sense to use s.) The surface shapes are then described by

$$(\Psi - \Psi_{mn})^2 = \frac{4B_{mn}}{m\frac{d\iota}{d\Psi}}\left[s + \sin^2\left(\frac{m\theta - n\varphi}{2}\right)\right]. \tag{33}$$

so

$$\Psi - \Psi_{mn} = \pm\sqrt{\frac{4B_{mn}}{m\frac{d\iota}{d\Psi}}\left[s + \sin^2\left(\frac{m\theta - n\varphi}{2}\right)\right]}. \tag{34}$$

To see the shapes of the perturbed surfaces, fix s (which labels the perturbed surfaces) and vary θ and/or φ. If $s > 0$, we can access any θ and φ, while Ψ oscillates. This corresponds to intact flux surfaces that are now rippled. If $s < 0$, we can no longer access all θ and φ because the argument of the square root would become negative. This case corresponds to the interior of the island. The minimum s for which there are any solutions is $s = -1$. The separatrix is $s = 0$, in which case we can just barely access all θ and φ before the radicand would become negative.

To plot the surfaces including the islands, we can rearrange the above formula as

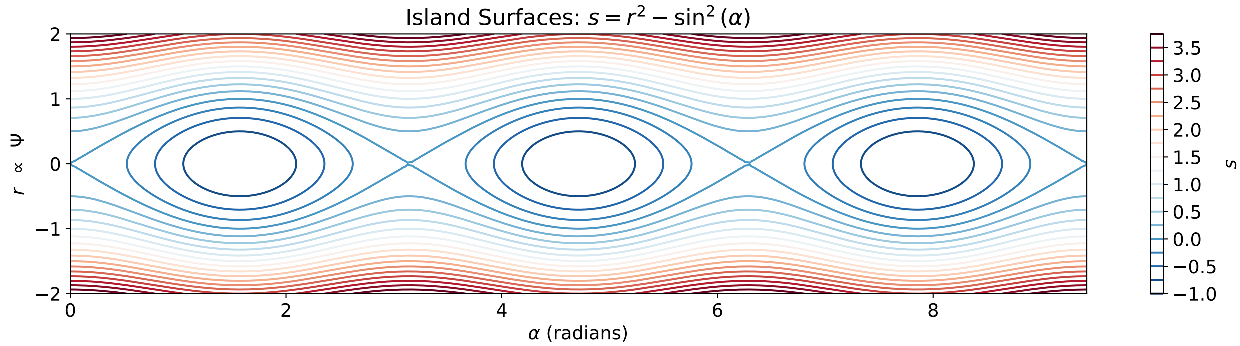
$$s = (\Psi - \Psi_{mn})^2\frac{m\frac{d\iota}{d\Psi}}{2B_{mn}} - \sin^2\left(\frac{m\theta - n\varphi}{2}\right) = r^2 - \sin^2\alpha \tag{35}$$

where

$$r = (\Psi - \Psi_{mn})\sqrt{\frac{m\frac{d\iota}{d\Psi}}{2B_{mn}}} \tag{36}$$

$$\alpha = \frac{m\theta - n\varphi}{2} \tag{37}$$

and the results look as follows:



The full width of the island corresponds to the maximum variation in Ψ on the separatrix, which is

$$\Delta\Psi = 4\sqrt{\frac{B_{mn}}{m\frac{d\iota}{d\Psi}}}. \tag{38}$$

To get this expression there is a factor of 2 from the difference between Ψ along the $+$ root Ψ compared to along the $-$ root. To get the full width w in real space, we can divide by $|\nabla\Psi|$: $w = \Delta\Psi/|\nabla\Psi|$. This yield our main result:

$$\boxed{w = \frac{4}{|\nabla\Psi|}\sqrt{\frac{B_{mn}}{m\frac{d\iota}{d\Psi}}}.} \tag{39}$$