Relationships between bounce-averaged drifts and the longitudinal invariant

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1 Overview

In this note we will show the equivalence of two ways to define omnigenity: "the bounce-averaged radial drift is zero" and "J is a flux function," where $J = \oint v_{||} d\ell$ is the longitudinal adiabatic invariant. We will also show that J is indeed conserved by the bounce-averaged motion.

To prove both of these facts, we will first derive a general result

$$\langle \mathbf{v}_d \cdot \nabla \alpha \rangle = \frac{m}{Ze\tau_b} \frac{\partial J}{\partial \gamma}, \tag{1}$$

$$\langle \mathbf{v}_d \cdot \nabla \gamma \rangle = -\frac{m}{Ze\tau_b} \frac{\partial J}{\partial \alpha}, \tag{2}$$

where $\mathbf{v}_d$ indicates the magnetic drifts, $\langle \ldots \rangle$ indicates a bounce averaged, and the other symbols are defined in the next section. Therefore, if we take γ to be a radial coordinate and α to be a field line label, we see $\langle \mathbf{v}_d \cdot \nabla \gamma \rangle = 0 \Leftrightarrow \partial J / \partial \alpha = 0$.

2 Definitions

We take the magnetic field to be given by

$$\mathbf{B} = \nabla \alpha \times \nabla \gamma \tag{3}$$

for some coordinates α and γ, which are otherwise left general. The field does not need to satisfy MHD equilibrium or any other condition, other than being divergence-free. We will use independent variables (α, γ, ℓ), where ℓ is arclength along the field, satisfying $\mathbf{B} \cdot \nabla \ell = B$. Note that the Jacobian of these coordinates is

$$\sqrt{g} = \frac{1}{\nabla \alpha \times \nabla \gamma \cdot \nabla \ell} = \frac{1}{\mathbf{B} \cdot \nabla \ell} = \frac{1}{B}. \tag{4}$$

We can write the parallel velocity as

$$|v_{||}| = \sqrt{v_{||}^2} = \sqrt{v^2 - v_\perp^2} = v\sqrt{1 - \frac{v_\perp^2}{v^2}} = v\sqrt{1 - \frac{mv_\perp^2}{2B}\frac{2}{mv^2}B} = v\sqrt{1 - \frac{\mu}{W}B} = v\sqrt{1 - \lambda B}, \tag{5}$$

where $\lambda = \mu / W$ is the ratio of magnetic moment $\mu = mv_\perp^2 / (2B)$ to kinetic energy $W = mv^2/2$, and m is the particle mass. The J invariant can then be written as

$$J = 2\int_{\ell_-}^{\ell_+} |v_{||}| \, d\ell = 2v \int_{\ell_-}^{\ell_+} \sqrt{1 - \lambda B} d\ell. \tag{6}$$

Here, ℓ_- and ℓ_+ are the bounce points, i.e. the values of ℓ at which $v_{||} = 0$. The time for a particle to complete a full bounce is

$$\tau_b = \int dt = 2\int_{\ell_-}^{\ell_+} \frac{d\ell}{|v_{||}|}. \tag{7}$$

The bounce average of any quantity q is

$$\langle q \rangle = \frac{2}{\tau_b}\int_{\ell_-}^{\ell_+} \frac{q\, d\ell}{|v_{||}|} = \frac{\int_{\ell_-}^{\ell_+} \frac{q\,d\ell}{|v_{||}|}}{\int_{\ell_-}^{\ell_+} \frac{d\ell}{|v_{||}|}}. \tag{8}$$

3 Expression for the drifts

To proceed, we will use the following handy formula for the drifts:

$$\mathbf{v}_d = \frac{mv_{||}}{ZeB}\nabla\times\left(v_{||}\mathbf{b}\right) + (\text{small term} \parallel \text{to } \mathbf{B}), \tag{9}$$

where Ze is the particle charge. We must be careful to define which velocity coordinates are considered fixed while taking the gradient. If ∇ is performed at fixed μ and W, we get the magnetic drifts (∇B and curvature drift). If ∇ is performed at fixed μ and total energy $W + Ze\Phi$ for an electrostatic potential Φ, we get the magnetic and $\mathbf{E}\times\mathbf{B}$ drifts. Let us now prove this result for the former case - throughout this note we will neglect electric fields and time derivatives for simplicity.

To prove (9), we first write

$$\frac{mv_{||}}{ZeB}\nabla\times\left(v_{||}\mathbf{b}\right) = \frac{mv_{||}^2}{ZeB}\nabla\times\mathbf{b} + \frac{mv_{||}}{ZeB}\left(\nabla v_{||}\right)\times\mathbf{b}. \tag{10}$$

In the second term,

$$\nabla v_{||} = \nabla\left(v\sqrt{1-\lambda B}\right) = \frac{v}{2\sqrt{1-\lambda B}}\left(-\lambda\nabla B\right) = -\frac{v^2\lambda\nabla B}{2v_{||}} \tag{11}$$

so we get a term

$$\frac{mv_{||}}{ZeB}\left(\nabla v_{||}\right)\times\mathbf{b} = -\frac{mv^2\lambda}{2ZeB}\nabla B\times\mathbf{b} = \frac{mv_\perp^2}{2ZeB^2}\mathbf{b}\times\nabla B, \tag{12}$$

which is the ∇B drift. For the first right-hand side term of (10), we can use the identity

$$\mathbf{b}\times\boldsymbol{\kappa} = \mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b}) = \mathbf{b}\times[(\nabla\times\mathbf{b})\times\mathbf{b}] = (\nabla\times\mathbf{b})\,\mathbf{b}\cdot\mathbf{b} - \mathbf{b}\mathbf{b}\cdot\nabla\times\mathbf{b} = \nabla\times\mathbf{b} - \mathbf{b}\mathbf{b}\cdot\nabla\times\mathbf{b}. \tag{13}$$

Therefore the first right-hand-side term of (10) is

$$\frac{mv_{||}^2}{ZeB}\nabla\times\mathbf{b} = \frac{mv_{||}^2}{ZeB}\mathbf{b}\times\boldsymbol{\kappa} + \frac{mv_{||}^2}{ZeB}\mathbf{b}\mathbf{b}\cdot\nabla\times\mathbf{b}, \tag{14}$$

which is the curvature drift plus a parallel term. Thus we have demonstrated (9) for the case of fixed W. Note that the parallel term in (9) does not affect $\mathbf{v}_d\cdot\nabla\alpha$ or $\mathbf{v}_d\cdot\nabla\gamma$ since $\mathbf{B}\cdot\nabla\alpha = 0$ and $\mathbf{B}\cdot\nabla\gamma = 0$.

4 Derivative of J

Next let us compute a derivative of J. In principle we should include contributions from the movement of the endpoints, although these contributions end up vanishing in this case:

$$\frac{\partial J}{\partial\gamma} = v\int_{\ell_-}^{\ell_+}\frac{d\ell}{\sqrt{1-\lambda B}}\left(-\lambda\frac{\partial B}{\partial\gamma}\right) + \left[2\left|v_{||}\right|\frac{\partial\ell_+}{\partial\gamma}\right]_{\ell_+} - \left[2\left|v_{||}\right|\frac{\partial\ell_-}{\partial\gamma}\right]_{\ell_-}. \tag{15}$$

The boundary terms vanish since $v_{||} = 0$ at $\ell = \ell_+$ and $\ell = \ell_-$. We are left with

$$\frac{\partial J}{\partial\gamma} = v\int_{\ell_-}^{\ell_+}\frac{d\ell}{\sqrt{1-\lambda B}}\left(-\lambda\frac{\partial B}{\partial\gamma}\right). \tag{16}$$

5 Bounce averaged drift

Now let us compute the bounce-averaged drifts, to compare to the preceding equation.

$$\langle \mathbf{v}_d \cdot \nabla \alpha \rangle = \frac{m}{Ze} \frac{2}{\tau_b} \int_{\ell_-}^{\ell_+} \frac{d\ell}{|v_{||}|} \frac{|v_{||}|}{B} \left[\nabla \times \left(|v_{||}| \mathbf{b} \right) \right] \cdot \nabla \alpha \tag{17}$$

We apply a vector identity and write out the divergence in general coordinates:

$$\begin{aligned} \left[\nabla \times \left(|v_{||}| \mathbf{b} \right) \right] \cdot \nabla \alpha &= \nabla \cdot \left(|v_{||}| \mathbf{b} \times \nabla \alpha \right) + |v_{||}| \mathbf{b} \cdot \nabla \times \nabla \alpha & (18) \\ &= \nabla \cdot \left(|v_{||}| \mathbf{b} \times \nabla \alpha \right) & (19) \\ &= \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial \ell} \left(\sqrt{g} |v_{||}| \mathbf{b} \times \nabla \alpha \cdot \nabla \ell \right) + \frac{\partial}{\partial \gamma} \left(\sqrt{g} |v_{||}| \mathbf{b} \times \nabla \alpha \cdot \nabla \gamma \right) \right] & (20) \\ &= B \frac{\partial}{\partial \ell} \left(\sqrt{g} |v_{||}| \mathbf{b} \times \nabla \alpha \cdot \nabla \ell \right) + B \frac{\partial}{\partial \gamma} \left(|v_{||}| \right) & (21) \end{aligned}$$

Thus we have

$$\langle \mathbf{v}_d \cdot \nabla \alpha \rangle = \frac{m}{Ze} \frac{2}{\tau_b} \int_{\ell_-}^{\ell_+} \frac{d\ell}{B} \left[B \frac{\partial}{\partial \ell} \left(\sqrt{g} |v_{||}| \mathbf{b} \times \nabla \alpha \cdot \nabla \ell \right) + B \frac{\partial}{\partial \gamma} \left(|v_{||}| \right) \right]. \tag{22}$$

The first term is an integral of a derivative:

$$\int_{\ell_-}^{\ell_+} d\ell \frac{\partial}{\partial \ell} \left(\sqrt{g} |v_{||}| \mathbf{b} \times \nabla \alpha \cdot \nabla \ell \right) = \left[\sqrt{g} |v_{||}| \mathbf{b} \times \nabla \alpha \cdot \nabla \ell \right]_{\ell_+} - \left[\sqrt{g} |v_{||}| \mathbf{b} \times \nabla \alpha \cdot \nabla \ell \right]_{\ell_-} = 0. \tag{23}$$

Therefore we are left with

$$\langle \mathbf{v}_d \cdot \nabla \alpha \rangle = \frac{m}{Ze} \frac{2}{\tau_b} \int_{\ell_-}^{\ell_+} d\ell \frac{\partial}{\partial \gamma} \left(|v_{||}| \right) = \frac{mv}{Ze\tau_b} \int_{\ell_-}^{\ell_+} \frac{d\ell}{\sqrt{1 - \lambda B}} \left(-\lambda \frac{\partial B}{\partial \gamma} \right) \tag{24}$$

Comparing to (16), we see

$$\langle \mathbf{v}_d \cdot \nabla \alpha \rangle = \frac{m}{Ze\tau_b} \frac{\partial J}{\partial \gamma}. \tag{25}$$

Since we can also write the field as

$$\mathbf{B} = \nabla (-\gamma) \times \nabla \alpha, \tag{26}$$

we can make the simultaneous replacements $\{\alpha \to -\gamma, \gamma \to \alpha\}$. Hence, it is also true that

$$\langle \mathbf{v}_d \cdot \nabla \gamma \rangle = -\frac{m}{Ze\tau_b} \frac{\partial J}{\partial \alpha}. \tag{27}$$

We have thus proved (1)-(2).

6 Conservation of J

Note that $J = J(\alpha, \gamma, \mu, W)$. Since μ and W are conserved, J varies along a particle trajectory only through variation of α and γ. Therefore the total time derivative of J along a bounce-averaged particle trajectory is

$$\frac{dJ}{dt} = \frac{d\alpha}{dt} \frac{\partial J}{\partial \alpha} + \frac{d\gamma}{dt} \frac{\partial J}{\partial \gamma} = \langle \mathbf{v}_d \cdot \nabla \alpha \rangle \frac{\partial J}{\partial \alpha} + \langle \mathbf{v}_d \cdot \nabla \gamma \rangle \frac{\partial J}{\partial \gamma} = \frac{m}{Ze\tau_b} \left(\frac{\partial J}{\partial \gamma} \frac{\partial J}{\partial \alpha} - \frac{\partial J}{\partial \alpha} \frac{\partial J}{\partial \gamma} \right) = 0. \tag{28}$$

Hence J is indeed conserved under the bounce-averaged motion.

7 Further reading

Derivations of the relations (1)-(2) can be found in many papers. One is Catto & Hazeltine, "Bumpy torus transport in the low collisionality frequency limit", Physics of Fluids 24, 290 (1981); see equations (1)-(19) and Appendix A. Another is Calvo et al, "The effect of tangential drifts on neoclassical transport in stellarators close to omnigeneity," Plasma Physics and Controlled Fusion 59, 055014 (2017); see eq (34)-(36) and Appendix A.