Sixteenth Homework: MATH 410
Due Friday, 18 December 2020 (but not collected!)

1. Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be bounded, positive sequences in $\mathbb{R}$.

   (a) Prove that
   $$\limsup_{k\to\infty}(a_k b_k) \le \Big(\limsup_{k\to\infty}\, a_k\Big)\Big(\limsup_{k\to\infty}\, b_k\Big)\,.$$

   (b) Give an example for which equality does not hold above.

2. Determine the set of $a \in \mathbb{R}$ for which the following formal infinite series converge. Give your reasoning.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{n3^n}$

   (b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k^2+1}{k^4+1}\right)^a$

3. Let $[a,b] \subset \mathbb{R}$ be a closed, bounded interval. Let $f : [a,b] \to [a,b]$. Suppose there exists an $M \in (0,1)$ such that
   $$|f(x) - f(y)| \le M\,|x-y| \quad \text{for every } x, y \in [a,b]\,.$$
   Let $x_0 \in [a,b]$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ by
   $$x_{n+1} = f(x_n) \quad \text{for every } n \in \mathbb{N}\,.$$
   Show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. (Hint: Consider $x_{n+1} - x_n = f(x_n) - f(x_{n-1})$.)

4. Let $f : (a,b) \to \mathbb{R}$ be differentiable at a point $c \in (a,b)$ with $f'(c) > 0$. Show that there exists a $\delta > 0$ such that
   $$x \in (c-\delta, c) \subset (a,b) \implies f(x) < f(c)\,,$$
   $$x \in (c, c+\delta) \subset (a,b) \implies f(c) < f(x)\,,$$

5. Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be Riemann integrable over $[a,b]$. Prove that $f+g$ is Riemann integrable over $[a,b]$.

6. Let $\alpha \in (0,1]$ and $K \in \mathbb{R}_+$ such that the function $f : [a,b] \to \mathbb{R}$ satisfy the Hölder bound
   $$|f(x) - f(y)| < K\,|x-y|^{\alpha} \qquad \text{for every } x, y \in [a,b]\,.$$

   (a) Show that f is uniformly continuous over $[a,b]$.

   (b) Show that for every partition P of $[a,b]$ one has
   $$0 \le U(f,P) - L(f,P) < |P|^{\alpha} K\,(b-a)\,.$$

7. Prove that every countable subset of $\mathbb{R}$ has measure zero.

More Problems on the back of this Page.

8. For every $n \in \mathbb{Z}_+$ define $h_n(x) = nx(1+nx)^{-2}$ for every $x \in [0, \infty)$.
   (a) Prove that $h_n \to 0$ pointwise over $[0, \infty)$.
   (b) Prove that this limit is not uniform over $[0, \infty)$.
   (c) Prove that this limit is uniform over $[\delta, \infty)$ for every $\delta > 0$.

9. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that there exists $p \in (a, b)$ such that
$$f(p) = \frac{1}{e^b - e^a} \int_a^b f(x) e^x \, \mathrm{d}x \, .$$

10. Consider a function f defined by
$$f(x) = \sum_{k=0}^{\infty} \frac{1}{4^k} \sin(3^k x) \, ,$$
for every $x \in \mathbb{R}$ for which the above series converges.
   (a) Show that f is defined for every $x \in \mathbb{R}$.
   (b) Show that the series converges uniformly over $\mathbb{R}$.
   (c) Show that f is continuously differentiable over $\mathbb{R}$ and that
$$f'(x) = \sum_{k=0}^{\infty} \frac{3^k}{4^k} \cos(3^k x) \, .$$

11. For every $n \in \mathbb{Z}_+$ define $f_n(x) = n(1+nx)^{-2}$ for every $x \in [0, \infty)$.
   (a) Prove for every $\delta > 0$ that
$$\lim_{n\to\infty} f_n = 0 \qquad \text{uniformly over } [\delta, \infty) \, .$$
   (b) Prove for every $\delta > 0$ that
$$\lim_{n\to\infty} \int_0^\delta f_n = 1 \, .$$
   (c) Let $g : [0, 1] \to \mathbb{R}$ be continuous. Show that
$$\lim_{n\to\infty} \int_0^1 f_n g = g(0) \, .$$

12. Given that for every $x > -1$ and every $n \in \mathbb{Z}_+$ we have
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \log(1+x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n} \, ,$$
prove that
$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \quad \text{for every } x \in (-1, 1] \, ,$$
that the series converges uniformly over every $[-R, R] \subset (-1, 1)$, and that the series diverges for every real $x \notin (-1, 1]$.