Fourteenth Homework: MATH 410
Due Friday, 4 December 2020

1. Exercise 1 of Section 6.5 in the text.
2. Exercise 5 of Section 6.5 in the text.
3. Exercise 1 of Section 6.6 in the text.
4. Exercise 3 of Section 6.6 in the text.
5. Exercise 7 of Section 6.6 in the text.
6. Exercise 3 of Section 7.2 in the text.
7. Exercise 4 of Section 7.2 in the text.
8. Exercise 5 of Section 7.2 in the text.
9. Exercise 9 of Section 7.2 in the text.
10. Let $f : [a, b] \to \mathbb{R}$. Let $F : [a, b] \to \mathbb{R}$ be a primitive of f over $[a, b]$. Let $g : [a, b] \to \mathbb{R}$ such that $g(x) = f(x)$ at all but a finite number of points of $[a, b]$. Show that F is also a primitive of g over $[a, b]$.
11. Let $f : [0, 3] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} x & \text{for } 0 \leq x < 1\,, \\ -x & \text{for } 1 \leq x < 2\,, \\ 1 & \text{for } 2 \leq x \leq 3\,. \end{cases}$$
Find F, the primitive of f over $[0, 3]$ specified by $F(0) = 1$.
12. The assumption that G is increasing over $[a, b]$ in Proposition 11.2 of the Notes can be weakened to the assumption that G is nondecreasing over $[a, b]$. Prove this. The proof can be very similar to that given for Proposition 11.2 except you will have to work harder to show that $F(G)$ is a primitive of $f(G)g$ over $[a, b]$. Specifically, because G^{-1} may not exist, you will need to replace the partition $G^{-1}(P)$ in the proof of Proposition 11.2 with a more complicated partition.
13. Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable and nonnegative over $[a, b]$. Prove that if $\int_a^b g > 0$ then there exists $p \in (a, b)$ such that
$$\int_a^b fg = f(p) \int_a^b g\,.$$
(This strenghthens the integral mean-value theorem given as Theorem 11.3 in the notes.)
14. When $q \in \mathbb{N}$ the binomial expansion yields
$$(1 + x)^q = \sum_{k=0}^{q} \frac{q!}{k!(q - k)!}\, x^k = 1 + \sum_{k=1}^{q} \frac{q(q - 1) \cdots (q - k + 1)}{k!}\, x^k\,.$$
Now let $q \in \mathbb{R} - \mathbb{N}$. Let $f(x) = (1 + x)^q$ for every $x > -1$. Then
$$f^{(k)}(x) = q(q - 1) \cdots (q - k + 1)(1 + x)^{q-k} \quad \text{for every } x > -1 \text{ and } k \in \mathbb{Z}_+\,.$$
The formal Taylor series of f about 0 is therefore
$$1 + \sum_{k=1}^{\infty} \frac{q(q - 1) \cdots (q - k + 1)}{k!}\, x^k\,.$$
Prove this series converges absolutely to $(1 + x)^q$ when $|x| < 1$ and diverges when $|x| > 1$. (This formula is Newton's extension of the binomial expansion to powers q that are real.)

15. Show that for every $q > -1$ one has
$$2^q = 1 + \sum_{k=1}^{\infty} \frac{q(q-1)\cdots(q-k+1)}{k!}\,,$$
while for every $q \leq -1$ the above series diverges. (Hint: This is the case $x = 1$ for the series in the previous problem.)