Eleventh Homework: MATH 410
Due Monday, 9 November 2020

1. Prove Proposition 8.6 on page 46 in the notes.
2. Give a counterexample to each of the following false assertions.
   (a) If $f:\mathbb{R}\to\mathbb{R}$ is differentiable and increasing over $\mathbb{R}$ then $f'>0$ over $\mathbb{R}$.
   (b) If $f:(a,b)\to\mathbb{R}$ is continuous then f has a minimum or a maximum over (a,b).
3. Let $D\subset\mathbb{R}$ and $f:D\to\mathbb{R}$ be uniformly continuous over D. Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence contained in D. Show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a convergent sequence.
4. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable. Prove it is continuous.
5. Let $f(x)=\sinh(x)$ for every $x\in\mathbb{R}$. Show that
$$\sinh(x)=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\,x^{2k+1}\quad\text{for every } x\in\mathbb{R}\,.$$
6. Evaluate the following limit. Give your reasoning.
$$\lim_{x\to 3}\frac{x^4-81}{x^2-9}\,.$$
7. Suppose that $f:(a,b)\to\mathbb{R}$ is twice differentiable and that $f'':(a,b)\to\mathbb{R}$ is bounded over (a,b). Show that there exists an $M\in\mathbb{R}_+$ such that for every $x,y\in(a,b)$ we have
$$\big|f'(x)-f'(y)\big|\leq M\,|x-y|\,.$$
8. Prove that for every $x>0$ we have
$$1+\tfrac{3}{2}x<(1+x)^{\frac{3}{2}}<1+\tfrac{3}{2}x+\tfrac{3}{8}x^2\,.$$
9. Let $D\subset\mathbb{R}$. A function $f:D\to\mathbb{R}$ is said to be Hölder continuous of order $\alpha\in(0,1]$ if there exists a $C\in\mathbb{R}_+$ such that for every $x,y\in D$ we have
$$|f(x)-f(y)|\leq C\,|x-y|^{\alpha}\,.$$
   Show that if $f:D\to\mathbb{R}$ is Hölder continuous of order α for some $\alpha\in(0,1]$ then it is uniformly continuous over D.
10. Let $\alpha\in(0,1)$. Let $f:[0,\infty)\to\mathbb{R}$ be defined by $f(x)=x^{\alpha}$. Show that f is uniformly continuous over $[0,\infty)$. Hint: Use the previous problem after showing that
$$\big|x^{\alpha}-y^{\alpha}\big|\leq|x-y|^{\alpha}\quad\text{for every } x,y\in[0,\infty)\,.$$
11. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable. Suppose the equation $f'(x)=0$ has at most one solution over $x\in\mathbb{R}$. Show the equation $f(x)=0$ has at most two solutions over $x\in\mathbb{R}$.
12. Let $D\subset\mathbb{R}$ and $f:D\to\mathbb{R}$. Write negations of the following assertions.
   (a) "For all sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ contained in D we have
$$\lim_{k\to\infty}|x_k-y_k|=0\quad\Longrightarrow\quad\lim_{k\to\infty}\big|f(x_k)-f(y_k)\big|=0\,.\text{"}$$
   (b) "For every $\epsilon>0$ there exists a $\delta>0$ such that for all points $x,y\in D$ we have
$$|x-y|<\delta\quad\Longrightarrow\quad\big|f(x)-f(y)\big|<\epsilon\,.\text{"}$$