Sixth Homework: MATH 410
Due Wednesday, 7 October 2020

1. Prove Proposition 4.9 in the class notes.
2. Prove Proposition 4.12 in the class notes.
3. Prove Proposition 4.13 in the class notes.
4. Do the exercise on page 64 of the class notes.
5. Prove that for every nonzero $x \in \mathbb{R}$ we have the inequality
$$1 + \tfrac{4}{3}x < (1+x)^{\frac{4}{3}}\,.$$
6. Consider the real sequence $\{b_k\}_{k\in\mathbb{N}}$ given by
$$b_k = (-1)^k \left(3 + \frac{1}{(k+1)^2}\right) \quad \text{for every } k \in \mathbb{N}\,,$$
where $\mathbb{N} = \{0, 1, 2, \cdots\}$.
   (a) Give the first three terms of the subsequence $\{b_{3k}\}_{k\in\mathbb{N}}$.
   (b) Give the first three terms of the subsequence $\{b_{2^k-1}\}_{k\in\mathbb{N}}$.
   (c) Compute $\limsup_{k\to\infty} b_k$ and $\liminf_{k\to\infty} b_k$. Justify your answers.
7. Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be bounded sequences in $\mathbb{R}$.
   (a) Prove that
$$\limsup_{k\to\infty}(a_k + b_k) \leq \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k\,.$$
   (b) Give an example for which equality does not hold above.
8. Determine all the values of $a \in \mathbb{R}$ for which
$$\sum_{n=2}^{\infty} \frac{1}{\log(n)}\, a^n \quad \text{converges}\,.$$
9. Determine all the values of $a \in \mathbb{R}$ for which
$$\sum_{k=0}^{\infty} \left(\frac{2k+3}{k^4+1}\right)^a \quad \text{converges}\,.$$
10. Determine all the values of $a \in \mathbb{R}$ for which
$$\sum_{m=1}^{\infty} \frac{1}{m^2}\big(2 + (-1)^m\big)^m a^m \quad \text{converges}\,.$$
11. Give a counterexample to each of the following false assertions.
   (a) A sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ is convergent if the sequence $\{a_k^2\}_{k\in\mathbb{N}}$ is convergent.
   (b) Every convergent series in $\mathbb{R}$ is absolutely convergent.
   (c) A countable union of closed subsets of $\mathbb{R}$ is closed.
12. Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$.
Write the negations of the following assertions.
   (a) "For every $m \in \mathbb{R}$ one has $b_j > m$ frequently as $j \to \infty$."
   (b) "Every sequence in A has a subsequence that converges to a limit in A."