Fourth Homework: MATH 410
Due Monday, 28 September 2020

1. Exercise 1 of Section 9.1 in the text.
2. Exercise 2 of Section 9.1 in the text.
3. Exercise 3 of Section 9.1 in the text.
4. Exercise 4 of Section 9.1 in the text.
5. Consider the infinite series
$$\sum_{k=1}^{\infty} \frac{1}{k(k+3)} .$$
Find an expression for its partial sums. Use this expression to determine if the series converges or diverges. If it converges then find its sum.
6. Consider a formal infinite series of the form
$$\sum_{k=1}^{\infty} k r^k ,$$
for some $r \in \mathbb{R}$. Find all the values of r for which this series converges and evaluate the sum. (Hint: Find an explicit expression for the partial sums and evaluate the limit. The explicit expression may be derived from the partial sums of a geometric series.)
7. Prove Proposition 3.3 in the notes.
8. The proof of Proposition 3.4 in the notes argues that the direct comparison test applies whenever the limit comparison test applies, and that the limit comparison test applies whenever the ratio comparison test applies.
   (a) Can you find an example where the direct comparison test applies but the limit comparison test fails to apply?
   (b) Can you find an example where the limit comparison test applies but the ratio comparison test fails to apply?
9. Use a comparison test to determine whether the following series converge or diverge
$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{k^2}$$
You must compare with a series for which the result is known by page 37 of the notes.
10. Let $\{a_k\}$ be a nonincreasing, positive sequence. Prove that
$$\sum_{k=1}^{\infty} a_k \quad \text{converges} \quad \iff \quad \sum_{k=0}^{\infty} 5^k a_{5^k} \quad \text{converges} .$$
11. Complete the proof of Proposition 3.10 in the notes by
    (a) completing the proof of its convergence assertion,
    (b) proving of its divergence assertion.
12. Complete the proof of Proposition 3.11 in the notes by showing that
$$\lim_{k\to\infty} s_{2k} = s \quad \text{and} \quad \lim_{k\to\infty} s_{2k+1} = s \quad \implies \quad \lim_{k\to\infty} s_k = s .$$