Quiz 11 Solutions, Math 246, Professor David Levermore
Thursday, 10 December 2020

(1) [5] Consider the system

$$x' = -4x + 2y\,, \qquad y' = 5x - y - 3x^2\,.$$

Its stationary points are $(0,0)$ and $(1,2)$. Classify the type and stability of each of these stationary points. (You do not have to sketch anything.)

Solution. The Jacobian matrix is $\boldsymbol{\partial}\mathbf{f}(x,y) = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 5-6x & -1 \end{pmatrix}$.

- At $(0,0)$ the coefficient matrix of its linearization is $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(0,0) = \begin{pmatrix} -4 & 2 \\ 5 & -1 \end{pmatrix}$, which has characterisitc polynomial
$$p_{\mathbf{A}}(\zeta) = \zeta^2 + 5\zeta - 6 = (\zeta + 6)(\zeta - 1)\,.$$
The eigenvalues of $\mathbf{A}$ are -6 and 1. Because these are real with opposite sign, the stationary point $(0,0)$ is a *saddle* and thereby is *unstable*, but not repelling.

- At $(1,2)$ the coefficient matrix of its linearization is $\mathbf{B} = \boldsymbol{\partial}\mathbf{f}(1,2) = \begin{pmatrix} -4 & 2 \\ -1 & -1 \end{pmatrix}$, which has characteristic polynomial
$$p_{\mathbf{B}}(\zeta) = \zeta^2 + 5\zeta + 6 = (\zeta + 2)(\zeta + 3)\,.$$
The eigenvalues of $\mathbf{B}$ are -2 and -3. Because these are both negative, the stationary point $(1,2)$ is a *nodal sink* and thereby is *attracting*.

(2) [5] Consider the planar system

$$u' = 3u - v\,, \qquad v' = 2u + v - 5u^2\,.$$

Its stationary points are $(0,0)$ and $(1,3)$. Classify the type and stability of each of these stationary points. (You do not have to sketch anything.)

Solution. The Jacobian matrix is $\boldsymbol{\partial}\mathbf{f}(u,v) = \begin{pmatrix} \partial_u f & \partial_v f \\ \partial_u g & \partial_v g \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2-10u & 1 \end{pmatrix}$.

- At $(0,0)$ the coefficient matrix of its linearization is $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(0,0) = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$, which has characterisitc polynomial
$$p_{\mathbf{A}}(\zeta) = \zeta^2 - 4\zeta + 5 = (\zeta - 2)^2 + 1^2\,.$$
The eigenvalues of $\mathbf{A}$ are $2 \pm i$. Because this is a conjugate pair with positive real part, and because $a_{21} = 2 > 0$, the stationary point $(0,0)$ is a *counterclockwise spiral source* and thereby is *repelling*.

- At $(1,3)$ the coefficient matrix of its linearization is $\mathbf{B} = \boldsymbol{\partial}\mathbf{f}(1,3) = \begin{pmatrix} 3 & -1 \\ -8 & 1 \end{pmatrix}$, which has characterisitc polynomial
$$p_{\mathbf{B}}(\zeta) = \zeta^2 - 4\zeta - 5 = (\zeta - 5)(\zeta + 1)\,.$$
The eigenvalues of $\mathbf{B}$ are -1 and 5. Because these are real with opposite sign, the stationary point $(1,3)$ is a *saddle* and thereby is *unstable*, but not repelling.