Quiz 10 Solutions, Math 246, Professor David Levermore
Thursday, 3 December 2020

(1) [3] Consider the system $\mathbf{x}' = \mathbf{C}\mathbf{x}$ where $\mathbf{C} = \begin{pmatrix} 5 & -3 \\ 6 & -1 \end{pmatrix}$.

(a) [1] Classify its phase-plane portrait.
(b) [1] Determine the stability of the origin for this system.
(c) [1] Sketch its phase-plane portrait.

Solution (a). The characteristic polynomial of $\mathbf{C}$ is

$$p(z) = z^2 - \text{tr}(\mathbf{C})z + \det(\mathbf{C}) = z^2 - 4z + \big(5 \cdot (-1) - 6 \cdot (-3)\big)$$
$$= z^2 - 4z + 13 = (z-2)^2 + 3^2\,.$$

Because this has the conjugate pair of roots $2 \pm i3$, the phase-plane portrait of the system $\mathbf{x}' = \mathbf{C}\mathbf{x}$ is a spiral source. Because the c_{21} entry is positive, the phase-plane portrait is a *counterclockwise spiral source*.

Solution (b). The origin is *repelling* for a spiral source.

Solution (c). Your sketch should show a curve that spirals away from the origin in a counterclockwise fashion.

(2) [3] Consider the system $\mathbf{x}' = \mathbf{B}\mathbf{x}$ where the 2×2 matrix $\mathbf{B}$ has eigenpairs

$$\left(-2\,, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)\,, \qquad \left(-1\,, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)\,.$$

(a) [1] Classify its phase-plane portrait.
(b) [1] Determine the stability of the origin for this system.
(c) [1] Sketch its phase-plane portrait.

Solution (a). Because $\mathbf{B}$ has two negative eigenvalues, the phase-plane portrait of the system $\mathbf{x}' = \mathbf{B}\mathbf{x}$ is a *nodal sink*.

Solution (b). The origin is *attracting* for a nodal sink.

Solution (c). Your sketch should show the line through the origin and the point $(2, 3)$, and the line through the origin and the point $(1, -2)$. The four half-lines on either side of the origin are the four eigenorbits. Each of these orbits should have an arrow on it pointing towards the origin. The two lines separate the plane into four regions. Your sketch should show at least one representative orbit in each of these four regions. Because -1 is the eigenvalue with the smaller magnitude, each representative orbit should emerge from the origin tangent to the line through the point $(1, -2)$ and should bend to become more parallel to the line through the point $(2, 3)$ further away from the origin. Each representative orbit should have an arrow on it pointing towards the origin.

(3) [4] Consider the system

$$u' = -4u + 2v - 9v^2\,, \qquad v' = u + 4v\,.$$

(a) [2] This system has two stationary points. Find them.
(b) [2] Find a nonconstant function $H(u,v)$ such that every orbit of this system satisfies $H(u,v) = c$ for some constant c. (No sketch is needed.)

Solution (a). The stationary points satisfy

$$0 = -4u + 2v - 9v^2\,, \qquad 0 = u + 4v\,.$$

The second equation is satisfied if and only if $u = -4v$, whereby the first equation becomes $0 = 18v - 9v^2 = 9v(2-v)$, which has solutions $v = 0$ and $v = 2$. Therefore the stationary points are $(0,0)$ and $(-8,2)$.

Solution (b). The system is *Hamiltonian* because

$$\partial_u\big(-4u + 2v - 9v^2\big) + \partial_v(u + 4v) = -4 + 4 = 0\,,$$

whereby the orbit equation is *exact*. Therefore there exists $H(u,v)$ such that

$$\partial_v H(u,v) = -4u + 2v - 9v^2\,, \qquad -\partial_u H(u,v) = u + 4v\,.$$

By integrating the second equation we find that

$$H(u,v) = -\tfrac{1}{2}u^2 - 4uv + h(v)\,.$$

By substituting this into the first equation we see that

$$-4u + h'(v) = -4u + 2v - 9v^2\,,$$

whereby $h'(v) = 2v - 9v^2$. By taking $h(v) = v^2 - 3v^3$, we obtain

$$H(u,v) = -\tfrac{1}{2}u^2 - 4uv + v^2 - 3v^3\,.$$

Remark. The stationary points of the system are critical points of the Hamiltonian $H(u,v)$ — i.e. points where $\partial_u H(u,v) = 0$ and $\partial_v H(u,v) = 0$. The nature of these critical points can be studied with the Hessian matrix of $H(u,v)$, which is

$$\boldsymbol{\partial}^2 H(u,v) = \begin{pmatrix} \partial_{uu}H(u,v) & \partial_{uv}H(u,v) \\ \partial_{vu}H(u,v) & \partial_{vv}H(u,v) \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -4 & 2 - 18v \end{pmatrix}.$$

- At the critical point $(0,0)$ we see that

$$\det\big(\boldsymbol{\partial}^2 H(0,0)\big) = \det\begin{pmatrix} -1 & -4 \\ -4 & 2 \end{pmatrix} = -18 < 0\,,$$

  whereby the critical point $(0,0)$ is a saddle point of $H(u,v)$.
- At the critical point $(-8,2)$ we see that

$$\det\big(\boldsymbol{\partial}^2 H(-8,2)\big) = \det\begin{pmatrix} -1 & -4 \\ -4 & 34 \end{pmatrix} = -50 > 0\,,$$

$$\operatorname{tr}\big(\boldsymbol{\partial}^2 H(-8,2)\big) = \operatorname{tr}\begin{pmatrix} -1 & -4 \\ -4 & 34 \end{pmatrix} = 33 > 0\,,$$

  whereby the critical point $(-8,2)$ is a local minimizer of $H(u,v)$.

Therefore, with the sign conventions that we have adopted, we see that:
- the point $(0,0)$ is a saddle point in the phase-plane portrait;
- the point $(-8,2)$ is a clockwise center in the phase-plane portrait.

Remark. The phase-plane portrait can be filled out by plotting some level sets of the Hamiltonian $H(u,v)$. The most important level set to sketch is that associated with the saddle point $(0,0)$ is the set of all points (u,v) that satisfy $H(u,v) = H(0,0) = 0$, — i.e. that satisfy

$$-\tfrac{1}{2}u^2 - 4uv + v^2 - 3v^3 = 0\,.$$

Because this equation is quadratic in u, we may solve for u as a function of v. For example, after multiplying it by -2 and completing the square it becomes

$$(u + 4v)^2 - 18v^2 + 6v^3 = 0\,,$$

whereby

$$u = -4v \pm \sqrt{18v^2 - 6v^3} \qquad \text{for } v \le 3\,.$$

These two curves may be sketched in the phase-plane using techniques from calculus. They intersect where $v = 0$ and where $v = 3$, which is at the points $(0,0)$ and $(-12,3)$. Sketching other level sets is slightly harder. Alternatively, the MATLAB command "contour" can be used to plot a few level sets. Because $H(-8,2) = 12$, a good idea of the phase-plane portrait can be obtained by plotting the three level sets

$$H(u,v) = 4\,, \quad H(u,v) = 0\,, \quad H(u,v) = -4\,.$$

The orbits around $(-8,2)$ must be clockwise. From that fact and from the fact that $(0,0)$ is a saddle, the direction of all the orbits can be figured out. Try it. The orbits on the level set $H(u,v) = 0$ are separatrices, which separate the different behavior seen for orbits with $H(u,v) < 0$ and that seen for orbits with $H(u,v) > 0$.