Quiz 9 Solutions, Math 246, Professor David Levermore
Thursday, 12 November 2020

(1) [2] Recast the equation $y'''' - \sin(y+t)y'' + e^{y'}y = 0$ as a first-order system of ordinary differential equations.

Solution. Because the equation is fourth order, the first-order system must have dimension four. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \sin(x_1+t)x_3 - e^{x_2}x_1 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix}.$$

Remark. Your answer should contain both a first-order system and a dictionary that relates the variables of the system to y and some of its derivatives. The variable y should not appear in your first-order system. It should only appear in the dictionary as shown on the right-hand side above.

(2) [3] Determine the interval of definition for the solution of the initial-value problem

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{t^2-16}\begin{pmatrix} e^{2t} & \cos(2t) \\ t^2 & e^{-3t} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \log(t^2) \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} x(-1) \\ y(-1) \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}.$$

Give your reasoning.

Solution. This first-order nonhomogeneous linear system is already in normal form. Notice that

- ⋄ the coefficient matrix is undefined at $t = \pm 4$ and is continuous elsewhere;
- ⋄ the forcing vector is undefined at $t = 0$ and is continuous elsewhere;
- ⋄ the initial time is $t = -1$.

Therefore the interval of definition is $(-4, 0)$ because

- the initial time -1 is in $(-4, 0)$,
- the coefficient matrix and the forcing vector are continuous over $(-4, 0)$,
- the coefficient matrix is undefined at $t = -4$,
- the forcing vector is undefined at $t = 0$.

Remark. All four reasons must be given for full credit.

- ∘ The first two are why a (unique) solution exists over the interval $(-4, 0)$.
- ∘ The last two are why this solution does not exist over a larger interval.

(3) [5] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} -t^4 \\ 2 - t^6 \end{pmatrix}$.

(a) [2] Compute their Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.

Solution (a). The Wronskian is

$$\begin{aligned} \mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) &= \det\begin{pmatrix} 1 & -t^4 \\ 2t^2 & 2 - t^6 \end{pmatrix} \\ &= 1 \cdot (2 - t^6) - 2t^2 \cdot (-t^4) = 2 - t^6 + 2t^6 = 2 + t^6 \,. \end{aligned}$$

Solution (b). Let

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & -t^4 \\ 2t^2 & 2 - t^6 \end{pmatrix} .$$

From Part (a) we see that $\det\big(\mathbf{\Psi}(t)\big) = 2 + t^6 \neq 0$, whereby $\mathbf{\Psi}(t)$ is invertible. If $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ then $\mathbf{\Psi}(t)$ is a fundamental matrix for this system. It thereby satisfies

$$\mathbf{\Psi}'(t) = \mathbf{A}(t)\mathbf{\Psi}(t) \,,$$

which implies that

$$\begin{aligned} \mathbf{A}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 0 & -4t^3 \\ 4t & -6t^5 \end{pmatrix} \begin{pmatrix} 1 & -t^4 \\ 2t^2 & 2 - t^6 \end{pmatrix}^{-1} \\ &= \frac{1}{2 + t^6} \begin{pmatrix} 0 & -4t^3 \\ 4t & -6t^5 \end{pmatrix} \begin{pmatrix} 2 - t^6 & t^4 \\ -2t^2 & 1 \end{pmatrix} \\ &= \frac{1}{2 + t^6} \begin{pmatrix} 8t^5 & -4t^3 \\ 8t + 8t^7 & -2t^5 \end{pmatrix} . \end{aligned}$$