Quiz 8 Solutions, Math 246, Professor David Levermore
Thursday, 5 November 2020

Short Table: $\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$ for $s > a$, $\quad \mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$.

(1) [3] Find $x(t) = \mathcal{L}^{-1}[X](t)$ where $X(s) = e^{-2s}\dfrac{21}{(s-3)(s+4)}$.

Solution. By the partial fraction identity

$$J(s) = \frac{21}{(s-3)(s+4)} = \frac{3}{s-3} + \frac{-3}{s+4},$$

and by the first table entry with $n = 0$ and $a = 3$ and with $n = 0$ and $a = -4$ we have

$$\begin{aligned} j(t) = \mathcal{L}^{-1}\left[\frac{21}{(s-3)(s+4)}\right](t) &= 3\mathcal{L}^{-1}\left[\frac{1}{s-3}\right](t) - 3\mathcal{L}^{-1}\left[\frac{1}{s+4}\right](t) \\ &= 3e^{3t} - 3e^{-4t}. \end{aligned}$$

Therefore by the second table entry with $c = 2$ we have

$$\begin{aligned} x(t) = \mathcal{L}^{-1}[X](t) = \mathcal{L}^{-1}\big[e^{-2s}J(s)\big](t) &= u(t-2)j(t-2) \\ &= u(t-2)\big(3e^{3(t-2)} - 3e^{-4(t-2)}\big) \\ &= 3u(t-2)\big(e^{3t-6} - e^{-4t+8}\big). \end{aligned}$$

(2) [3] Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^2 + 8\mathrm{D} + 16$, where $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$.

Solution. The Green function $g(t)$ is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t),$$

where $p(s)$ is the characteristic polynomial of L, which is

$$p(s) = s^2 + 8s + 16 = (s+4)^2.$$

The first table entry with $n = 1$ and $a = -4$ gives

$$\mathcal{L}[t\,e^{-4t}](s) = \frac{1}{(s+4)^2} \quad \text{for } s > -4,$$

which implies that

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+4)^2}\right](t) = t\,e^{-4t}.$$

Remark. Because

$$p(s) = s^2 + 8s + 16\,,$$

the natural fundamental set for L can be found by

$$\begin{aligned} N_1(t) = g(t) \qquad &= t\,e^{-4t}\,, \\ N_0(t) = \mathrm{D}N_1(t) + 8g(t) \quad &= \big(e^{-4t} - 4t\,e^{-4t}\big) + 8t\,e^{-4t} = (1+4t)e^{-4t}\,. \end{aligned}$$

(3) [4] Find $F(s) = \mathcal{L}[f](s)$ for

$$f(t) = \begin{cases} 3t & \text{for } 0 \le t < 2\,, \\ 6e^{2-t} & \text{for } 2 \le t\,. \end{cases}$$

Solution. The piecewise-defined function $f(t)$ can be expressed as

$$f(t) = 3t + u(t-2)\big(6e^{2-t} - 3t\big)\,.$$

This has the form

$$f(t) = 3t + u(t-2)j(t-2)\,,$$

where

$$j(t) = 6e^{2-(t+2)} - 3(t+2) = 6e^{-t} - 3t - 6\,.$$

The second table entry with $c = 2$ shows that

$$\mathcal{L}\big[u(t-2)j(t-2)\big](s) = e^{-2s}\mathcal{L}[j](s) = e^{-2s}\mathcal{L}\Big[6e^{-t} - 3t - 6\Big](s)\,,$$

whereby

$$\begin{aligned} F(s) = \mathcal{L}[f](s) &= \mathcal{L}[3t](s) + e^{-2s}\mathcal{L}\Big[6e^{-t} - 3t - 6\Big](s) \\ &= 3\mathcal{L}[t](s) + e^{-2s}\Big(6\mathcal{L}[e^{-t}](s) - 3\mathcal{L}[t](s) - 6\mathcal{L}[1](s)\Big)\,. \end{aligned}$$

The first table entry with $n = 0$ and $a = -1$, with $n = 1$ and $a = 0$, and with $n = 0$ and $a = 0$ gives

$$\mathcal{L}[e^{-t}](s) = \frac{1}{s+1}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,, \qquad \mathcal{L}[1](s) = \frac{1}{s}\,,$$

whereby

$$F(s) = \frac{3}{s^2} + e^{-2s}\left(\frac{6}{s+1} - \frac{3}{s^2} + \frac{6}{s}\right)\,.$$

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