# Quiz 7 Solutions, Math 246, Professor David Levermore
# Thursday, 29 October 2020

**Short Table:** $\mathcal{L}[t^n](s) = \dfrac{n!}{s^{n+1}}$ for $s > 0$, $\qquad \mathcal{L}[\cos(bt)](s) = \dfrac{s}{s^2+b^2}$ for $s > 0$.

(1) [4] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for the function $f(t) = u(t-5)e^{-3t}$, where $u$ is the unit step function.

**Solution.** By the definitions of the Laplace transform and the unit step function

$$\begin{aligned}\mathcal{L}[f](s) = \lim_{T\to\infty}\int_0^T e^{-st} f(t)\,\mathrm{d}t &= \lim_{T\to\infty}\int_0^T e^{-st}u(t-5)e^{-3t}\,\mathrm{d}t \\ &= \lim_{T\to\infty}\int_5^T e^{-st}e^{-3t}\,\mathrm{d}t = \lim_{T\to\infty}\int_5^T e^{-(s+3)t}\,\mathrm{d}t\,.\end{aligned}$$

For $s > -3$ and $T > 5$

$$\int_5^T e^{-(s+3)t}\,\mathrm{d}t = -\frac{e^{-(s+3)t}}{s+3}\bigg|_5^T = \frac{e^{-(s+3)5}}{s+3} - \frac{e^{-(s+3)T}}{s+3}\,,$$

whereby

$$\begin{aligned}\mathcal{L}[f](s) = \lim_{T\to\infty}\int_5^T e^{-(s+3)t}\,\mathrm{d}t &= \lim_{T\to\infty}\left[\frac{e^{-(s+3)5}}{s+3} - \frac{e^{-(s+3)T}}{s+3}\right] \\ &= \frac{e^{-(s+3)5}}{s+3} \qquad \text{for } s > -3\,.\end{aligned}$$

For $s \leq -3$ we have $e^{-(s+3)t} \geq 1$, so for $T > 5$

$$\int_5^T e^{-(s+3)t}\,\mathrm{d}t \geq \int_5^T \mathrm{d}t = T - 5\,,$$

whereby $\mathcal{L}[f](s)$ is undefined for $s \leq -3$ because

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\int_5^T e^{-(s+3)t}\,\mathrm{d}t \geq \lim_{T\to\infty}(T-5) = \infty \qquad \text{for } s \leq -3\,.$$

**Remark.** You must give a reason why $\mathcal{L}[f](s)$ is undefined for $s \leq -3$ for full credit. You must use the definition of the Laplace transform for any credit.

(2) [1] Find the exponential order of $h(t) = u(t-4)t^2e^{-3t}\sin(5t)$.

**Solution.** Because the exponential order of its factors $u(t-4)$, $t^2$, $e^{-3t}$ and $\sin(5t)$ are 0, 0, $-3$, and 0 respectively, the exponential order of $h(t) = u(t-4)t^2e^{-3t}\sin(5t)$ is $0 + 0 + (-3) + 0 = -3$.

**Remark.** Your reasoning must be given for any credit.

(3) [5] Find the Laplace transform $X(s)$ of the solution $x(t)$ of the initial-value problem

$$x'' + 4x' + 29x = 7\cos(3t)\,, \qquad x(0) = 2\,, \quad x'(0) = -4\,.$$

DO NOT solve for $x(t)$, just $X(s)$! You should refer to the short table above.

**Solution.** The Laplace transform of the differential equation is

$$\mathcal{L}[x''](s) + 4\mathcal{L}[x'](s) + 29\mathcal{L}[x](s) = 7\mathcal{L}[\cos(3t)](s)\,,$$

where the initial conditions give

$$\begin{aligned}
\mathcal{L}[x](s) &= X(s)\,,\\
\mathcal{L}[x'](s) &= s\mathcal{L}[x](s) - x(0) = sX(s) - 2\,,\\
\mathcal{L}[x''](s) &= s\mathcal{L}[x'](s) - x'(0) = s\big(sX(s) - 2\big) + 4 = s^2X(s) - 2s + 4\,.
\end{aligned}$$

The second item in the table with $b = 3$ gives

$$\mathcal{L}[\cos(3t)](s) = \frac{s}{s^2 + 3^2} = \frac{s}{s^2 + 9}\,.$$

By placing these into the Laplace transform of the differential equation we get

$$\big(s^2X(s) - 2s + 4\big) + 4\big(sX(s) - 2\big) + 29X(s) = 7\,\frac{s}{s^2 + 9}\,,$$

which yields

$$(s^2 + 4s + 29)X(s) - 2s - 4 = \frac{7s}{s^2 + 9}\,,$$

whereby

$$X(s) = \frac{1}{s^2 + 4s + 29}\left(2s + 4 + \frac{7s}{s^2 + 9}\right).$$