Quiz 6 Solutions, Math 246, Professor David Levermore
Thursday, 15 October 2020

(1) [3] Give the degree, characteristic, and multiplicity for the forcing term of the equation

$$v'' + 6v' + 45v = 4t^2 e^{-3t} \cos(6t)\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 + 6\mathrm{D} + 45$. Its characteristic polynomial is $p(z) = z^2 + 6z + 45 = (z+3)^2 + 6^2$, which has conjugate pair of roots $-3 \pm i6$.

The forcing term $4t^2 e^{-3t}\cos(6t)$ has degree $d = 2$, characteristic $\mu + i\nu = -3 + i6$, and multiplicity $m = 1$.

(2) [3] Find a particular real solution of the equation

$$y'' - 4y' + 4y = 32e^{2t}\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 - 4\mathrm{D} + 4$. Its characteristic polynomial is $p(z) = z^2 - 4z + 4 = (z-2)^2$, which has the double root 2.

Its forcing has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = 2$, and multiplicity $m = 2$. A particular solution $y_P(t)$ should be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Below we show that each of these methods gives the particular solution

$$y_P(t) = 16t^2 e^{2t}\,.$$

Key Identity Evaluations. Because $m = m + d = 2$ and $\mu + i\nu = 2$, we need to evaluate the second derivative of the Key Identity at $z = 2$. Because $p(z) = z^2 - 4z + 4$, the Key Identity and its first two derivatives with respect to z are

$$\begin{aligned}
\mathrm{L}(e^{zt}) &= (z^2 - 4z + 4)e^{zt}\,,\\
\mathrm{L}(t\,e^{zt}) &= (z^2 - 4z + 4)t\,e^{zt} + (2z - 4)e^{zt}\,,\\
\mathrm{L}(t^2 e^{zt}) &= (z^2 - 4z + 4)t^2 e^{zt} + 2(2z - 4)t\,e^{zt} + 2e^{zt}\,.
\end{aligned}$$

By evaluating the second derivative of the Key Identity at $z = 2$ we obtain

$$\mathrm{L}(t^2 e^{2t}) = (2^2 - 4\cdot 2 + 4)t^2 e^{2t} + 2(2\cdot 2 - 4)t\,e^{2t} + 2e^{2t} = 2e^{2t}\,.$$

Therefore a particular solution of $\mathrm{L}(y) = 32e^{2t}$ is

$$y_P(t) = 16t^2 e^{2t}\,.$$

Zero Degree Formula. This formula can be used because $d = 0$. For a forcing in the phasor form

$$f(t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)\,,$$

it gives the particular solution

$$y_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right)\,.$$

For this problem $f(t) = 32e^{2t}$ and $p(z) = z^2 - 4z + 4$, so that $\mu + i\nu = 2$, $\alpha - i\beta = 32$, $m = 2$, and $p''(z) = 2$, whereby

$$y_P(t) = t^2 e^{2t} \frac{32}{p''(2)} = \frac{32}{2} t^2 e^{2t} = 16t^2 e^{2t} .$$

Undetermined Coefficients. Because $m = m + d = 2$ and $\mu + i\nu = 2$, there is a particular solution of $\mathrm{L}(y) = 32e^{2t}$ in the form

$$y_P(t) = At^2 e^{2t} .$$

By taking derivatives we get

$$\begin{aligned} y_P'(t) &= At^2 \cdot 2e^{2t} + A2t \cdot e^{2t} \\ &= 2At^2 e^{2t} + 2At\, e^{2t} , \\ y_P''(t) &= At^2 \cdot 4e^{2t} + 2A2t \cdot 2e^{2t} + A2 \cdot e^{2t} \\ &= 4At^2 e^{2t} + 8At\, e^{2t} + 2A\, e^{2t} , \end{aligned}$$

whereby

$$\begin{aligned} \mathrm{L}\big(y_P(t)\big) &= y_P''(t) - 4y_P'(t) + 4y_P(t) \\ &= \big[4At^2 e^{2t} + 8At\, e^{2t} + 2A\, e^{2t}\big] - 4\big[2At^2 e^{2t} + 2At\, e^{2t}\big] + 4At^2 e^{2t} \\ &= \big(4 - 4 \cdot 2 + 4\big) At^2 e^{2t} + \big(8 - 4 \cdot 2\big) At\, e^{2t} + 2Ae^{2t} = 2Ae^{2t} . \end{aligned}$$

By setting $2Ae^{2t} = 32e^{2t}$ we see that $2A = 32$, whereby $A = 16$. Therefore a particular solution of $\mathrm{L}(y) = 32e^{2t}$ is

$$y_P(t) = 16t^2 e^{2t} .$$

(3) [4] Find a particular real solution of the equation

$$\ddot{x} + 2\dot{x} + 5x = 10\cos(t) .$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 + 2\mathrm{D} + 5$. Its characteristic polynomial is $p(z) = z^2 + 2z + 5 = (z + 1)^2 + 2^2$, which has the roots $-1 \pm i2$.

Its forcing has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = i$, and multiplicity $m = 0$. A particular solution $y_P(t)$ should be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Below we show that each of these methods gives the particular solution

$$x_P(t) = 2\cos(t) + \sin(t) .$$

Key Identity Evaluations. Because $m = m + d = 0$ and $\mu + i\nu = i$, we need to evaluate the Key Identity at $z = i$. Because $p(z) = z^2 + 2z + 5$, the Key Identity is

$$\mathrm{L}(e^{zt}) = (z^2 + 2z + 5)e^{zt} ,$$

By evaluating this at $z = i$ we obtain

$$\mathrm{L}(e^{it}) = (i^2 + 2i + 5)e^{it} = (4 + i2)e^{it} ,$$

Because the forcing has the phasor form $10\cos(t) = \mathrm{Re}(10e^{it})$ and

$$\mathrm{L}\bigg(\frac{10e^{it}}{4+i2}\bigg) = 10e^{it}\,,$$

a particular solution of $\mathrm{L}(x) = 10\cos(t)$ is given by

$$\begin{aligned} x_P(t) &= \mathrm{Re}\bigg(\frac{10e^{it}}{4+i2}\bigg) = 5\,\mathrm{Re}\bigg(\frac{e^{it}}{2+i}\bigg) = 5\,\mathrm{Re}\bigg(\frac{2-i}{2-i}\cdot\frac{e^{it}}{2+i}\bigg) \\ &= \frac{5}{2^2+1^2}\,\mathrm{Re}\big((2-i)e^{it}\big) = \mathrm{Re}\Big((2-i)\big(\cos(t)+i\sin(t)\big)\Big) \\ &= 2\cos(t) + \sin(t)\,. \end{aligned}$$

Zero Degree Formula. This formula can be used because $d = 0$. For a forcing in the phasor form

$$f(t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)\,,$$

it gives the particular solution

$$x_P(t) = t^m e^{\mu t}\,\mathrm{Re}\bigg(\frac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\bigg)\,.$$

For this problem $f(t) = 10\cos(t)$ and $p(z) = z^2+2z+5$, so that $\mu+i\nu = i$, $\alpha - i\beta = 10$, $m = 0$, and $p(i) = i^2 + 2i + 5 = 4 + i2$, whereby

$$\begin{aligned} x_P(t) &= \mathrm{Re}\bigg(\frac{10e^{it}}{4+i2}\bigg) = 5\,\mathrm{Re}\bigg(\frac{e^{it}}{2+i}\bigg) = 5\,\mathrm{Re}\bigg(\frac{2-i}{2-i}\cdot\frac{e^{it}}{2+i}\bigg) \\ &= \frac{5}{2^2+1^2}\,\mathrm{Re}\big((2-i)e^{it}\big) = \mathrm{Re}\Big((2-i)\big(\cos(t)+i\sin(t)\big)\Big) \\ &= 2\cos(t) + \sin(t)\,. \end{aligned}$$

Undetermined Coefficients. Because $m = m + d = 0$ and $\mu + i\nu = i$, there is a particular solution of $\mathrm{L}(x) = 10\cos(t)$ in the form

$$x_P(t) = A\cos(t) + B\sin(t)\,.$$

By taking derivatives we get

$$\dot{x}_P(t) = -A\sin(t) + B\cos(t)\,, \qquad \ddot{x}_P(t) = -A\cos(t) - B\sin(t)\,.$$

whereby

$$\begin{aligned} \mathrm{L}\big(x_P(t)\big) &= \ddot{x}_P(t) + 2\dot{x}_P(t) + 5x_P(t) \\ &= \big[-A\cos(t) - B\sin(t)\big] + 2\big[-A\sin(t) + B\cos(t)\big] \\ &\quad + 5\big[A\cos(t) + B\sin(t)\big] \\ &= \big(4A + 2B\big)\cos(t) + \big(4B - 2A\big)\sin(t)\,. \end{aligned}$$

By setting $\mathrm{L}\big(x_p(t)\big) = 10\cos(t)$ we see that

$$4A + 2B = 10\,, \qquad 4B - 2A = 0\,.$$

This linear algebraic system has solution $A = 2$, $B = 1$, whereby

$$x_P(t) = 2\cos(t) + \sin(t)\,.$$