Quiz 5 Solutions, Math 246, Professor David Levermore
Thursday, 8 October 2020

(1) [3] Given that $x = c_1 e^{2t} + c_2 t\, e^{2t}$ is a general solution of the equation $x'' - 4x' + 4x = 0$, find the natural fundamental set of solutions to this equation associated with $t = 0$.

Solution. The general initial-value problem associated with $t = 0$ is

$$x'' - 4x' + 4x = 0\,, \qquad x(0) = x_0\,, \quad x'(0) = x_1\,.$$

We are given that $x(t) = c_1 e^{2t} + c_2 t\, e^{2t}$ is a general solution of the differential equation. Its derivative is

$$x'(t) = c_1 2e^{2t} + c_2\big(e^{2t} + 2t\, e^{2t}\big)\,,$$

so the initial conditions of the general initial-value problem yield

$$\begin{aligned} x_0 &= x(0) = c_1 e^0 + c_2 0\, e^0 = c_1\,, \\ x_1 &= x'(0) = c_1 2e^0 + c_2\big(e^0 + 0\, e^0\big) = 2c_1 + c_2\,. \end{aligned}$$

The solution of this linear algebraic system is $c_1 = x_0$ and $c_2 = x_1 - 2x_0$. Therefore the solution of the general initial-value problem is

$$\begin{aligned} x &= x_0 e^{2t} + (x_1 - 2x_0) t\, e^{2t} \\ &= \big(e^{2t} - 2te^{2t}\big) x_0 + t\, e^{2t} x_1\,. \end{aligned}$$

We thereby see that the natural fundamental set of solutions associated with $t = 0$ is

$$N_0(t) = (1 - 2t)e^{2t}\,, \qquad N_1(t) = t\, e^{2t}\,.$$

Remark. This is a second-order, homogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 - 4\mathrm{D} + 4$. Its characteristic polynomial is $p(z) = z^2 - 4z + 4 = (z - 2)^2$, which has roots 2, 2. Therefore e^{2t} and $t\, e^{2t}$ are a fundamental set of solutions for it and $x = c_1 e^{2t} + c_2 t\, e^{2t}$ is a general solution of it.

(2) [5] Give a real general solution to the equation

$$\big(\mathrm{D}^2 - 6\mathrm{D} + 13\big)^2 (\mathrm{D} + 4)^3 y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a seventh-order, homogeneous, linear differential equation with constant coefficients. Its linear operator is $\mathrm{L} = \big(\mathrm{D}^2 - 6\mathrm{D} + 13\big)^2 (\mathrm{D} + 4)^3$. Its characteristic polynomial is

$$p(z) = (z^2 - 6z + 13)^2 (z + 4)^3 = \big((z - 3)^2 + 2^2\big)^2 (z + 4)^3\,,$$

whch has roots

$$3 + i2\,, \quad 3 + i2\,, \quad 3 - i2\,, \quad 3 - i2\,, \quad -4\,, \quad -4\,, \quad -4\,.$$

A fundamental set of seven real-valued solutions is built as follows.

- ⋄ The conjugate pair of double roots $3 \pm i2$ contributes

$$e^{3t}\cos(2t)\,, \quad e^{3t}\sin(2t)\,, \quad t\, e^{3t}\cos(2t)\,, \quad \text{and} \quad t\, e^{3t}\sin(2t)\,.$$

- ⋄ The triple real root -4 contributes

$$e^{-4t}\,, \quad t\, e^{-4t}\,, \quad \text{and} \quad t^2 e^{-4t}\,.$$

Therefore a real general solution of the equation $\mathrm{L}y = 0$ is

$$y = c_1 e^{3t}\cos(2t) + c_2 e^{3t}\sin(2t) + c_3 t\, e^{3t}\cos(2t) + c_4 t\, e^{3t}\sin(2t) \\ + c_5 e^{-4t} + c_6 t\, e^{-4t} + c_7 t^2 e^{-4t}\,.$$

(3) [2] Give a real general solution to the equation

$$v'' + 4v' + 3v = 30e^{2t}\,,$$

given that $v = 2e^{2t}$ is a solution to it and that e^{-t} and e^{-3t} are a fundamental set of solutions to the associated homogeneous equation.

Solution. Because we are given that e^{-t} and e^{-3t} are a fundamental set of solutions to the associated homogeneous equation, a real general solution of the associated homogeneous equation is

$$v_H(t) = c_1 e^{-t} + c_2 e^{-3t}\,.$$

Because we are given that a particular solution is $v_P(t) = 2e^{2t}$, a real general solution of the nonhomogeneous equation is

$$v(t) = c_1 e^{-t} + c_2 e^{-3t} + 2e^{2t}\,.$$

Remark. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 + 4\mathrm{D} + 3$. Its characteristic polynomial is $p(z) = z^2 + 4z + 3 = (z+1)(z+3)$, which has roots -1 and -3. Therefore e^{-t} and e^{-3t} are a fundamental set of solutions to the associated homogeneous equation.

Remark. The forcing $36e^{2t}$ has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = 2$, and multiplicity $m = 0$. Therefore you should be able to find a particular solution of the equation by using either Key Identity Evaluations, the Zero Degree Formula, Undetermined Coefficients, or the Green Function. Try all four methods! Do they give the same result?