Quiz 4 Solutions, Math 246, Professor David Levermore
Thursday, 1 October 2020

(1) [3] Determine the interval of definition for the solution to the initial-value problem

$$y''' - 6t\,y'' + \frac{e^t}{6-t}\,y' - \frac{3}{\sin(t)}\,y = \frac{\cos(2t)}{5+t}\,, \qquad y(4) = y'(4) = y''(4) = -3\,.$$

Solution. This nonhomogeneous linear equation for y is already in normal form. Notice that

- ⋄ the coefficient of y'' is continuous everywhere;
- ⋄ the coefficient of y' is undefined at $t = 6$ and is continuous elsewhere;
- ⋄ the coefficient of y is undefined at $t = n\pi$ for every integer n and is continuous elsewhere;
- ⋄ the forcing is undefined at $t = -5$ and is continuous elsewhere;
- ⋄ the initial time is $t = 4$.

Therefore the interval of definition is $(\pi, 6)$ because

- the initial time 4 is in $(\pi, 6)$,
- all the coefficients and the forcing are continuous over $(\pi, 6)$,
- the coefficient of y is undefined at $t = \pi$,
- the coefficient of y' is undefined at $t = 6$.

Remark. All four reasons must be given for full credit.

- ∘ The first two are why a (unique) solution exists over the interval $(\pi, 6)$.
- ∘ The last two are why this solution does not exist over a larger interval.

(2) [3] Let $U_1(t) = \cos(3t)$ and $U_2(t) = \sin(3t)$. Compute their Wronskian $\mathrm{Wr}[U_1, U_2](t)$. (Evaluate the determinant and simplify.)

Solution. Because $U_1'(t) = -3\sin(3t)$ and $U_2'(t) = 3\cos(3t)$, the Wronskian is

$$\begin{aligned}\mathrm{Wr}[U_1, U_2](t) &= \det\begin{pmatrix} U_1(t) & U_2(t) \\ U_1'(t) & U_2'(t)\end{pmatrix} = \det\begin{pmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t)\end{pmatrix} \\ &= \cos(3t)\big(3\cos(3t)\big) - \big(-3\sin(3t)\big)\sin(3t) \\ &= 3\big(\cos(3t)\big)^2 + 3\big(\sin(3t)\big)^2 = 3\,.\end{aligned}$$

Remark. It is easily checked that $U_1(t) = \cos(3t)$ and $U_2(t) = \sin(3t)$ are solutions of $u'' + 9u = 0$. Becasue their Wronskian is nonzero, they comprise a fundamental set of solutions for the homogeneous second-order equation $u'' + 9u = 0$.

Remark. Because $U_1(t) = \cos(3t)$ and $U_2(t) = \sin(3t)$ solve the homogeneous second-order equation $u''+9u = 0$, the Abel Theorem states that $w(t) = \mathrm{Wr}[U_1, U_2](t)$ should solve the homogeneous first-order equation $w' = 0$. This gives a check on the Wronskian calculation above, because $w = 3$ solves $w' = 0$.

(3) [4] Given that $\cos(3t)$ and $\sin(3t)$ are solutions of $u'' + 9u = 0$, solve the general initial-value problem associated with $t = 0$ — namely, solve

$$u'' + 9u = 0\,, \qquad u(0) = u_0\,, \quad u'(0) = u_1\,.$$

Solution. This is a homogeneous linear equation with constant coefficients. Because we are given that $\cos(3t)$ and $\sin(3t)$ are solutions to it, we can use the method of linear superposition to seek the solution of the general initial-value problem in the form

$$u(t) = c_1\,\cos(3t) + c_2\,\sin(3t)\,.$$

It follows that

$$u'(t) = -3c_1\,\sin(3t) + 3c_2\,\cos(3t)\,.$$

Then the initial conditions yield

$$u_0 = u(0) = c_1\,\cos(0) + c_2\,\sin(0) = c_1\,,$$
$$u_1 = u'(0) = -3c_1\,\sin(0) + 3c_2\,\cos(0) = 3c_2\,.$$

It follows that

$$c_1 = u_0\,, \qquad c_2 = \tfrac{1}{3}u_1\,.$$

Therefore the solution of the general initial-value problem is

$$u = u_0\,\cos(3t) + u_1\,\tfrac{1}{3}\sin(3t)\,.$$