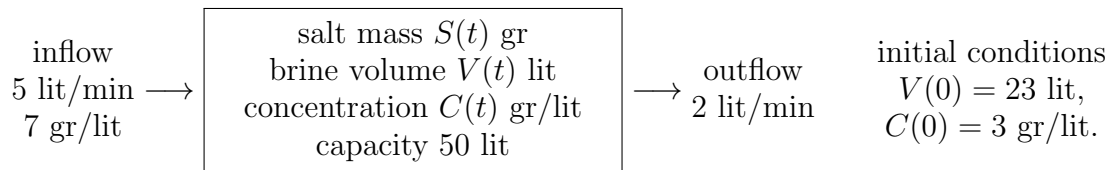


**Quiz 3 Solutions, Math 246, Professor David Levermore**  
**Thursday, 17 September 2020**

- (1) [5] A tank with a capacity of 50 liters initially contains 23 liters of brine (salt water) with a salt concentration of 3 grams per liter (gr/lit). At time  $t = 0$  brine with a salt concentration of 7 grams per liter (gr/lit) begins to flow into the tank at a constant rate of 5 liters per minute (lit/min) and the well-stirred mixture flows out of the tank at a constant rate of 2 liters per minute (lit/min).
- (a) [1] Find  $V(t)$ , the volume (lit) of brine in the tank as a function of time.
- (b) [4] Give an initial-value problem that governs  $S(t)$ , the grams of salt in the tank for  $t > 0$  until the tank overflows. (Do not solve the initial-value problem!)

**Picture.** Let  $S(t)$  be the amount (gr) of salt and  $V(t)$  be the volume (lit) of brine in the tank at time  $t$  minutes. We have the following (optional) picture.



We want to find  $V(t)$  and give an initial-value problem that governs  $S(t)$ .

**Solution (a).** Because the brine flows in at a rate of 5 lit/min while it flows out at a rate of 2 lit/min, the rate at which  $V(t)$  changes will be

$$\frac{dV}{dt} = (\text{flow rate in}) - (\text{flow rate out}) = 5 - 2 = 3 \quad \text{lit/min}.$$

Because  $V(0) = 23$  we thereby see that

$$V(t) = 23 + 3t.$$

**Remark.** Because the tank capacity is 50 lit, it overflows when  $V(t) = 23 + 3t = 50$ , which happens at  $t = 9$  min.

**Solution (b).** Because the mixture is well-stirred, we have  $C(t) = S(t)/V(t)$  gr/lit. Therefore the rate at which  $S(t)$  changes will be

$$\begin{aligned} \frac{dS}{dt} &= (\text{concentration in}) \cdot (\text{flow rate in}) - (\text{concentration out}) \cdot (\text{flow rate out}) \\ &= 7 \cdot 5 - C \cdot 2 = 35 - \frac{S}{V} \cdot 2 = 35 - \frac{2}{23 + 3t} S \quad \text{gr/min}. \end{aligned}$$

Because  $S(0) = C(0) \cdot V(0) = 3 \cdot 23 = 69$  gr, the initial-value problem governing  $S$  is

$$\frac{dS}{dt} = 35 - \frac{2}{23 + 3t} S, \quad S(0) = 69.$$

**Remark.** This initial-value problem governs the amount of salt in the tank beginning at  $t = 0$  and ending when the tank overflows at  $t = 9$ . So from the point of view of the word problem the interval of definition for its solution is  $[0, 9]$ .

(2) [5] Consider the solution  $r(t)$  of the initial-value problem

$$\ddot{r} = -\frac{4}{r^3}, \quad r(0) = 2, \quad \dot{r}(0) = 3.$$

Find the reduced initial-value problem satisfied by  $r(t)$ .

**Solution.** Because this second-order initial-value problem is [autonomous](#), it can be reduced to a first-order autonomous initial-value problem by the [autonomous reduction method](#). Its [auxiliary initial-value problem](#) is

$$v \frac{dv}{dr} = -\frac{4}{r^3}, \quad v(2) = 3.$$

This differential equation is [separable](#). It has the [separated differential form](#)

$$2v \, dv = -2 \frac{4}{r^3} \, dr,$$

which integrates to

$$v^2 = \frac{4}{r^2} + c.$$

The initial condition  $v(2) = 3$  implies that

$$3^2 = \frac{4}{2^2} + c,$$

which yields  $c = 9 - 1 = 8$ . Because  $v(2) = 3 > 0$  we obtain

$$v = \sqrt{\frac{4}{r^2} + 8}.$$

Therefore the [reduced initial-value problem](#) satisfied by  $r(t)$  is

$$\dot{r} = \sqrt{\frac{4}{r^2} + 8}, \quad r(0) = 2.$$