Quiz 3 Solutions, Math 246, Professor David Levermore
Thursday, 17 September 2020

(1) [5] A tank with a capacity of 50 liters initially contains 23 liters of brine (salt water) with a salt concentration of 3 grams per liter (gr/lit). At time $t = 0$ brine with a salt concentration of 7 grams per liter (gr/lit) begins to flow into the tank at a constant rate of 5 liters per minute (lit/min) and the well-stirred mixture flows out of the tank at a constant rate of 2 liters per minute (lit/min).

(a) [1] Find $V(t)$, the volume (lit) of brine in the tank as a function of time.

(b) [4] Give an initial-value problem that governs $S(t)$, the grams of salt in the tank for $t > 0$ until the tank overflows. (Do not solve the initial-value problem!)

Picture. Let $S(t)$ be the amount (gr) of salt and $V(t)$ be the volume (lit) of brine in the tank at time t minutes. We have the following (optional) picture.

inflow 5 lit/min 7 gr/lit	$\longrightarrow$	salt mass $S(t)$ gr brine volume $V(t)$ lit concentration $C(t)$ gr/lit capacity 50 lit	$\longrightarrow$	outflow 2 lit/min	initial conditions $V(0) = 23$ lit, $C(0) = 3$ gr/lit.

We want to find $V(t)$ and give an initial-value problem that governs $S(t)$.

Solution (a). Because the brine flows in at a rate of 5 lit/min while it flows out at a rate of 2 lit/min, the rate at which $V(t)$ changes will be

$$\frac{\mathrm{d}V}{\mathrm{d}t} = (\text{flow rate in}) - (\text{flow rate out}) = 5 - 2 = 3 \quad \text{lit/min}\,.$$

Because $V(0) = 23$ we thereby see that

$$V(t) = 23 + 3t\,.$$

Remark. Because the tank capacity is 50 lit, it overflows when $V(t) = 23 + 3t = 50$, which happens at $t = 9$ min.

Solution (b). Because the mixture is well-stirred, we have $C(t) = S(t)/V(t)$ gr/lit. Therefore the rate at which $S(t)$ changes will be

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= (\text{concentration in}) \cdot (\text{flow rate in}) - (\text{concentration out}) \cdot (\text{flow rate out}) \\ &= 7 \cdot 5 - C \cdot 2 = 35 - \frac{S}{V} \cdot 2 = 35 - \frac{2}{23 + 3t}\, S \quad \text{gr/min}\,.\end{aligned}$$

Because $S(0) = C(0) \cdot V(0) = 3 \cdot 23 = 69$ gr, the initial-value problem governing S is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 35 - \frac{2}{23 + 3t}\, S\,, \qquad S(0) = 69\,.$$

Remark. This initial-value problem governs the amount of salt in the tank beginning at $t = 0$ and ending when the tank overflows at $t = 9$. So from the point of view of the word problem the interval of definition for its solution is $[0, 9]$.

(2) [5] Consider the solution $r(t)$ of the initial-value problem

$$\ddot{r} = -\frac{4}{r^3}, \qquad r(0) = 2, \quad \dot{r}(0) = 3.$$

Find the reduced initial-value problem satisfied by $r(t)$.

Solution. Because this second-order initial-value problem is autonomous, it can be reduced to a first-order autonomous initial-value problem by the autonomous reduction method. Its auxiliary initial-value problem is

$$v\,\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{4}{r^3}, \qquad v(2) = 3.$$

This differential equation is separable. It has the separated differential form

$$2v\,dv = -2\,\frac{4}{r^3}\,\mathrm{d}r,$$

which integrates to

$$v^2 = \frac{4}{r^2} + c.$$

The initial condition $v(2) = 3$ implies that

$$3^2 = \frac{4}{2^2} + c,$$

which yields $c = 9 - 1 = 8$. Because $v(2) = 3 > 0$ we obtain

$$v = \sqrt{\frac{4}{r^2} + 8}.$$

Therefore the reduced initial-value problem satisfied by $r(t)$ is

$$\dot{r} = \sqrt{\frac{4}{r^2} + 8}, \qquad r(0) = 2.$$