Quiz 2 Solutions, Math 246, Professor David Levermore
Thursday, 10 September 2020

(1) [5] Find the explicit solution of the initial-value problem

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos(x)\,\frac{v^2+9}{2v}\,, \qquad v(0) = -3\,.$$

Solution. This is a nonautonomous, separable equation in normal form with factors

$$f(x) = \cos(x)\,, \qquad g(v) = \frac{v^2+9}{2v}\,.$$

The factor $f(x)$ is continuous everywhere. The factor $g(v)$ is undefind at $v = 0$ and is continuous elsewhere. Because $v^2 + 9 > 0$, it has no stationary points. Its separated differential form is

$$\frac{2v}{v^2+9}\,\mathrm{d}v = \cos(x)\,\mathrm{d}x\,,$$

whereby

$$\int \frac{2v}{v^2+9}\,\mathrm{d}v = \int \cos(x)\,\mathrm{d}x\,.$$

Upon integrating both sides we find the implicit general solution

$$\log(v^2+9) = \sin(x) + c\,.$$

Imposing the initial condition $v(0) = -3$ implies that

$$\log\big((-3)^2+9\big) = \sin(0) + c\,,$$

whereby $c = \log(18)$. Therefore an implicit solution of the initial-value problem is

$$\log(v^2+9) = \sin(x) + \log(18)\,,$$

which can also be expressed as

$$v^2 + 9 = 18\exp(\sin(x))\,.$$

Upon solving this for v we find that

$$v = \pm\sqrt{18\exp(\sin(x)) - 9} = \pm 3\sqrt{2\exp(\sin(x)) - 1}\,.$$

The initial condtion $v(0) = -3$ then implies that the solution is

$$v = -3\sqrt{2\exp(\sin(x)) - 1}\,.$$

Remark. Because the differental equation is undefined at $v = 0$ while the inital condition is $v(0) = -3 < 0$, the solution of the initial-value problem must satisfy $v < 0$ over its interval of definition. This means that the interval of definition of the solution will be the interval containing the initial point $x = 0$ over which $2\exp(\sin(x)) - 1 > 0$. This inequality holds if and only if

$$\sin(x) > \log(\tfrac{1}{2}) = -\log(2)\,.$$

Because $0 < \log(2) < 1$, we see that the interval of definition for the solution is

$$-\sin^{-1}(\log(2)) < x < \pi + \sin^{-1}(\log(2))\,.$$

(2) [5] Consider the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{(u+3)^3(u-1)(u-5)^2}{(u^2+4)^2(u+7)}\,.$$

(a) [3] Sketch its phase-line portrait and identify each stationary point as being either stable, unstable, or semistable. (You do not have to find the solution!)

(b) [2] How does $u(t)$ behave as $t \to \infty$ if $u(2) = 0$? if $u(-2) = 4$?

Solution (a). This equation is autonomous. Its right-hand side is undefined at $u = -7$ and is differentiable elsewhere. Its stationary points are found by setting

$$\frac{(u+3)^3(u-1)(u-5)^2}{(u^2+4)^2(u+7)} = 0\,.$$

Therefore the stationary points are $u = -3$, $u = 1$, and $u = 5$. A sign analysis of the right-hand side shows that the phase-line portrait is

$$\begin{array}{ccccccccc} - & & + & & - & & + & & + \\ \to\to\to\to & \circ & \to\to\to\to & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \to\to\to\to & \bullet & \to\to\to\to \quad u \\ & -7 & & -3 & & 1 & & 5 & \\ & \text{undefined} & & \text{stable} & & \text{unstable} & & \text{semistable} & \end{array}$$

Remark. Here the terms stable, unstable, and semistable are applied to solutions. The point $u = -7$ is not a solution, so these terms should not be applied to it.

Solution (b). Because the initial condition $u(2) = 0$ has the initial value 0, which lies in the interval $(-3, 1)$, the phase-line portrait shows that

$$\lim_{t\to\infty} u(t) = -3\,.$$

Because the initial condition $u(-2) = 4$ has the initial value 4, which lies in the interval $(1, 5)$, the phase-line portrait shows that

$$\lim_{t\to\infty} u(t) = 5\,.$$