

Quiz 2 Solutions, Math 246, Professor David Levermore
Thursday, 10 September 2020

(1) [5] Find the explicit solution of the initial-value problem

$$\frac{dv}{dx} = \cos(x) \frac{v^2 + 9}{2v}, \quad v(0) = -3.$$

Solution. This is a nonautonomous, separable equation in normal form with factors

$$f(x) = \cos(x), \quad g(v) = \frac{v^2 + 9}{2v}.$$

The factor $f(x)$ is continuous everywhere. The factor $g(v)$ is undefined at $v = 0$ and is continuous elsewhere. Because $v^2 + 9 > 0$, it has no stationary points. Its separated differential form is

$$\frac{2v}{v^2 + 9} dv = \cos(x) dx,$$

whereby

$$\int \frac{2v}{v^2 + 9} dv = \int \cos(x) dx.$$

Upon integrating both sides we find the implicit general solution

$$\log(v^2 + 9) = \sin(x) + c.$$

Imposing the initial condition $v(0) = -3$ implies that

$$\log((-3)^2 + 9) = \sin(0) + c,$$

whereby $c = \log(18)$. Therefore an implicit solution of the initial-value problem is

$$\log(v^2 + 9) = \sin(x) + \log(18),$$

which can also be expressed as

$$v^2 + 9 = 18 \exp(\sin(x)).$$

Upon solving this for v we find that

$$v = \pm \sqrt{18 \exp(\sin(x)) - 9} = \pm 3 \sqrt{2 \exp(\sin(x)) - 1}.$$

The initial condition $v(0) = -3$ then implies that the solution is

$$v = -3 \sqrt{2 \exp(\sin(x)) - 1}.$$

Remark. Because the differential equation is undefined at $v = 0$ while the initial condition is $v(0) = -3 < 0$, the solution of the initial-value problem must satisfy $v < 0$ over its interval of definition. This means that the interval of definition of the solution will be the interval containing the initial point $x = 0$ over which $2 \exp(\sin(x)) - 1 > 0$. This inequality holds if and only if

$$\sin(x) > \log\left(\frac{1}{2}\right) = -\log(2).$$

Because $0 < \log(2) < 1$, we see that the interval of definition for the solution is

$$-\sin^{-1}(\log(2)) < x < \pi + \sin^{-1}(\log(2)).$$

(2) [5] Consider the equation

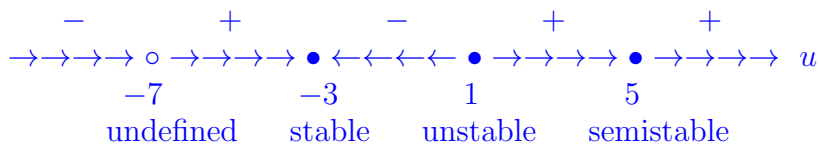
$$\frac{du}{dt} = \frac{(u+3)^3(u-1)(u-5)^2}{(u^2+4)^2(u+7)}.$$

- (a) [3] Sketch its phase-line portrait and identify each stationary point as being either stable, unstable, or semistable. (You do not have to find the solution!)
 (b) [2] How does $u(t)$ behave as $t \rightarrow \infty$ if $u(2) = 0$? if $u(-2) = 4$?

Solution (a). This equation is autonomous. Its right-hand side is undefined at $u = -7$ and is differentiable elsewhere. Its stationary points are found by setting

$$\frac{(u+3)^3(u-1)(u-5)^2}{(u^2+4)^2(u+7)} = 0.$$

Therefore the stationary points are $u = -3$, $u = 1$, and $u = 5$. A sign analysis of the right-hand side shows that the phase-line portrait is



Remark. Here the terms stable, unstable, and semistable are applied to solutions. The point $u = -7$ is not a solution, so these terms should not be applied to it.

Solution (b). Because the initial condition $u(2) = 0$ has the initial value 0, which lies in the interval $(-3, 1)$, the phase-line portrait shows that

$$\lim_{t \rightarrow \infty} u(t) = -3.$$

Because the initial condition $u(-2) = 4$ has the initial value 4, which lies in the interval $(1, 5)$, the phase-line portrait shows that

$$\lim_{t \rightarrow \infty} u(t) = 5.$$