**Quiz 1 Solutions, Math 246, Professor David Levermore**
**Thursday, 3 September 2020**

(1) [4] For each of the following ordinary differential equations, determine its order and whether it is linear or nonlinear. If it is nonlinear, write down term that makes it so.

(a) $v''' + e^v v' = 2\cos(3t)v + 4e^{5t}$

(b) $u'''' + 3\sin(2t)u' = 2\cos(4t)$

**Solution (a).** The equation is *third order* because $v'''$ is the highest order deriviative that appears. It is *nonlinear* because of the $e^v v'$ term.

**Solution (b).** The equation is *fourth order* because $u''''$ is the highest order derivative that appears. It is *linear* and is already in normal form.

(2) [2] What is the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{e^x}{x^2 - 4}\, w = \frac{\sin(x)}{x^2 - 36}\,, \qquad w(3) = -7\,.$$

(You do not need to solve the differential equation, but you must give your reasoning!)

**Solution.** This is a nonhomogeneous linear equation that is already in normal form. Observe that

– its coefficient $\dfrac{e^x}{x^2 - 4}$ is undefined at $x = \pm 2$ and is continuous elsewhere;

– its forcing $\dfrac{\sin(x)}{x^2 - 36}$ is undefined at $x = \pm 6$ and is continuous elsewhere.

The initial time is $x = 3$. The situation is pictured on the $x$-axis as follows.

$$- - \circ - - + - - + - - + - - \circ - - + - - + - - + - - \circ - - \bullet - - + - - + - - \circ - \rightarrow x$$
$$-6 \quad -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

Therefore the interval of definition for the solution is $(2, 6)$ because:

- the initial time $x = 3$ is in $(2, 6)$,
- the coefficient and forcing are both continuous over $(2, 6)$,
- the coefficient is undefined at $x = 2$,
- the forcing is undefined at $z = 6$.

**Remark.** All four reasons had to be given for full credit. The first two reasons are why a unique solution exists over $(2, 6)$, The third reason is why the left endpoint is $x = 2$. The fourth reason is why the right endpoint $x = 6$.

(3) [4] Solve the initial-value problem

$$(1+t)\,\frac{\mathrm{d}y}{\mathrm{d}t} + 2\,y = 6t^2\,, \qquad y(0) = 5\,.$$

**Solution.** This is a nonhomogeneous linear equation. Its normal form is

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2}{1+t}\,y = \frac{6t^2}{1+t}\,.$$

The coefficient and forcing are both undefined at $t = -1$ and continuous elsewhere. The initial time is $t = 0$. The interval of definition of the solution is thereby $(-1, \infty)$. An integrating factor is $e^{A(t)}$ where $A'(t) = 2/(1+t)$. Setting

$$A(t) = 2\log(|1+t|) = \log\big((1+t)^2\big)\ ,$$

we obtain $e^{A(t)} = (1+t)^2$. The equation thereby has the intgrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1+t)^2 y\big) = (1+t)^2 \cdot \frac{6t^2}{1+t} = 6t^2 + 6t^3\,.$$

Integrating both sides yields

$$(1+t)^2 y = 2t^3 + \tfrac{3}{2}t^4 + c\,.$$

Imposing the initial condition $y(0) = 5$ gives

$$(1+0)^2 5 = 20^3 + \tfrac{3}{2}0^4 + c\,,$$

whereby $c = 5$. Therefore the solution of the initial=value problem is

$$y = \frac{2t^3 + \frac{3}{2}t^4 + 5}{(1+t)^2}\,.$$

**Remark.** The interval of definition of this solution is indeed $(-1, \infty)$.