Math 246, Professor David Levermore
Group Work Exercises for Discussion
Wednesday, 11 November 2020

Set A of Group Work Exercises [3]

A.1. Recast the pendulum equation $y'' = -9\sin(y)$ as a first-order system.

A.2. Recast the equation $x'''' = \cos(x + x'') - e^{x'}x'''$ as a first-order system.

A.3. Determine the interval of definition for the solution of the initial-value problem

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{t^2 - 9}\begin{pmatrix} \log(t^2) & \cos(2t) \\ \sin(t) & e^{3t} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t^2 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}.$$

Set B of Group Work Exercises [3]

Two interconnected tanks contain brine (salt water). At $t = 0$ the first tank contains 23 liters and the second contains 32 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 17 grams of salt in the first tank and 29 grams in the second.

B.1. Give an initial-value problem that governs how many grams of salt are in each tank as a function of time.

B.2. Give the interval of definition for the solution of the above initial-value problem.

B.3. Express the above initial-value problem in the form

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(0) = \mathbf{x}^I.$$

Give $\mathbf{A}(t)$, $\mathbf{f}(t)$, and $\mathbf{x}^I$.

Set C of Group Work Exercises [4]

Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} -2t^2 \\ 4 - t^4 \end{pmatrix}$.

C.1. Compute their Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

C.2. Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.

C.3. Give the natural fundamental matrix for $t_I = 2$ of the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.

C.4. Solve the initial-value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(0) = \mathbf{0},$$

where

$$\mathbf{f}(t) = \begin{pmatrix} 0 \\ 2t \end{pmatrix}.$$