Final Exam Sample Problems, Math 246, Fall 2020

(1) Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = (9 - y^2)y^2$.
  (a) Find all of its stationary points and classify their stability.
  (b) Sketch its phase-line portrait in the interval $-5 \le y \le 5$.
  (c) If $y_1(0) = -1$, how does the solution $y_1(t)$ behave as $t \to \infty$?
  (d) If $y_2(0) = 4$, how does the solution $y_2(t)$ behave as $t \to \infty$?
  (e) Evaluate
$$\lim_{t\to\infty} \big(y_2(t) - y_1(t)\big)\,.$$

(2) Solve each of the following initial-value problems and give the interval of definition of each solution.

  (a) $x' = \dfrac{t}{(t^2+1)x}\,, \qquad x(0) = -3\,.$
  (b) $\dfrac{\mathrm{d}y}{\mathrm{d}t} + \dfrac{2ty}{1+t^2} = t^2\,, \qquad y(0) = 1\,.$
  (c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{e^x y + 2x}{2y + e^x} = 0\,, \qquad y(0) = 0\,.$

(3) Determine constants a and b such that the following differential equation is exact. Then find a general solution in implicit form.
$$\big(ye^x + y^3\big)\,\mathrm{d}x + \big(ae^x + bxy^2\big)\,\mathrm{d}y = 0\,.$$

(4) Consider the following Matlab function m-file.

```
function [t,y] = solveit(ti, yi, tf, n)
t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = ti; y(1) = yi; h = (tf - ti)/n;
for i = 1:n
t(i + 1) = t(i) + h; y(i + 1) = y(i) + h*((t(i))^4 + (y(i))^2);
end
```

Suppose that the input values are ti = 1, yi = 1, tf = 5, and n = 40.
  (a) What initial-value problem is being approximated numerically?
  (b) What numerical method is being used?
  (c) What is the step size?
  (d) What are the output values of t(2), y(2), t(3), and y(3)?

(5) Let $y(t)$ be the solution of the initial-value problem
$$y' = 4t(y + y^2)\,, \qquad y(0) = 1\,.$$
  (a) Use two steps of the explicit Euler method to approximate $y(1)$.
  (b) Use one step of the Runge-trapeziodal method to approximate $y(1)$.
  (c) Use one step of the Runge-midpoint method to approximate $y(1)$.

(6) Consider the following Matlab commands.

[t,y] = ode45(@(t,y) y.*(y−1).*(2−y), [0,3], -0.5:0.5:2.5); plot(t,y)

The following questions need not be answered in Matlab format!

(a) What is the differential equation being solved numerically?
(b) Give the initial condition for each solution being approximated?
(c) Over what time interval are the solutions being approximated?
(d) Sketch each of these solutions over this time interval on a single graph. Label the initial value of each solution clearly.
(e) What is the limiting behavior of each solution as $t \to \infty$?

(7) Suppose we are using the Runge-midpoint method to numerically approximate the solution of an initial-value problem over the time interval $[1, 9]$. By what factor would we expect the error to decrease when we increase the number of time steps taken from 400 to 2000?

(8) A NASA engineer has used the Runge-Kutta method to approximate the solution of an initial-value problem over the time interval $[2, 10]$ with 800 uniform time steps.

(a) How many uniform time steps are needed to reduce the global error by a factor of $\frac{1}{256}$?
(b) What is the order of a numerical method that reduces the global error by a factor of $\frac{1}{256}$ when the step size is halved?

(9) Give an explicit real-valued general solution of the following equations.

(a) $y'' - 2y' + 5y = t\,e^t + \cos(2t)$
(b) $\ddot{u} - 3\dot{u} - 10u = t\,e^{-2t}$
(c) $v'' + 9v = \cos(3t)$
(d) $w'''' + 13w'' + 36w = 9\sin(t)$

(10) Solve the following initial-value problems.

(a) $w'' + 4w' + 20w = 5e^{2t}, \qquad w(0) = 3\,, \quad w'(0) = -7.$
(b) $y'' - 4y' + 4y = \dfrac{e^{2t}}{3+t}, \qquad y(0) = 0\,, \quad y'(0) = 5.$

Evaluate any definite integrals that arise.

(11) Given that $y_1(t) = t$ and $y_2(t) = t^{-2}$ solve the associated homogeneous equation, find a general solution of

$$t^2 y'' + 2t\,y' - 2y = \frac{3}{t^2} + 5t\,, \qquad \text{for } t > 0\,.$$

(12) Given that t^2 and $t^2\log(t)$ solve the associated homogeneous differential equation, consider the initial-value problem

$$t^2 x'' - 3t\,x' + 4x = t^2\log(t)\,, \qquad x(1) = 0\,, \quad x'(1) = 0\,.$$

(a) Give the interval of definition of its solution.
(b) Compute $\mathrm{Wr}[t^2, t^2\log(t)]$.
(c) Find $x(t)$. Evaluate any definite integrals that arise.

(13) Give an explicit real-valued general solution of the equation

$$\ddot{h} + 2\dot{h} + 5h = 0\,.$$

Sketch a typical solution for $t \geq 0$. If this equation governs a spring-mass system, is the system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

(14) When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.5 m. (Gravitational acceleration is 9.8 m/sec^2.) At $t = 0$ the mass is set in motion from 0.3 meters below its rest (equilibrium) position with a upward velocity of 2 m/sec. It is acted upon by an external force of $2\cos(5t)$. Neglect damping and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (Do not solve this initial-value problem; just write it down!)

(15) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial-value problem

$$y'' + 4y' + 8y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 4\,.$$

where

$$f(t) = \begin{cases} 4 & \text{for } 0 \leq t < 2\,, \\ t^2 & \text{for } 2 \leq t\,. \end{cases}$$

You may refer to the table of Laplace transforms on the last page. (Do not take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

(16) Let $x(t)$ be the solution of the initial-value problem

$$x'' + 10x' + 29x = f(t)\,, \qquad x(0) = 3\,, \quad x'(0) = -7\,,$$

where the forcing $f(t)$ is given by

$$f(t) = \begin{cases} t^2 & \text{for } 0 \leq t < 1\,, \\ e^{1-t} & \text{for } 1 \leq t < \infty\,. \end{cases}$$

(a) Find the Laplace transform $F(s)$ of the forcing $f(t)$.
(b) Find the Laplace transform $X(s)$ of the solution $x(t)$.
(DO NOT take the inverse Laplace transform to find $x(t)$; just solve for $X(s)$!)

You may refer to the table of Laplace transforms on the last page.

(17) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by

$$\text{(a)}\quad Y(s) = \frac{e^{-3s}4}{s^2 - 6s + 5}\,, \qquad\qquad \text{(b)}\quad Y(s) = \frac{e^{-2s}s}{s^2 + 4s + 8}\,.$$

You may refer to the table of Laplace transforms on the last page.

(18) Consider the real vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3 + t^4 \end{pmatrix}$.

(a) Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.
(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the linear system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.
(c) Give a general solution to the system you found in part (b).

(19) Two interconnected tanks, each with a capacity of 60 liters, contain brine (salt water). At $t = 0$ the first tank contains 22 liters and the second contains 17 liters. Brine with a salt concentration of 6 grams per liter flows into the first tank at 7 liters per hour. Well-stirred brine flows from the first tank into the second at 8 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 2 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 31 grams of salt in the first tank and 43 grams in the second.

(a) Determine the volume of brine in each tank as a function of time.
(b) Give an initial-value problem that governs the amount of salt in each tank as a function of time. (Do not solve the IVP.)
(c) Give the interval of definition for the solution of this initial-value problem.

(20) Give a real, vector-valued general solution of the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for

$$\text{(a)}\quad \mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}, \qquad \text{(b)}\quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

(21) Sketch the phase-plane portrait of the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for

$$\text{(a)}\quad \mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}, \qquad \text{(b)}\quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

(22) What answer will be produced by the following Matlab command?

```
>> A = [1 4; 3 2]; [vect, val] = eig(sym(A))
```

You do not have to give the answer in Matlab format.

(23) A real 2×2 matrix $\mathbf{B}$ has the eigenpairs

$$\left(2, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right) \quad \text{and} \quad \left(-1, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right).$$

(a) Give a general solution to the linear planar system $\mathbf{x}' = \mathbf{B}\mathbf{x}$.
(b) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ that diagonalize $\mathbf{B}$.
(c) Compute $e^{t\mathbf{B}}$.
(d) Find $\mathbf{B}$.
(e) Sketch a phase-plane portrait for this system and identify its type. Classify the stability of the origin. Carefully mark all sketched orbits with arrows!

(24) Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}^{\mathrm{I}}$ for the following $\mathbf{A}$ and $\mathbf{x}^{\mathrm{I}}$.

(a) $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ -5 & -7 \end{pmatrix}, \qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$

(b) $\mathbf{A} = \begin{pmatrix} 8 & -5 \\ 5 & -2 \end{pmatrix}, \qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$

(c) $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}, \qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$

(25) Consider the system

$$\dot{x} = 2xy\,, \qquad \dot{y} = 9 - 9x - y^2\,.$$

(a) Find all of its stationary points.
(b) Find all of its semistationary orbits.
(c) Find a nonconstant function $H(x, y)$ such that every orbit of the system satisfies $H(x, y) = c$ for some constant c.
(d) Classify the type and stability of each stationary point.
(e) Sketch the stationary points plus the level set $H(x, y) = c$ for each value of c that corresponds to a stationary point that is a saddle. Carefully mark all sketched orbits with arrows!

(26) Consider the system

$$\dot{p} = -9p + 3q\,, \qquad \dot{q} = 4p - 8q + 10p^2\,.$$

(a) This system has two stationary points. Find them.
(b) Find the Jacobian matrix at each stationary point.
(c) Classify the type and stability of each stationary point.
(d) Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. Carefully mark all sketched orbits with arrows!

(27) Consider the system

$$u' = -5v\,, \qquad v' = u - 4v - u^2\,.$$

(a) Find all of its stationary points.
(b) Compute the Jacobian matrix at each stationary point.
(c) Classify the type and stability of each stationary point.
(d) Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. Carefully mark all sketched orbits with arrows!

(28) Consider the system

$$\dot{p} = p(3 - 3p + 2q)\,, \qquad \dot{q} = q(6 - p - q)\,.$$

(a) Find all of its stationary points.
(b) Compute the Jacobian matrix at each stationary point.
(c) Classify the type and stability of each stationary point.
(d) Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. Carefully mark all sketched orbits with arrows!
(e) Add the orbits of all semistationary solutions to the phase-plane portrait sketched for part (d). Carefully mark these sketched orbits with arrows!
(f) Why do solutions that start in the first quadrant stay in the first quadrant?

Table of Laplace Transforms

	$h(t) = \mathcal{L}^{-1}[H](t)$		$H(s) = \mathcal{L}[h](s)$	
1.	$t^n e^{at}$	for $n \geq 0$	$\dfrac{n!}{(s-a)^{n+1}}$	for $s > a$
2.	$e^{at}\cos(bt)$		$\dfrac{s-a}{(s-a)^2+b^2}$	for $s > a$
3.	$e^{at}\sin(bt)$		$\dfrac{b}{(s-a)^2+b^2}$	for $s > a$
4.	$e^{at}\cosh(bt)$		$\dfrac{s-a}{(s-a)^2-b^2}$	for $s > a+\lvert b\rvert$
5.	$e^{at}\sinh(bt)$		$\dfrac{b}{(s-a)^2-b^2}$	for $s > a+\lvert b\rvert$
6.	$t^n j(t)$	for $n \geq 0$	$(-1)^n J^{(n)}(s)$	where $J(s) = \mathcal{L}[j](s)$
7.	$j'(t)$		$s\,J(s) - j(0)$	where $J(s) = \mathcal{L}[j](s)$
8.	$e^{at} j(t)$		$J(s-a)$	where $J(s) = \mathcal{L}[j](s)$
9.	$u(t-c)j(t-c)$	for $c \geq 0$	$e^{-cs}J(s)$	where $J(s) = \mathcal{L}[j](s)$
10.	$\delta(t-c)j(t)$	for $c \geq 0$	$e^{-cs}j(c)$	

Here a, b, and c are real numbers; n is an integer; $j(t)$ is any function that is nice enough; $u(t)$ is the unit step (Heaviside) function; $\delta(t)$ is the unit impulse (Dirac delta).