Solutions of the Sample Problems for the Third Exam
Math 246, Fall 2020, Professor David Levermore

(1) Compute the Laplace transform of $f(t) = t\,e^{3t}u(t-2)$ from its definition.

Solution. The definition of the Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} t\, e^{3t} u(t-2)\,\mathrm{d}t = \lim_{T\to\infty} \int_2^T t\, e^{-(s-3)t}\,\mathrm{d}t\,.$$

This limit diverges to $+\infty$ for $s \leq 3$ because in that case for every $T > 2$ we have

$$\int_2^T t\, e^{-(s-3)t}\,\mathrm{d}t \geq \int_2^T t\,\mathrm{d}t = \frac{T^2}{2} - 2\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

For $s > 3$ an integration by parts shows that

$$\begin{aligned}
\int_2^T t\, e^{-(s-3)t}\,\mathrm{d}t &= -t\,\frac{e^{-(s-3)t}}{s-3}\bigg|_2^T + \int_2^T \frac{e^{-(s-3)t}}{s-3}\,\mathrm{d}t \\
&= \left(-\,t\,\frac{e^{-(s-3)t}}{s-3} - \frac{e^{-(s-3)t}}{(s-3)^2}\right)\bigg|_2^T \\
&= \left(-\,T\,\frac{e^{-(s-3)T}}{s-3} - \frac{e^{-(s-3)T}}{(s-3)^2}\right) + \left(2\,\frac{e^{-(s-3)2}}{s-3} + \frac{e^{-(s-3)2}}{(s-3)^2}\right).
\end{aligned}$$

Hence, for $s > 3$ we have that

$$\begin{aligned}
\mathcal{L}[f](s) &= \lim_{T\to\infty} \left[\left(-\,T\,\frac{e^{-(s-3)T}}{s-3} - \frac{e^{-(s-3)T}}{(s-3)^2}\right) + \left(2\,\frac{e^{-(s-3)2}}{s-3} + \frac{e^{-(s-3)2}}{(s-3)^2}\right)\right] \\
&= \frac{e^{-(s-3)2}}{(s-3)^2} + 2\,\frac{e^{-(s-3)2}}{s-3} - \lim_{T\to\infty}\left(T\,\frac{e^{-(s-3)T}}{s-3} + \frac{e^{-(s-3)T}}{(s-3)^2}\right) \\
&= \frac{e^{-(s-3)2}}{(s-3)^2} + 2\,\frac{e^{-(s-3)2}}{s-3}\,.
\end{aligned}$$

(2) Consider the following MATLAB commands.

```
>> syms t y(t) s Y
>> f = heaviside(t)*t^2 + heaviside(t − 3)*(3*t − t^2);
>> diffeqn = diff(y, 2) − 6*diff(y, 1) + 10*y(t) == f;
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, ...
            [laplace(y(t), t, s), y(0), subs(diff(y(t), t), t, 0)], [Y, 2, 3]);
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) Give the initial-value problem for $y(t)$ that is being solved.
(b) Find the Laplace transform $Y(s)$ of the solution $y(t)$.

DO NOT take the inverse Laplace transform of $Y(s)$ to find $y(t)$, just solve for $Y(s)$! You may refer to the table on the last page.

Solution (a). The initial-value problem for $y(t)$ that is being solved is

$$y'' - 6y' + 10y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 3\,,$$

where the forcing $f(t)$ can be expressed either as

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 3\,, \\ 3t & \text{for } 3 \le t\,, \end{cases}$$

or in terms of the unit step function as $f(t) = t^2 + u(t-3)(3t - t^2)$.

Solution (b). The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) - 6\mathcal{L}[y'](s) + 10\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = s\,Y(s) - 2\,, \\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2 Y(s) - 2s - 3\,. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, we first write $f(t)$ as

$$f(t) = t^2 + u(t-3)(3t - t^2) = t^2 + u(t-3)j(t-3)\,,$$

where by setting $j(t-3) = 3t - t^2$ we see by the shifty step method that

$$j(t) = 3(t+3) - (t+3)^2 = 3t + 9 - t^2 - 6t - 9 = -t^2 - 3t\,.$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 2$ and with $a = 0$ and $n = 1$ shows that

$$\mathcal{L}[t^2](s) = \frac{2}{s^3}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,,$$

whereby item 6 with $c = 3$ and $j(t) = -t^2 - 3t$ shows that

$$\mathcal{L}\big[u(t-3)j(t-3)\big](s) = e^{-3s}\mathcal{L}[j](s) = -e^{-3s}\mathcal{L}\big[t^2 + 3t\big](s) = -e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore

$$\mathcal{L}[f](s) = \mathcal{L}\big[t^2 + u(t-3)j(t-3)\big](s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2 Y(s) - 2s - 3\big) - 6\big(sY(s) - 2\big) + 10Y(s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right),$$

which becomes

$$(s^2 - 6s + 10)Y(s) - 2s + 9 = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 - 6s + 10}\left(2s - 9 + \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right)\right).$$

(3) Find $Y(s) = \mathcal{L}[y](s)$ where $y(t)$ solves the initial-value problem

$$y'' + 4y' + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \leq t < 2\pi\,, \\ t - 2\pi & \text{for } t \geq 2\pi\,. \end{cases}$$

DO NOT take the inverse Laplace transform of $Y(s)$ to find $y(t)$, just solve for $Y(s)$! You may refer to the table on the last page.

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 13\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = s\,Y(s) - 4\,, \\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - 4s - 1\,. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned} f(t) &= \big(1 - u(t-2\pi)\big)\cos(t) + u(t-2\pi)(t-2\pi) \\ &= \cos(t) + u(t-2\pi)\big(t - 2\pi - \cos(t)\big) \\ &= \cos(t) + u(t-2\pi)j(t-2\pi)\,, \end{aligned}$$

where by setting $j(t-2\pi) = t - 2\pi - \cos(t)$ we see by the shifty step method that

$$j(t) = (t + 2\pi) - 2\pi - \cos(t + 2\pi) = t - \cos(t)\,.$$

Here we have used the fact that $\cos(t)$ is 2π-periodic. Referring to the table on the last page, item 6 with $c = 2\pi$ shows that

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}[\cos(t)](s) + \mathcal{L}[u(t-2\pi)j(t-2\pi)](s) \\ &= \mathcal{L}[\cos(t)](s) + e^{-2\pi s}\mathcal{L}[j(t)](s) \\ &= \mathcal{L}[\cos(t)](s) + e^{-2\pi s}\mathcal{L}[t - \cos(t)](s)\,. \end{aligned}$$

Then item 2 with $a = 0$ and $b = 1$, and item 1 with $n = 1$ and $a = 1$ imply that

$$\mathcal{L}[f](s) = \frac{s}{s^2+1} + e^{-2\pi s}\left(\frac{1}{s^2} - \frac{s}{s^2+1}\right)\,.$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 4s - 1\big) + 4\big(sY(s) - 4\big) + 13Y(s) = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,,$$

which becomes

$$(s^2 + 4s + 13)Y(s) - 4s - 1 - 16 = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 4s + 13}\left(4s + 17 + \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\right)\,.$$

(4) Find $X(s) = \mathcal{L}[x](s)$ where $x(t)$ solves the initial-value problem

$$x'' + 4x = \delta(t-3)\,, \qquad x(0) = 5\,, \quad x'(0) = 0\,.$$

DO NOT take the inverse Laplace transform of $X(s)$ to find $x(t)$, just solve for $X(s)$! You may refer to the table on the last page.

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[x''](s) + 4\mathcal{L}[x](s) = \mathcal{L}[\delta(t-3)](s)\,,$$

where

$$\begin{aligned}\mathcal{L}[x](s) &= X(s)\,,\\ \mathcal{L}[x'](s) &= s\,\mathcal{L}[x](s) - x(0) = s\,X(s) - 5\,,\\ \mathcal{L}[x''](s) &= s\,\mathcal{L}[x'](s) - x'(0) = s^2X(s) - 5s - 0\,.\end{aligned}$$

Referring to the table on the last page, item 7 with $c = 3$ and $h(t) = 1$ shows that

$$\mathcal{L}[\delta(t-3)](s) = e^{-3s}\,.$$

The Laplace transform of the initial-value problem then becomes

$$(s^2+4)X(s) - 5s = e^{-3s}\,.$$

Hence, $X(s)$ is given by

$$X(s) = \frac{5s + e^{-3s}}{s^2+4}\,.$$

Remark. You should be able to take the inverse Laplace transform to obtain

$$x(t) = \mathcal{L}^{-1}[X](t) = \mathcal{L}^{-1}\left[\frac{5s + e^{-3s}}{s^2+4}\right](t) = 5\cos(2t) + \tfrac{1}{2}u(t-3)\sin(2(t-3))\,.$$

(5) Find the inverse Laplace transforms of the following functions.

(a) $F(s) = \dfrac{2}{(s+5)^2}$,

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}$,

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}$.

You may refer to the table on the last page.

Solution (a). Referring to the table on the last page, item 1 with $n = 1$ and $a = -5$ gives

$$\mathcal{L}[t\,e^{-5t}](s) = \frac{1}{(s+5)^2}\,.$$

Therefore we conclude that

$$\mathcal{L}^{-1}\left[\frac{2}{(s+5)^2}\right](t) = 2\mathcal{L}^{-1}\left[\frac{1}{(s+5)^2}\right](t) = 2t\,e^{-5t}\,.$$

Solution (b). Because the denominator factors as $(s-3)(s+2)$, we have the partial fraction identity

$$\frac{3s}{s^2-s-6} = \frac{3s}{(s-3)(s+2)} = \frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2} \,.$$

Referring to the table on the last page, item 1 with $n=0$ and $a=3$, and with $n=0$ and $a=-2$ gives

$$\mathcal{L}[e^{3t}](s) = \frac{1}{s-3}\,, \qquad \mathcal{L}[e^{-2t}](s) = \frac{1}{s+2}\,.$$

Therefore we conclude that

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{3s}{s^2-s-6}\right](t) &= \mathcal{L}^{-1}\left[\frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}\right](t) \\
&= \tfrac{9}{5}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right](t) + \tfrac{6}{5}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t) \\
&= \tfrac{9}{5}e^{3t} + \tfrac{6}{5}e^{-2t}\,.
\end{aligned}$$

Solution (c). Complete the square in the denominator to get $(s-2)^2+1$. Referring to the table on the last page, item 2 with $a=2$ and $b=1$ gives

$$\mathcal{L}[e^{2t}\cos(t)](s) = \frac{s-2}{(s-2)^2+1}\,.$$

Item 6 with $c=3$ and $j(t)=e^{2t}\cos(t)$ then gives

$$\mathcal{L}[u(t-3)e^{2(t-3)}\cos(t-3)](s) = e^{-3s}\frac{s-2}{(s-2)^2+1}\,.$$

Therefore we conclude that

$$\mathcal{L}^{-1}\left[e^{-3s}\frac{s-2}{s^2-4s+5}\right](t) = u(t-3)e^{2(t-3)}\cos(t-3)\,.$$

(6) For each of the following differential operators compute its Green function $g(t)$ and its natural fundamental set for $t=0$.

(a) $\mathrm{L} = \mathrm{D}^4 + 8\mathrm{D}^2 - 9\,,$

(b) $\mathrm{L} = (\mathrm{D}-2)^3\,.$

You may refer to the table on the last page.

Solution (a). The characteristic polynomial of $\mathrm{L} = \mathrm{D}^4+8\mathrm{D}^2-9$ is $p(s) = s^4+8s^2-9$. Therefore its Green function $g(t)$ is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s^4+8s^2-9}\right](t)\,.$$

Because $p(s)$ factors as $p(s) = (s^2-1)(s^2+9)$ we have the partial fraction identity

$$\frac{1}{s^4+8s^2-9} = \frac{1}{(s^2-1)(s^2+9)} = \frac{\frac{1}{10}}{s^2-1} + \frac{-\frac{1}{10}}{s^2+9}\,.$$

Because $s^2 - 1$ factors as $s^2 - 1 = (s-1)(s+1)$ we have the partial fraction identity

$$\frac{1}{s^2 - 1} = \frac{1}{(s-1)(s+1)} = \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}}{s+1}\,.$$

By combining the above partial fraction identities we obtain

$$\frac{1}{s^4 + 8s^2 - 9} = \tfrac{1}{20}\frac{1}{s-1} - \tfrac{1}{20}\frac{1}{s+1} - \tfrac{1}{10}\frac{1}{s^2+9}\,.$$

Referring to the table on the last page, item 1 with $a = 1$ and $n = 0$ and with $a = -1$ and $n = 0$ gives

$$\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) = e^t\,, \qquad \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = e^{-t}\,,$$

while item 3 with $a = 0$ and $b = 3$ gives

$$\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right](t) = \sin(3t)\,.$$

Therefore the Green function $g(t)$ is given by

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^4 + 8s^2 - 9}\right](t) \\ &= \tfrac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) - \tfrac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) - \tfrac{1}{30}\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right](t) \\ &= \tfrac{1}{20}e^t - \tfrac{1}{20}e^{-t} - \tfrac{1}{30}\sin(3t)\,. \end{aligned}$$

Then because we see the characteristic polynomial as

$$p(s) = s^4 + 0s^3 + 8s^2 + 0s - 9\,,$$

the natrual fundamental set for $t = 0$ is found by

$$\begin{aligned} N_3(t) &= g(t) = \tfrac{1}{20}e^t - \tfrac{1}{20}e^{-t} - \tfrac{1}{30}\sin(3t)\,, \\ N_2(t) &= N_3'(t) + 0g(t) = \tfrac{1}{20}e^t + \tfrac{1}{20}e^{-t} - \tfrac{1}{10}\cos(3t)\,, \\ N_1(t) &= N_2'(t) + 8g(t) \\ &\quad = \tfrac{1}{20}e^t - \tfrac{1}{20}e^{-t} + \tfrac{3}{10}\sin(3t) + \tfrac{8}{20}e^t - \tfrac{8}{20}e^{-t} - \tfrac{8}{30}\sin(3t)\,, \\ &\quad = \tfrac{9}{20}e^t - \tfrac{9}{20}e^{-t} + \tfrac{1}{30}\sin(3t)\,, \\ N_0(t) &= N_1'(t) + 0g(t) = \tfrac{9}{20}e^t + \tfrac{9}{20}e^{-t} + \tfrac{1}{10}\cos(3t)\,. \end{aligned}$$

Remark. The calculation of the natural fundamental set is a bit simpler if the Green function is expressed in terms of hyperbolic functions. It becomes

$$\begin{aligned} N_3(t) &= g(t) = \tfrac{1}{10}\sinh(t) - \tfrac{1}{30}\sin(3t)\,, \\ N_2(t) &= N_3'(t) + 0g(t) = \tfrac{1}{10}\cosh(t) - \tfrac{1}{10}\cos(3t)\,, \\ N_1(t) &= N_2'(t) + 8g(t) \\ &\quad = \tfrac{1}{10}\sinh(t) + \tfrac{3}{10}\sin(3t) + \tfrac{8}{10}\sinh(t) - \tfrac{8}{30}\sin(3t)\,, \\ &\quad = \tfrac{9}{10}\sinh(t) + \tfrac{1}{30}\sin(3t)\,, \\ N_0(t) &= N_1'(t) + 0g(t) = \tfrac{9}{10}\cosh(t) + \tfrac{1}{10}\cos(3t)\,. \end{aligned}$$

Solution (b). The characteristic polynomial of $\mathrm{L} = (\mathrm{D}-2)^3$ is $p(s) = (s-2)^3$. Therefore its Green function $g(t)$ is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s-2)^3}\right](t)\,.$$

Referring to the table on the last page, item 1 with $a = 2$ and $n = 2$ gives

$$g(t) = \tfrac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s-2)^3}\right](t) = \tfrac{1}{2}t^2e^{2t}\,.$$

Then because by the binomial expansion we see the characteristic polynomial as

$$\begin{aligned} p(s) = (s-2)^3 &= s^3 + 3(-2)s^2 + 3(-2)^2s + (-2)^3 \\ &= s^3 - 6s^2 + 12s - 8\,, \end{aligned}$$

the natrual fundamental set for $t = 0$ is found by

$$\begin{aligned} N_2(t) &= g(t) = \tfrac{1}{2}t^2e^{2t}\,, \\ N_1(t) &= N_2'(t) - 6g(t) \\ &= \big(t\,e^{2t} + t^2e^{2t}\big) - \tfrac{6}{2}t^2e^{2t} = t\,e^{2t} - 2t^2e^{2t}\,, \\ N_0(t) &= N_1'(t) + 12g(t) \\ &= \big(e^{2t} - 2t\,e^{2t} - 4t^2e^{2t}\big) + \tfrac{12}{2}t^2e^{2t} = e^{2t} - 2t\,e^{2t} + 2t^2e^{2t}\,. \end{aligned}$$

(7) Recast the equation $u''' + t^2u' - 3u = \sinh(2t)$ as a first-order system of ordinary differential equations.

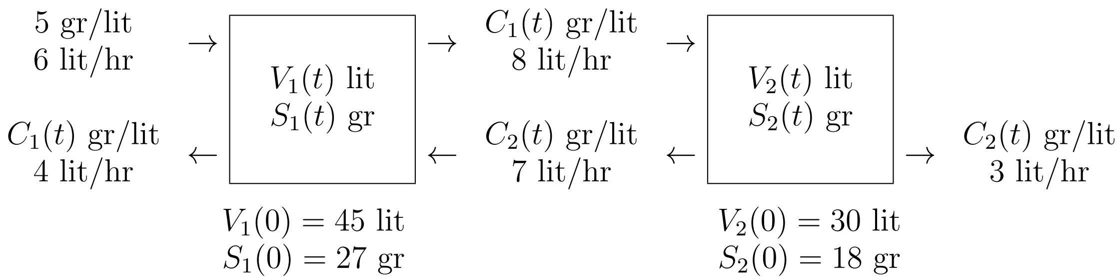
Solution. Because the equation is third order, the first-order system must have dimension three. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \sinh(2t) + 3x_1 - t^2x_2 \end{pmatrix}\,, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}\,.$$

(8) Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 45 liters and the second contains 30 liters. Brine with a salt concentration of 5 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 8 liters per hour, from the second into the first at 7 liters per hour, from the first into a drain at 4 liter per hour, and from the second into a drain at 3 liters per hour. At $t = 0$ there are 27 grams of salt in the first tank and 18 grams in the second.

(a) Give an initial-value problem that governs the amount of salt in each tank as a function of time.
(b) Give the interval of definition for the solution of this initial-value problem.

Solution (a). Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t hours. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t hours. Because mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.



We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will be $V_1(t) = 45 + t$ liters of brine in the first tank and $V_2(t) = 30 - 2t$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = 5 \cdot 6 + \frac{S_2}{30 - 2t}\,7 - \frac{S_1}{45 + t}\,8 - \frac{S_1}{45 + t}\,4\,, \qquad S_1(0) = 27\,,$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{S_1}{45 + t}\,8 - \frac{S_2}{30 - 2t}\,7 - \frac{S_2}{30 - 2t}\,3\,, \qquad S_2(0) = 18\,.$$

You could leave the answer in the above form. However, it can be simplified to

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = 30 + \frac{7}{30 - 2t}\,S_2 - \frac{12}{45 + t}\,S_1\,, \qquad S_1(0) = 27\,,$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{8}{45 + t}\,S_1 - \frac{5}{15 - t}\,S_2\,, \qquad S_2(0) = 18\,.$$

Solution (b). This first-order system of differential equations is *linear.*

◇ Its coefficients are undefined either at $t = -45$ or at $t = 15$ and are continuous elsewhere.
◇ Its forcing is constant, so is continuous everywhere.
◇ Its initial time is $t = 0$.

Therefore the natural interval of definition for the solution of this initial-value problem is $(-45, 15)$ because:

- the initial time $t = 0$ is in $(-45, 15)$;
- all the coefficients and the forcing are continuous over $(-45, 15)$;
- two coefficients are undefined at $t = -45$;
- two coefficients are undefined at $t = 15$.

However, it could also be argued that the interval of definition for the solution of this initial-value problem is $[0, 15)$ because the word problem starts at $t = 0$.

(9) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices
(a) $\mathbf{A}^{\mathrm{T}}$,
(b) $\overline{\mathbf{A}}$,
(c) $\mathbf{A}^{\mathrm{H}}$,
(d) $5\mathbf{A} - \mathbf{B}$,
(e) $\mathbf{AB}$,
(f) $\mathbf{B}^{-1}$.

Solution (a). The transpose of $\mathbf{A}$ is

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} -i2 & 2+i \\ 1+i & -4 \end{pmatrix}.$$

Solution (b). The conjugate of $\mathbf{A}$ is

$$\overline{\mathbf{A}} = \begin{pmatrix} i2 & 1-i \\ 2-i & -4 \end{pmatrix}.$$

Solution (c). The Hermitian transpose of $\mathbf{A}$ is

$$\mathbf{A}^{\mathrm{H}} = \begin{pmatrix} i2 & 2-i \\ 1-i & -4 \end{pmatrix}.$$

Solution (d). The difference of $5\mathbf{A}$ and $\mathbf{B}$ is given by

$$5\mathbf{A} - \mathbf{B} = \begin{pmatrix} -i10 & 5+i5 \\ 10+i5 & -20 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} -7-i10 & -1+i5 \\ 2+i5 & -27 \end{pmatrix}.$$

Solution (e). The product of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -i2\cdot 7 + (1+i)\cdot 8 & -i2\cdot 6 + (1+i)\cdot 7 \\ (2+i)\cdot 7 - 4\cdot 8 & (2+i)\cdot 6 - 4\cdot 7 \end{pmatrix} \\ &= \begin{pmatrix} 8-i6 & 7-i5 \\ -18+i7 & -16+i6 \end{pmatrix}. \end{aligned}$$

Solution (f). Observe that it is clear that $\mathbf{B}$ has an inverse because

$$\det(\mathbf{B}) = \det\begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = 7 \cdot 7 - 6 \cdot 8 = 49 - 48 = 1.$$

Then the inverse of $\mathbf{B}$ is given by

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix}.$$

(10) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) Give a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b).

(d) For the system found in part (b), solve the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}\,, \qquad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(e) For the $\mathbf{A}(t)$ found in part (b), give the Green matrix for the system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,.$$

Solution (a).

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} = 3t^4 + 9 - 2t^4 = t^4 + 9\,.$$

Solution (b). Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}$. Because $\frac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, we have

$$\begin{aligned} \mathbf{A}(t) &= \frac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t}\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix}\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{t^4+9}\begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix}\begin{pmatrix} 3 & -t^2 \\ -2t^2 & t^4+3 \end{pmatrix} = \frac{1}{t^4+9}\begin{pmatrix} 8t^3 & 6t-2t^5 \\ 12t & -4t^3 \end{pmatrix}. \end{aligned}$$

Solution (c). Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a fundamental matrix for the system found in part (b) is simply given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}.$$

Solution (d). Because a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b) was given in part (c), the solution of the initial-value problem is

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{\Psi}(t)\mathbf{\Psi}(1)^{-1}\mathbf{x}(1) = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\frac{1}{10}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3t^4+9-2t^2 \\ 6t^2-6 \end{pmatrix}. \end{aligned}$$

Alternative Solution (d). Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 3 \end{pmatrix} .$$

The initial condition then implies that

$$\mathbf{x}(1) = c_1 \begin{pmatrix} 4 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4c_1 + c_2 \\ 2c_1 + 3c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

from which we see that $c_1 = \frac{3}{10}$ and $c_2 = -\frac{1}{5}$. The solution of the initial-value problem is thereby

$$\mathbf{x}(t) = \tfrac{3}{10} \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix} - \tfrac{1}{5} \begin{pmatrix} t^2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}t^4 - \frac{1}{5}t^2 + \frac{9}{10} \\ \frac{3}{5}t^2 - \frac{3}{5} \end{pmatrix} .$$

Solution (e). Because a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b) was given in part (c), the Green matrix for the nonhomogeneous system is

$$\begin{aligned} \mathbf{G}(t,s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} s^4+3 & s^2 \\ 2s^2 & 3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \frac{1}{s^4+9} \begin{pmatrix} 3 & -s^2 \\ -2s^2 & s^4+3 \end{pmatrix} \\ &= \frac{1}{s^4+9} \begin{pmatrix} 3t^4 + 9 - 2t^2s^2 & t^2(s^4+3) - (t^4+3)s^2 \\ 6t^2 - 6s^2 & 3s^4 + 9 - 2t^2s^2 \end{pmatrix} . \end{aligned}$$

(11) Compute $e^{t\mathbf{A}}$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

Solution (a) by Two-by-Two Formula. Because

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} ,$$

the characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z-1)^2 - 2^2 .$$

This is a difference of squares with $\mu = 1$ and $\nu = 2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cosh(2t)\mathbf{I} + \frac{\sinh(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[\cosh(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(2t)}{2} \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix} . \end{aligned}$$

Solution (a) by the Natural Fundamental Set Method. Because

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix},$$

the characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z+1)(z-3).$$

Below we show that the natural fundamental set of solutions for $t = 0$ associated with $p(\mathrm{D}) = \mathrm{D}^2 - 2\mathrm{D} + 3$ is

$$N_0(t) = \frac{e^{3t} + 3e^{-t}}{4}, \qquad N_1(t) = \frac{e^{3t} - e^{-t}}{4}.$$

Then

$$\begin{aligned} e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A} &= \frac{e^{3t} + 3e^{-t}}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{3t} - e^{-t}}{4}\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 2e^{3t} + 2e^{-t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}. \end{aligned}$$

From the Green Function. By the partial fraction identity

$$\frac{1}{s^2 - 2s + 3} = \frac{1}{(s-3)(s+1)} = \frac{\frac{1}{4}}{s-3} + \frac{-\frac{1}{4}}{s+1},$$

the Green function associated with $p(\mathrm{D}) = \mathrm{D}^2 - 2\mathrm{D} + 3$ is

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 - 2s + 3}\right](t) \\ &= \tfrac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right](t) - \tfrac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = \tfrac{1}{4}e^{3t} - \tfrac{1}{4}e^{-t}. \end{aligned}$$

Then, because the characteristic polynomial is $p(s) = s^2 - 2s + 3$, the natural fundamental set is

$$\begin{aligned} N_1(t) &= g(t) = \tfrac{1}{4}e^{3t} - \tfrac{1}{4}e^{-t}, \\ N_0(t) &= N_1'(t) - 2g(t) = \left(\tfrac{3}{4}e^{3t} + \tfrac{1}{4}e^{-t}\right) - \left(\tfrac{2}{4}e^{3t} - \tfrac{2}{4}e^{-t}\right) = \tfrac{1}{4}e^{3t} + \tfrac{3}{4}e^{-t}. \end{aligned}$$

From the General Initial-Value Problem. The general initial-value problem associated with $p(\mathrm{D}) = \mathrm{D}^2 - 2\mathrm{D} + 3$ is

$$y'' - 2y' - 3y = 0, \qquad y(0) = y_0, \quad y'(0) = y_1.$$

This has the general solution $y(t) = c_1e^{3t} + c_2e^{-t}$. Because $y'(t) = 3c_1e^{3t} - c_2e^{-t}$, the general initial conditions yield

$$y_0 = y(0) = c_1 + c_2, \qquad y_1 = y'(0) = 3c_1 - c_2.$$

This system can be solved to obtain

$$c_1 = \frac{y_0 + y_1}{4}, \qquad c_2 = \frac{3y_0 - y_1}{4}.$$

The solution of the general initial-value problem is thereby

$$y(t) = \frac{y_0 + y_1}{4}\,e^{3t} + \frac{3y_0 - y_1}{4}\,e^{-t} = \frac{e^{3t} + 3e^{-t}}{4}\,y_0 + \frac{e^{3t} - e^{-t}}{4}\,y_1.$$

Therefore the associated natural fundamental set of solutions is

$$N_0(t) = \frac{e^{3t} + 3e^{-t}}{4}, \qquad N_1(t) = \frac{e^{3t} - e^{-t}}{4}.$$

Solution (a) by Eigen Methods. Because

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix},$$

the characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z+1)(z-3).$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -1 and 3. Because

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix},$$

we can read off that $\mathbf{A}$ has the eigenpairs

$$\left(-1, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right), \qquad \left(3, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

Set

$$\mathbf{V} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Because $\det(\mathbf{V}) = 4$, we see that

$$\begin{aligned} e^{t\mathbf{A}} = \mathbf{V} e^{t\mathbf{D}} \mathbf{V}^{-1} &= \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & -2e^{-t} \\ e^{3t} & 2e^{3t} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{-t} + 2e^{3t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{-t} + 2e^{3t} \end{pmatrix}. \end{aligned}$$

Solution (b) by Two-by-Two Formula. Because

$$\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix},$$

the characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z-4)^2.$$

This is a perfect square with $\mu = 4$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} = e^{4t}\Big[\mathbf{I} + t\,(\mathbf{A} - 4\mathbf{I})\Big] &= e^{4t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}\right] \\ &= e^{4t} \begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix}. \end{aligned}$$

Solution (b) by the Natural Fundamental Set Method. Because

$$\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix},$$

the characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z-4)^4 .$$

Below we show that the natural fundamental set of solutions for $t = 0$ associated with $p(\mathrm{D}) = \mathrm{D}^2 - 8\mathrm{D} + 16$ is

$$N_0(t) = (1-4t)e^{4t} , \qquad N_1(t) = t\,e^{4t} .$$

Then

$$e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A} = (1-4t)e^{4t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\,e^{4t}\begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$$
$$= e^{4t}\begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix} .$$

From the Green Function. Because $p(s) = s^2 - 8s + 16 = (s-4)^2$, the Green function associated with $p(\mathrm{D}) = \mathrm{D}^2 - 8\mathrm{D} + 16$ is

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 - 8s + 16}\right](t)$$
$$= \mathcal{L}^{-1}\left[\frac{1}{(s-4)^2}\right](t) = t\,e^{4t} .$$

Then, because the characteristic polynomial is $p(s) = s^2 - 8s + 16$, the natural fundamental set is

$$N_1(t) = g(t) = t\,e^{4t} ,$$
$$N_0(t) = N_1'(t) - 8g(t) = \left(e^{4t} + 4t\,e^{4t}\right) - 8t\,e^{4t} = e^{4t} - 4t\,e^{4t} .$$

From the General Initial-Value Problem. The general initial-value problem associated with $p(\mathrm{D}) = \mathrm{D}^2 - 8\mathrm{D} + 16$ is

$$y'' - 8y' + 16y = 0 , \qquad y(0) = y_0 , \quad y'(0) = y_1 .$$

This has the general solution $y(t) = c_1e^{4t} + c_2te^{4t}$. Because

$$y'(t) = 4c_1e^{3t} + 4c_2te^{-t} + c_2e^{4t} ,$$

the general initial conditions yield

$$y_0 = y(0) = c_1 , \qquad y_1 = y'(0) = 4c_1 + c_2 .$$

This system can be solved to obtain $c_1 = y_0$ and $c_2 = y_1 - 4y_0$. The solution of the general initial-value problem is thereby

$$y(t) = y_0\,e^{4t} + (y_1 - 4y_0)t\,e^{4t} = (1-4t)e^{4t}\,y_0 + t\,e^{4t}\,y_1 .$$

Therefore the associated natural fundamental set of solutions is

$$N_0(t) = (1-4t)e^{4t} , \qquad N_1(t) = t\,e^{4t} .$$

(12) Give the Green matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$ when

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

Solution (a). By the solution to part (a) of the previous problem

$$e^{t\mathbf{A}} = \frac{1}{4}\begin{pmatrix} 2e^{3t} + 2e^{-t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix} .$$

Therefore the Green matrix $\mathbf{G}(t,s)$ is given by

$$\mathbf{G}(t,s) = e^{t\mathbf{A}}e^{-s\mathbf{A}} = e^{(t-s)\mathbf{A}} = \frac{1}{4}\begin{pmatrix} 2e^{3(t-s)} + 2e^{-(t-s)} & 4e^{3(t-s)} - 4e^{-(t-s)} \\ e^{3(t-s)} - e^{-(t-s)} & 2e^{3(t-s)} + 2e^{-(t-s)} \end{pmatrix} .$$

Solution (b). By the solution to part (b) of the previous problem

$$e^{t\mathbf{A}} = e^{4t}\begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix} .$$

Therefore the Green matrix $\mathbf{G}(t,s)$ is given by

$$\mathbf{G}(t,s) = e^{t\mathbf{A}}e^{-s\mathbf{A}} = e^{(t-s)\mathbf{A}} = e^{4(t-s)}\begin{pmatrix} 1 + 2(t-s) & 4(t-s) \\ -(t-s) & 1 - 2(t-s) \end{pmatrix} .$$

(13) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix} .$$

Compute $e^{t\mathbf{A}}$ given that the characteristic polynomial of $\mathbf{A}$ is $p(z) = z^3 + 9z$ and that the natural fundamental set of solutions associated with $t = 0$ for $\mathrm{D}^3 + 9\mathrm{D}$ is

$$N_0(t) = 1 , \qquad N_1(t) = \tfrac{1}{3}\sin(3t) , \qquad N_2(t) = \tfrac{1}{9}\big(1 - \cos(3t)\big) .$$

Solution. The natural fundamental set method says that

$$e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A} + N_2(t)\mathbf{A}^2 .$$

Because $N_0(t) = 1$, $N_1(t) = \frac{1}{3}\sin(3t)$, $N_2(t) = \frac{1}{9}\big(1 - \cos(3t)\big)$, and

$$\begin{aligned} \mathbf{A}^2 &= \begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 8 - 2 & 2 & -1 + 4 + 1 \\ -4 + 0 + 4 & -8 & 4 - 0 - 2 \\ 2 + 0 - 2 & 4 & -2 - 0 + 1 \end{pmatrix} = \begin{pmatrix} -9 & 2 & 4 \\ 0 & -8 & 2 \\ 0 & 4 & -1 \end{pmatrix} , \end{aligned}$$

we see that

$$e^{t\mathbf{A}} = 1\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \tfrac{1}{3}\sin(3t)\begin{pmatrix}-1&-2&1\\4&0&-2\\-2&0&1\end{pmatrix} + \tfrac{1}{9}\big(1-\cos(3t)\big)\begin{pmatrix}-9&2&4\\0&-8&2\\0&4&-1\end{pmatrix}$$

$$= \begin{pmatrix}\cos(3t)-\frac{1}{3}\sin(3t) & -\frac{2}{3}\sin(3t)+\frac{2}{9}-\frac{2}{9}\cos(3t) & \frac{1}{3}\sin(3t)+\frac{4}{9}-\frac{4}{9}\cos(3t)\\ \frac{4}{3}\sin(3t) & \frac{1}{9}+\frac{8}{9}\cos(3t) & -\frac{2}{3}\sin(3t)+\frac{2}{9}-\frac{2}{9}\cos(3t)\\ -\frac{2}{3}\sin(3t) & \frac{4}{9}-\frac{4}{9}\cos(3t) & \frac{8}{9}+\frac{1}{9}\cos(3t)+\frac{1}{3}\sin(3t)\end{pmatrix}.$$

(14) Solve each of the following initial-value problems.

(a) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2&2\\5&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad \begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}.$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&1\\-4&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad \begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

Solution (a). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}2&2\\5&-1\end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - z - 12 = (z-\tfrac{1}{2})^2 - (\tfrac{7}{2})^2.$$

This is a difference of squares with $\mu = \frac{1}{2}$ and $\nu = \frac{7}{2}$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{\frac{1}{2}t}\left[\cosh\big(\tfrac{7}{2}t\big)\mathbf{I} + \frac{\sinh\big(\frac{7}{2}t\big)}{\frac{7}{2}}\big(\mathbf{A}-\tfrac{1}{2}\mathbf{I}\big)\right] \\ &= e^{\frac{1}{2}t}\left[\cosh\big(\tfrac{7}{2}t\big)\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{\sinh\big(\frac{7}{2}t\big)}{\frac{7}{2}}\begin{pmatrix}\frac{3}{2}&2\\5&-\frac{3}{2}\end{pmatrix}\right] \\ &= e^{\frac{1}{2}t}\begin{pmatrix}\cosh\big(\frac{7}{2}t\big)+\frac{3}{7}\sinh\big(\frac{7}{2}t\big) & \frac{4}{7}\sinh\big(\frac{7}{2}t\big)\\ \frac{10}{7}\sinh\big(\frac{7}{2}t\big) & \cosh\big(\frac{7}{2}t\big)-\frac{3}{7}\sinh\big(\frac{7}{2}t\big)\end{pmatrix}. \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned} \begin{pmatrix}x(t)\\y(t)\end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix}x(0)\\y(0)\end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix}1\\-1\end{pmatrix} \\ &= e^{\frac{1}{2}t}\begin{pmatrix}\cosh\big(\frac{7}{2}t\big)+\frac{3}{7}\sinh\big(\frac{7}{2}t\big) & \frac{4}{7}\sinh\big(\frac{7}{2}t\big)\\ \frac{10}{7}\sinh\big(\frac{7}{2}t\big) & \cosh\big(\frac{7}{2}t\big)-\frac{3}{7}\sinh\big(\frac{7}{2}t\big)\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} \\ &= e^{\frac{1}{2}t}\begin{pmatrix}\cosh\big(\frac{7}{2}t\big)-\frac{1}{7}\sinh\big(\frac{7}{2}t\big)\\ -\cosh\big(\frac{7}{2}t\big)+\frac{13}{7}\sinh\big(\frac{7}{2}t\big)\end{pmatrix}. \end{aligned}$$

Solution (b). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}1&1\\-4&1\end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 5 = (z-1)^2 + 2^2.$$

This is a sum of squares with $\mu = 1$ and $\nu = 2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[\cos(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix} . \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^t \begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) \\ -2\sin(2t) + \cos(2t) \end{pmatrix} . \end{aligned}$$

Remark. We could have used other methods to compute $e^{t\mathbf{A}}$ in each part of the above problem. Alternatively, we could have constructed a fundamental matrix $\mathbf{\Psi}(t)$ and then determined $\mathbf{c}$ so that $\mathbf{\Psi}(t)\mathbf{c}$ satisfies the initial conditions.

(15) Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution (a). We must find a general solution for the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 1 = (z-1)^2 .$$

This is a perfect square with $\mu = 2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} = e^t \big[\mathbf{I} + t\,(\mathbf{A} - \mathbf{I})\big] &= e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} 1 + 2t & -4t \\ t & 1 - 2t \end{pmatrix} . \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= c_1 e^t \begin{pmatrix} 1+2t \\ t \end{pmatrix} + c_2 e^t \begin{pmatrix} -4t \\ 1-2t \end{pmatrix} .$$

Solution (b) by Two-by-Two Formula. We must find a general solution for the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4^2 .$$

This is a sum of squares with $\mu = 0$ and $\nu = 4$. Hence,

$$e^{t\mathbf{A}} = \left[\cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}\mathbf{A} \right] = \left[\cos(4t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4} \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \right]$$
$$= \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} .$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= c_1 \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ \sin(4t) \end{pmatrix} + c_2 \begin{pmatrix} -\frac{5}{4}\sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} .$$

Solution (b) by Eigen Methods. We must find a general solution for the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $\pm i4$. Because

$$\mathbf{A} - i4\mathbf{I} = \begin{pmatrix} 2 - i4 & -5 \\ 4 & -2 - i4 \end{pmatrix} , \qquad \mathbf{A} + i4\mathbf{I} = \begin{pmatrix} 2 + i4 & -5 \\ 4 & -2 + i4 \end{pmatrix} ,$$

we can read off that $\mathbf{A}$ has the eigenpairs

$$\left(i4 , \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \right) , \qquad \left(-i4 , \begin{pmatrix} 1 - i2 \\ 2 \end{pmatrix} \right) .$$

Therefore the system has the complex-valued solution

$$\begin{aligned} e^{i4t}\begin{pmatrix} 1+i2 \\ 2 \end{pmatrix} &= \big(\cos(4t)+i\sin(4t)\big)\begin{pmatrix} 1+i2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(4t)-2\sin(4t)+i2\cos(4t)+i\sin(4t) \\ 2\cos(4t)+i2\sin(4t) \end{pmatrix} . \end{aligned}$$

By taking real and imaginary parts, we obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \mathrm{Re}\left(e^{i4t}\begin{pmatrix} 1+i2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} \cos(4t)-2\sin(4t) \\ 2\cos(4t) \end{pmatrix} , \\ \mathbf{x}_2(t) &= \mathrm{Im}\left(e^{i4t}\begin{pmatrix} 1+i2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 2\cos(4t)+\sin(4t) \\ 2\sin(4t) \end{pmatrix} . \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\begin{pmatrix} \cos(4t)-2\sin(4t) \\ 2\cos(4t) \end{pmatrix} + c_2\begin{pmatrix} 2\cos(4t)+\sin(4t) \\ 2\sin(4t) \end{pmatrix} .$$

Solution (c) by Two-by-Two Formula. We must find a general solution for the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} .$$

The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z-3)^2 + 4^2 .$$

This is a sum of squares with $\mu = 3$ and $\nu = 4$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t}\left[\cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}(\mathbf{A}-3\mathbf{I})\right] \\ &= e^{3t}\left[\cos(4t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4}\begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix}\right] \\ &= e^{3t}\begin{pmatrix} \cos(4t)+\frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t)-\frac{1}{2}\sin(4t) \end{pmatrix} . \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{3t}\begin{pmatrix} \cos(4t)+\frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t)-\frac{1}{2}\sin(4t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{3t}\begin{pmatrix} \cos(4t)+\frac{1}{2}\sin(4t) \\ -\frac{5}{4}\sin(4t) \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} \sin(4t) \\ \cos(4t)-\frac{1}{2}\sin(4t) \end{pmatrix} . \end{aligned}$$

Solution (c) by Eigen Methods. We must find a general solution for the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} .$$

The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z-3)^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $3 \pm i4$. Because

$$\mathbf{A} - (3+i4)\mathbf{I} = \begin{pmatrix} 2-i4 & 4 \\ -5 & -2-i4 \end{pmatrix}, \quad \mathbf{A} - (3-i4)\mathbf{I} = \begin{pmatrix} 2+i4 & 4 \\ -5 & -2+i4 \end{pmatrix},$$

we can read off that $\mathbf{A}$ has the eigenpairs

$$\left(3+i4\,, \begin{pmatrix} -2 \\ 1-i2 \end{pmatrix}\right), \qquad \left(3-i4\,, \begin{pmatrix} -2 \\ 1+i2 \end{pmatrix}\right).$$

Therefore the system has the complex-valued solution

$$\begin{aligned} e^{(3+i4)t} \begin{pmatrix} -2 \\ 1-i2 \end{pmatrix} &= e^{3t}\big(\cos(4t) + i\sin(4t)\big) \begin{pmatrix} -2 \\ 1-i2 \end{pmatrix} \\ &= e^{3t} \begin{pmatrix} -2\cos(4t) - i2\sin(4t) \\ \cos(4t) + 2\sin(4t) + i\sin(4t) - i2\cos(4t) \end{pmatrix}. \end{aligned}$$

By taking real and imaginary parts, we obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re}\left(e^{(3+i4)t} \begin{pmatrix} -2 \\ 1-i2 \end{pmatrix}\right) = e^{3t} \begin{pmatrix} -2\cos(4t) \\ \cos(4t) + 2\sin(4t) \end{pmatrix}, \\ \mathbf{x}_2(t) &= \operatorname{Im}\left(e^{(3+i4)t} \begin{pmatrix} -2 \\ 1+i2 \end{pmatrix}\right) = e^{3t} \begin{pmatrix} -2\sin(4t) \\ \sin(4t) - 2\cos(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} -2\cos(4t) \\ \cos(4t) + 2\sin(4t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -2\sin(4t) \\ \sin(4t) - 2\cos(4t) \end{pmatrix}.$$

(16) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 1.

Solution. The eigenvectors of $\mathbf{A}$ associated with 1 are all nonzero vectors $\mathbf{v}$ that satisfy $\mathbf{A}\mathbf{v} = \mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ that satisfy $(\mathbf{A}-\mathbf{I})\mathbf{v} = 0$, which is

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0\,.$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} v_1 - v_2 + v_3 &= 0\,, \\ v_1 \phantom{{}-v_2} - v_3 &= 0\,, \\ - v_2 + 2v_3 &= 0\,. \end{aligned}$$

We may solve this system either by elimination or by row reduction. By either method we find that its general solution is

$$v_1 = \alpha\,, \quad v_2 = 2\alpha\,, \quad v_3 = \alpha\,, \qquad \text{for any constant } \alpha\,.$$

The eigenvectors of $\mathbf{A}$ associated with 1 thereby have the form

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \text{for any nonzero constant } \alpha\,.$$

(17) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}\,.$$

(a) Find all the eigenvalues of $\mathbf{A}$.
(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.
(c) Diagonalize $\mathbf{A}$.
(d) Compute $e^{t\mathbf{A}}$.
(e) Compute $(s\mathbf{I} - \mathbf{A})^{-1}$ for every s where it is defined.

Solution (a). The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 15 = (z+3)(z-5)\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -3 and 5.

Solution (b) by the Cayley-Hamilton Method. We have

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}\,, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}\,.$$

Every nonzero column of $\mathbf{A} - 5\mathbf{I}$ has the form

$$\alpha_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{for some } \alpha_1 \neq 0\,.$$

These are all the eigenvectors associated with -3. Similarly, every nonzero column of $\mathbf{A} + 3\mathbf{I}$ has the form

$$\alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad \text{for some } \alpha_2 \neq 0\,.$$

These are all the eigenvectors associated with 5.

Solution (c). If we use the eigenpairs

$$\left(-3\,, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)\,, \qquad \left(5\,, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right)\,,$$

then set

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\,, \qquad \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}\,.$$

Because $\det(\mathbf{V}) = 1 \cdot 2 - (-2) \cdot 3 = 2 + 6 = 8$, we see that

$$\mathbf{V}^{-1} = \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}\,.$$

We conclude that $\mathbf{A}$ has the diagonalization

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} .$$

You do not have to multiply these matrices out. Had we started with different eigenpairs, the steps would be the same as above but we would obtain a different diagonalization.

Solution (d). Because $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ by part (c), we have

$$\begin{aligned} e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} &= \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{5t} \end{pmatrix} \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2e^{-3t} & -3e^{-3t} \\ 2e^{5t} & e^{5t} \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} 2e^{-3t} + 6e^{5t} & -3e^{-3t} + 3e^{5t} \\ -4e^{-3t} + 4e^{5t} & 6e^{-3t} + 2e^{5t} \end{pmatrix} . \end{aligned}$$

Solution (e). Because we see from part (a) that

$$\det(s\mathbf{I} - \mathbf{A}) = p_{\mathbf{A}}(s) = s^2 - 2s - 15 = (s+3)(s-5) ,$$

so whenever $s \neq -3$ and $s \neq 5$ we have

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} s-3 & -3 \\ -4 & s+1 \end{pmatrix}^{-1} = \frac{1}{(s+3)(s-5)} \begin{pmatrix} s+1 & 3 \\ 4 & s-3 \end{pmatrix} .$$

This is defined for every s except at $s = -3$ and $s = 5$.

Alternative Solution (e). Because $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ by part (c), we have

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})^{-1} = \mathbf{V}(s\mathbf{I} - \mathbf{D})^{-1}\mathbf{V}^{-1} &= \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s-5} \end{pmatrix} \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{s+3} & \frac{-3}{s+3} \\ \frac{2}{s-5} & \frac{1}{s-5} \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} \frac{2}{s+3} + \frac{6}{s-5} & \frac{-3}{s+3} + \frac{3}{s-5} \\ \frac{-4}{s+3} + \frac{4}{s-5} & \frac{6}{s+3} + \frac{2}{s-5} \end{pmatrix} . \end{aligned}$$

This is defined for every s except at $s = -3$ and $s = 5$.

Remark. Partial fraction identities show that the solutions given above are identical. The second approach is faster than the first for larger matrices.

(18) What answer will be produced by the following Matlab command?

$$\texttt{>> A = [1 4; 3 2]; [vect, val] = eig(sym(A))}$$

You do not have to give the answer in Matlab format.

Solution. The Matlab command will produce the eigenpairs of $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 3z - 10 = (z-5)(z+2)\,,$$

so its eigenvalues are 5 and -2. Because

$$\mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}, \qquad \mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix},$$

we can read off that the eigenpairs are

$$\left(5\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \qquad \left(-2\,, \begin{pmatrix} -4 \\ 3 \end{pmatrix}\right).$$

(19) A 3×3 matrix $\mathbf{A}$ has the eigenpairs

$$\left(-3\,, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \qquad \left(2\,, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right), \qquad \left(5\,, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right).$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{A}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution (a). One choice for $\mathbf{V}$ and $\mathbf{D}$ is

$$\mathbf{V} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Solution (b). Use the given eigenpairs to construct the special solutions

$$\mathbf{x}_1(t) = e^{-3t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3(t) = e^{5t}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$

Then a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t) \end{pmatrix} = \begin{pmatrix} e^{-3t} & -e^{2t} & e^{5t} \\ e^{-3t} & e^{2t} & -e^{5t} \\ 0 & e^{2t} & 2e^{5t} \end{pmatrix}.$$

Alternative Solution (b). Given the $\mathbf{V}$ and $\mathbf{D}$ from part (a), a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix} = \begin{pmatrix} e^{-3t} & -e^{2t} & e^{5t} \\ e^{-3t} & e^{2t} & -e^{5t} \\ 0 & e^{2t} & 2e^{5t} \end{pmatrix}.$$

Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[j'(t)](s) = sJ(s) - j(0) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s),\ c \geq 0,$$

and u is the unit step function.

$$\mathcal{L}[\delta(t-c)j(t)](s) = e^{-cs}j(c) \qquad \text{where } c \geq 0 \text{ and } \delta \text{ is the unit impulse.}$$