

**Sample Problems for the Third Exam**  
**Math 246, Fall 2020, Professor David Levermore**

- (1) Compute the Laplace transform of  $f(t) = te^{3t}u(t-2)$  from its definition.
- (2) Consider the following MATLAB commands.

```
>> syms t y(t) s Y
>> f = heaviside(t)*t^2 + heaviside(t - 3)*(3*t - t^2);
>> diffeqn = diff(y, 2) - 6*diff(y, 1) + 10*y(t) == f;
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, ...
                [laplace(y(t), t, s), y(0), subs(diff(y(t), t), t, 0)], [Y, 2, 3]);
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

- (a) Give the initial-value problem for  $y(t)$  that is being solved.  
 (b) Find the Laplace transform  $Y(s)$  of the solution  $y(t)$ .

DO NOT take the inverse Laplace transform of  $Y(s)$  to find  $y(t)$ , just solve for  $Y(s)$ !  
 You may refer to the table on the last page.

- (3) Find  $Y(s) = \mathcal{L}[y](s)$  where  $y(t)$  solves the initial-value problem

$$y'' + 4y' + 13y = f(t), \quad y(0) = 4, \quad y'(0) = 1,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \leq t < 2\pi, \\ t - 2\pi & \text{for } t \geq 2\pi. \end{cases}$$

DO NOT take the inverse Laplace transform of  $Y(s)$  to find  $y(t)$ , just solve for  $Y(s)$ !  
 You may refer to the table on the last page.

- (4) Find  $X(s) = \mathcal{L}[x](s)$  where  $x(t)$  solves the initial-value problem

$$x'' + 4x = \delta(t-3), \quad x(0) = 5, \quad x'(0) = 0.$$

DO NOT take the inverse Laplace transform of  $X(s)$  to find  $x(t)$ , just solve for  $X(s)$ !  
 You may refer to the table on the last page.

- (5) Find the inverse Laplace transforms of the following functions.

(a)  $F(s) = \frac{2}{(s+5)^2},$

(b)  $F(s) = \frac{3s}{s^2 - s - 6},$

(c)  $F(s) = \frac{(s-2)e^{-3s}}{s^2 - 4s + 5}.$

You may refer to the table on the last page.

- (6) For each of the following differential operators compute its Green function  $g(t)$  and its natural fundamental set for  $t = 0$ .

(a)  $L = D^4 + 8D^2 - 9$ ,

(b)  $L = (D - 2)^3$ .

You may refer to the table on the last page.

- (7) Recast the equation  $u''' + t^2u' - 3u = \sinh(2t)$  as a first-order system of ordinary differential equations.

- (8) Two interconnected tanks are filled with brine (salt water). At  $t = 0$  the first tank contains 45 liters and the second contains 30 liters. Brine with a salt concentration of 5 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 8 liters per hour, from the second into the first at 7 liters per hour, from the first into a drain at 4 liter per hour, and from the second into a drain at 3 liters per hour. At  $t = 0$  there are 27 grams of salt in the first tank and 18 grams in the second.

(a) Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(b) Give the interval of definition for the solution of this initial-value problem.

- (9) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

(a)  $\mathbf{A}^T$ ,

(b)  $\overline{\mathbf{A}}$ ,

(c)  $\mathbf{A}^H$ ,

(d)  $5\mathbf{A} - \mathbf{B}$ ,

(e)  $\mathbf{AB}$ ,

(f)  $\mathbf{B}^{-1}$ .

- (10) Consider the vector-valued functions  $\mathbf{x}_1(t) = \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix}$ ,  $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$ .

(a) Compute the Wronskian  $\text{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$ .

(b) Find  $\mathbf{A}(t)$  such that  $\mathbf{x}_1, \mathbf{x}_2$  is a fundamental set of solutions to the system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x},$$

wherever  $\text{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$ .

(c) Give a fundamental matrix  $\mathbf{\Psi}(t)$  for the system found in part (b).

(d) For the system found in part (b), solve the initial-value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(e) For the  $\mathbf{A}(t)$  found in part (b), give the Green matrix for the system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t).$$

(11) Compute  $e^{t\mathbf{A}}$  for the following matrices.

$$(a) \mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$$

$$(b) \mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$$

(12) Give the Green matrix for the system  $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$  when

$$(a) \mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$$

$$(b) \mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$$

(13) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix}.$$

Compute  $e^{t\mathbf{A}}$  given that the characteristic polynomial of  $\mathbf{A}$  is  $p(z) = z^3 + 9z$  and that the natural fundamental set of solutions associated with  $t = 0$  for  $D^3 + 9D$  is

$$N_0(t) = 1, \quad N_1(t) = \frac{1}{3} \sin(3t), \quad N_2(t) = \frac{1}{9}(1 - \cos(3t)).$$

(14) Solve each of the following initial-value problems.

$$(a) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$(b) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(15) Find a general solution for each of the following systems.

$$(a) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(b) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(c) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(16) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of  $\mathbf{A}$  associated with 1.

(17) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}.$$

- Find all the eigenvalues of  $\mathbf{A}$ .
- For each eigenvalue of  $\mathbf{A}$  find all of its eigenvectors.
- Diagonalize  $\mathbf{A}$ .
- Compute  $e^{t\mathbf{A}}$ .
- Compute  $(s\mathbf{I} - \mathbf{A})^{-1}$  for every  $s$  where it is defined.

(18) What answer will be produced by the following Matlab command?

$$>> \mathbf{A} = [1 \ 4; 3 \ 2]; [\text{vect}, \text{val}] = \text{eig}(\text{sym}(\mathbf{A}))$$

You do not have to give the answer in Matlab format.

(19) A  $3 \times 3$  matrix  $\mathbf{A}$  has the eigenpairs

$$\left(-3, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right).$$

- Give an invertible matrix  $\mathbf{V}$  and a diagonal matrix  $\mathbf{D}$  such that  $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$ . (You do not have to compute either  $\mathbf{V}^{-1}$  or  $e^{t\mathbf{A}}$ !)
- Give a fundamental matrix for the system  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ .

### Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \quad \text{for } s > a.$$

$$\mathcal{L}[e^{at} \cos(bt)](s) = \frac{s-a}{(s-a)^2 + b^2} \quad \text{for } s > a.$$

$$\mathcal{L}[e^{at} \sin(bt)](s) = \frac{b}{(s-a)^2 + b^2} \quad \text{for } s > a.$$

$$\mathcal{L}[j'(t)](s) = sJ(s) - j(0) \quad \text{where } J(s) = \mathcal{L}[j(t)](s).$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \quad \text{where } J(s) = \mathcal{L}[j(t)](s).$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \quad \text{where } J(s) = \mathcal{L}[j(t)](s).$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \quad \text{where } J(s) = \mathcal{L}[j(t)](s), c \geq 0, \text{ and } u \text{ is the unit step function.}$$

$$\mathcal{L}[\delta(t-c)j(t)](s) = e^{-cs}j(c) \quad \text{where } c \geq 0 \text{ and } \delta \text{ is the unit impulse.}$$