Sample Problems for the First In-Class Exam
Math 246, Fall 2020, Professor David Levermore

(1) (a) Sketch the graph that would be produced by the following Matlab command.

```
fplot(@(t) 2/t, [1,6])
```

(b) Sketch the graph that would be produced by the following Matlab commands.

```
[X, Y] = meshgrid(−5:0.1:5,−5:0.1:5)
contour(X, Y, X.ˆ2 + Y.ˆ2, [1, 9, 25])
axis square
```

(2) Find the explicit solution for each of the following initial-value problems and identify its interval of definition.

(a) $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{\cos(t) - z}{1 + t}, \quad z(0) = 2.$

(b) $\dfrac{\mathrm{d}u}{\mathrm{d}z} = e^u + 1\,, \quad u(0) = 0.$

(c) $\dfrac{\mathrm{d}v}{\mathrm{d}t} = -3t^2 e^{-v}\,, \quad v(2) = 0.$

(3) Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{t^2 - 4}\, x = \frac{1}{\sin(t)}\,, \qquad x(1) = 0\,.$$

(You do not need to solve this equation to answer this question, but your reasoning must be given!)

(4) Consider the following Matlab commands.

```
>> [T, Y] = meshgrid(−5.0:1.0:5.0,−5.0:1.0:5.0);
>> S = T.ˆ2 − Y.ˆ3;
>> L = sqrt(1 + S.ˆ2);
>> quiver(T, Y, 1./L, S./L, 0.5)
>> axis tight, xlabel 't', ylabel 'y'
```

(a) What is the differential equation being studied?
(b) What kind of graph will these Matlab commands produce?

(5) Consider the differential equation

$$\frac{dy}{dt} = \frac{y^2(y+2)(y-4)}{y-2}\,.$$

(a) Sketch its phase-line portrait over the interval $[-6, 6]$. Identify points where it is undefined. Identify its stationary points and classify each as being either stable, unstable, or semistable.
(b) For each stationary point identify the set of initial values $y(0)$ such that the solution $y(t)$ converges to that stationary point as $t \to \infty$.
(c) For each stationary point identify the set of initial values $y(0)$ such that the solution $y(t)$ converges to that stationary point as $t \to -\infty$.
(d) Identify all initial values $y(0)$ such that the interval of definition of the solution $y(t)$ is $(-\infty, \infty)$.
(e) Sketch a graph of y versus t showing several solution curves. The graph should show all of the stationary solutions as well as solution curves above and below each of them. Every value of y for which the equation is defined should lie on at least one sketched solution curve.

(6) In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population such that it would triple every five weeks. There are 85,000 mosquitoes in the area when a flock of birds arrives that eats 25,000 mosquitoes per week. Write down an initial-value problem that governs $M(t)$, the population of mosquitoes in the area after the flock of birds arrives. (You do not have to solve the initial-value problem!)

(7) A tank initially contains 100 liters of pure water. Beginning at time $t = 0$ brine (salt water) with a salt concentration of 2 grams per liter (gr/lit) flows into the tank at a constant rate of 3 liters per minute (lit/min) and the well-stirred mixture flows out of the tank at a the same rate. Let $S(t)$ denote the mass (gr) of salt in the tank at time $t \geq 0$.
(a) Write down an initial-value problem that governs $S(t)$.
(b) Is $S(t)$ an increasing or decreasing function of t? (Give your reasoning.)
(c) What is the behavior of $S(t)$ as $t \to \infty$? (Give your reasoning.)
(d) Derive an explicit formula for $S(t)$.
(e) How does the answer to part (a) change if the well-stirred mixture flows out of the tank at a constant rate of 2 liters per minute?

(8) A 2 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance of $v^2/40$ newtons ($=$ kg m/sec^2) where v is the downward velocity (m/sec) of the mass. The gravitational acceleration is 9.8 m/sec^2.
(a) What is the terminal velocity of the mass?
(b) Write down an initial-value problem that governs v as a function of time. (You do not have to solve it!)

(9) A puck with initial velocity $v_o > 0$ begins to slide on a surface that imparts a position-dependent frictional drag. Its position $x(t)$ is governed by the initial-value problem

$$\ddot{x} = -\frac{4}{(1+x)^2}\,\dot{x}\,, \qquad x(0) = 0, \quad \dot{x}(0) = v_o > 0\,.$$

This is a second-order autonomous initial-value problem.
(a) Find its reduced autonomous initial-value problem.
(b) Find the smallest initial velocity v_o for which $x(t) \to \infty$ as $t \to \infty$.

(10) Give an implicit general solution to each of the following differential equations.

(a) $\left(\frac{y}{x} + 3x\right) \mathrm{d}x + \big(\log(x) - y\big)\, \mathrm{d}y = 0\,.$

(b) $(x^2 + y^3 + 2x)\, \mathrm{d}x + 3y^2\, \mathrm{d}y = 0\,.$

(11) Suppose we are using the Runge-midpoint method to numerically approximate the solution of an initial-value problem over the time interval $[0, 5]$. By what factor would we expect the error to decrease when we increase the number of time steps taken from 500 to 2000?

(12) Consider the following Matlab function m-file.

```
function [t,y] = solveit(tI, yI, tF, n)
t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = tI; y(1) = yI; h = (tF - tI)/n;
for i = 1:n
z = t(i)^4 + y(i)^2;
t(i + 1) = t(i) + h;
y(i + 1) = y(i) + (h/2)*(z + t(i + 1)^4 + (y(i) + h*z)^2);
end
```

Suppose the input values are tI = 1, yI = 1, tF = 5, and n = 20.
(a) What is the initial-value problem being approximated numerically?
(b) What is the numerical method being used?
(c) What is the step size?
(d) What are the output values of t(2) and y(2)?

(13) Suppose we have used a numerical method to approximate the solution of an initial-value problem over the time interval $[1, 6]$ with 1000 uniform time steps. How many uniform time steps do we need to reduce the global error of our approximation by roughly a factor of $\frac{1}{81}$ if the method we had used was each of the following?
(a) Explicit Euler method
(b) Runge-trapezoidal method
(c) Runge-midpoint method
(d) Runge-Kutta method