Math 246 Final Exam
Professor David Levermore
begins 10:00am on Wednesday, 16 December 2020
due by noon on Thursday, 17 December 2020

- This is an "open-book exam". You may refer to the on-line text or the solutions to any of our past quizzes or group work.
- **Work on your own!** Do not consult with anyone or with any interactive resource.
- **On the top of the first page of your answers please hand copy and sign the University Honor Pledge.**

  University Honor Pledge: *I pledge on my honor that I have not given or received any unauthorized assistance on this examination.*
- **Answers should be handwritten.** Indicate where the answer to each part of each problem is located. Cross out work that you do not want considered.
- Use the **Table of Laplace Transforms** found at the end of the exam.
- **Your reasoning must be given for full credit!**
- Upload your answers to ELMS as a pdf file by **noon on Thursday, December 17.**

(1) [20] Find an explicit solution to each of the following initial-value problems. Identify their intervals of definition.

(a) [10] $x' = -\dfrac{4t}{(t^2+1)x}\,, \qquad x(0) = -2\,.$

(b) [10] $(1+t)\,v' + 4\,v = \dfrac{e^t}{(1+t)^3}\,, \qquad v(0) = 3\,.$

(2) [10] Consider the following Matlab commands.

[t,y] = ode45(@(t,y) y.*(y+4), [2,6], −2.0:4.0:2.0);
plot(t,y)

The following questions need not be answered in Matlab format!

(a) [2] What differential equation is being solved numerically?
(b) [1] Over what time interval are the solutions being approximated?
(c) [3] Give the initial condition for each solution being approximated.
(d) [4] Sketch each of these solutions over this time interval on a single graph. Label the initial value of each solution clearly.

(3) [20] Give an explicit real general solution to each of the following equations.

(a) [10] $u'' - 11u' + 24u = 12e^{2t}\,.$
(b) [10] $x'''' + 18x'' + 81x = 24\cos(3t)\,.$

(4) [10] Given that t^2 and t^4 solve the associated homogeneous differential equation, solve the initial-value problem

$$t^2v'' - 5t\,v' + 8v = \frac{8t^5}{1+t^2}\,, \qquad v(1) = 0\,, \quad v'(1) = 0\,.$$

Evaluate any definite integrals that arise.

(5) [20] Let $x(t)$ be the solution of the initial-value problem

$$x'' + 8x' + 20x = f(t)\,, \qquad x(0) = -3\,, \quad x'(0) = 5\,,$$

where the forcing $f(t)$ is given by

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 2\,, \\ 4e^{2-t} & \text{for } 2 \le t < \infty\,. \end{cases}$$

(a) [10] Find the Laplace transform $F(s)$ of the forcing $f(t)$.
(b) [10] Find the Laplace transform $X(s)$ of the solution $x(t)$.
(DO NOT take the inverse Laplace transform to find $x(t)$; just solve for $X(s)$!)
Use the **Table of Laplace Transforms** found at the end of the exam.

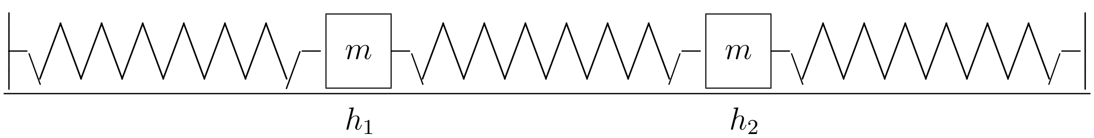
(6) [10] Find the function $y(t)$ whose Laplace transform is $Y(s) = \dfrac{e^{-5s}(s+9)}{s^2 - 4s - 21}$.
Use the **Table of Laplace Transforms** found at the end of the exam.

(7) [20] Solve the initial-value problem $\mathbf{x}' = \mathbf{B}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^{\mathrm{I}}$ for the following $\mathbf{B}$ and $\mathbf{x}^{\mathrm{I}}$.

(a) [10] $\mathbf{B} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$, $\qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

(b) [10] $\mathbf{B} = \begin{pmatrix} -2 & 5 \\ -1 & -4 \end{pmatrix}$, $\qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(8) [20] Two masses are connected by springs and slide along a frictionless horizontal track as illustrated by the following schematic diagram.



Their motion is governed by the second-order system

$$\ddot{h}_1 = -4h_1 - 6(h_1 - h_2)\,, \qquad \ddot{h}_2 = -6(h_2 - h_1) - 4h_2\,,$$

where h_1 and h_2 are the horizontal displacements of the masses from their respective equilibrium positions.

(a) [4] Recast this system as a first-order system in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 6 & 0 & 0 \\ 6 & -10 & 0 & 0 \end{pmatrix}.$$

(b) [8] Show that $\mathbf{A}$ has the eigenpairs

$$\left(i2\,, \begin{pmatrix} 1 \\ 1 \\ i2 \\ i2 \end{pmatrix} \right), \quad \left(-i2\,, \begin{pmatrix} 1 \\ 1 \\ -i2 \\ -i2 \end{pmatrix} \right), \quad \left(i4\,, \begin{pmatrix} 1 \\ -1 \\ i4 \\ -i4 \end{pmatrix} \right), \quad \left(-i4\,, \begin{pmatrix} 1 \\ -1 \\ -i4 \\ i4 \end{pmatrix} \right).$$

(c) [8] Give a fundamental set of real solutions for the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

(9) [10] Compute $e^{t\mathbf{C}}$ for

$$\mathbf{C} = \begin{pmatrix} 0 & -3 & 0 \\ 3 & 0 & 4 \\ 0 & -4 & 0 \end{pmatrix},$$

given that

- the characteristic polynomal of $\mathbf{C}$ is $p(z) = z^3 + 25z$,
- the Green function for $\mathrm{D}^3 + 25\mathrm{D}$ is $g(t) = \frac{1}{25}\big(1 - \cos(5t)\big)$.

(10) [20] A real 2×2 matrix $\mathbf{B}$ has the eigenpairs

$$\left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) \quad \text{and} \quad \left(-2, \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right).$$

(a) [4] Give a general solution to the linear planar system $\mathbf{x}' = \mathbf{B}\mathbf{x}$.
(b) [2] Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ that diagonalize $\mathbf{B}$.
(c) [8] Compute $e^{t\mathbf{B}}$.
(d) [6] Sketch a phase-plane portrait for this system and identify its type. Classify the stability of the origin. (Carefully mark all sketched orbits with arrows!)

(11) [20] Consider the system

$$x' = y - 4x + x^2, \qquad y' = 4y - 2xy.$$

(a) [10] Find a nonconstant function $H(x, y)$ such that every orbit of the system satisfies $H(x, y) = c$ for some constant c.
(b) [10] The stationary points are $(0, 0)$, $(4, 0)$, and $(2, 4)$. In the phase-plane sketch these stationary points plus the level set $H(x, y) = c$ for each value of c that corresponds to a stationary point that is a saddle. (No arrows are required!)

(12) [20] Consider the system

$$\dot{u} = -4u - v, \qquad \dot{v} = 2u - v - 3u^2.$$

(a) [3] This system has two stationary points. Find them.
(b) [3] Find the Jacobian matrix at each stationary point.
(c) [6] Classify the type and stability of each stationary point.
(d) [8] Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. (Carefully mark all sketched orbits with arrows!)

Table of Laplace Transforms

	$h(t) = \mathcal{L}^{-1}[H](t)$		$H(s) = \mathcal{L}[h](s)$	
1.	$t^n e^{at}$	for $n \geq 0$	$\dfrac{n!}{(s-a)^{n+1}}$	for $s > a$
2.	$e^{at}\cos(bt)$		$\dfrac{s-a}{(s-a)^2+b^2}$	for $s > a$
3.	$e^{at}\sin(bt)$		$\dfrac{b}{(s-a)^2+b^2}$	for $s > a$
4.	$e^{at}\cosh(bt)$		$\dfrac{s-a}{(s-a)^2-b^2}$	for $s > a+\lvert b\rvert$
5.	$e^{at}\sinh(bt)$		$\dfrac{b}{(s-a)^2-b^2}$	for $s > a+\lvert b\rvert$
6.	$t^n j(t)$	for $n \geq 0$	$(-1)^n J^{(n)}(s)$	where $J(s) = \mathcal{L}[j](s)$
7.	$j'(t)$		$s\,J(s) - j(0)$	where $J(s) = \mathcal{L}[j](s)$
8.	$e^{at}j(t)$		$J(s-a)$	where $J(s) = \mathcal{L}[j](s)$
9.	$u(t-c)j(t-c)$	for $c \geq 0$	$e^{-cs}J(s)$	where $J(s) = \mathcal{L}[j](s)$
10.	$\delta(t-c)j(t)$	for $c \geq 0$	$e^{-cs}j(c)$	

Here a, b, and c are real numbers; n is an integer; $j(t)$ is any function that is nice enough; $u(t)$ is the unit step (Heaviside) function; $\delta(t)$ is the unit impulse (Dirac delta).