Math 246 Exam 3 Solutions
Professor David Levermore
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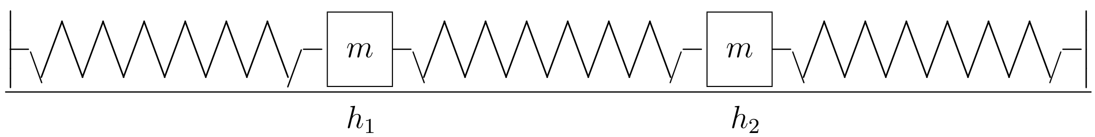
due by 4:00pm Friday, 20 November

(1) [6] Two masses are connected by springs and slide along a frictionless horizontal track as illustrated by the following schematic diagram.



Their motion is governed by the second-order system

$$\ddot{h}_1 = -4h_1 - 2(h_1 - h_2)\,, \qquad \ddot{h}_2 = -2(h_2 - h_1) - 3h_2\,,$$

where h_1 and h_2 are the horizontal displacements of the masses from their respective equilibrium positions. Recast this system as a first-order system in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

Solution. The second-order system is given in normal form. Because this system is two dimensional and is second order, the first-order system must have dimension at least four. One such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ -4x_1 - 2(x_1 - x_2) \\ -2(x_2 - x_1) - 3x_2 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \dot{h}_1 \\ \dot{h}_2 \end{pmatrix}.$$

This has the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 2 & 0 & 0 \\ 2 & -5 & 0 & 0 \end{pmatrix}.$$

Remark. There should be no h_1, h_2, $\dot{h}_1$, or $\dot{h}_2$ appearing in the first-order system. The only place these should appear is in the dictionary on the right that shows their relationship to the new variables. The first-order system should be expressed solely in terms of the new variables, which are x_1, x_2, x_3, and x_4 in the solution given above because the requested form was $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

Remark. The dyanamics of a general spring-mass system depicted in the schematic diagram is governed by the second-order system

$$m_1\ddot{h}_1 = -k_1h_1 - k_2(h_1 - h_2)\,, \qquad m_2\ddot{h}_2 = -k_2(h_2 - h_1) - k_3h_2\,,$$

where m_1 and m_2 are the respecitve masses and k_1, k_2 and k_3 are the respective spring coefficients. After dividing by the masses and comparing the result with the system given in the problem, we see that

$$\frac{k_1}{m_1} = 4\,, \qquad \frac{k_2}{m_1} = 2\,, \qquad \frac{k_2}{m_2} = 2\,, \qquad \frac{k_3}{m_2} = 3\,.$$

It follows that $m_1 = m_2$, which is why both masses are labeled with m in the diagram.

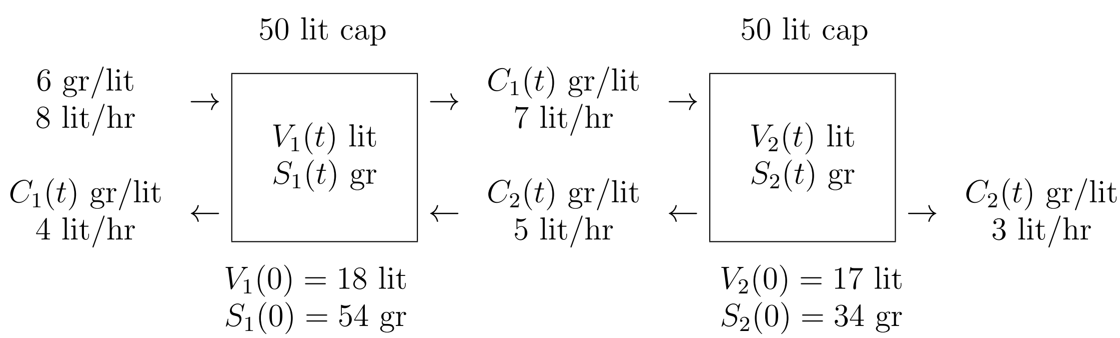
(2) [8] Two connected tanks, each with a capacity of 50 liters, contain brine (salt water). Initially the first tank contains 18 liters of brine with a salt concentration of 3 grams per liter and the second contains 17 liters of brine with a salt concentration of 2 grams per liter. At $t = 0$ brine with a salt concentration of 6 grams per liter flows into the first tank at 8 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 4 liter per hour, and from the second into a drain at 3 liters per hour.

(a) [2] Determine the volume (liters) of brine in each tank as a function of time.
(b) [4] Give an initial-value problem that governs the amount (grams) of salt in each tank as a function of time.
(c) [2] Give the interval of definition for the solution of this initial-value problem.

Remark. Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t hours. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t hours. Because the mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. At $t = 0$ we have

$$S_1(0) = C_1(0)V_1(0) = 3 \cdot 18 = 54 \quad \text{gr}, \qquad S_2(0) = C_2(0)V_2(0) = 2 \cdot 17 = 34 \quad \text{gr}.$$

We have the following picture.



Solution (a). We are asked to determine $V_1(t)$ and $V_2(t)$. The rates above give

$$V_1(t) = V_1(0) + (8 + 5 - 7 - 4)t = 18 + 2t \quad \text{lit},$$
$$V_2(t) = V_2(0) + (7 - 5 - 3)t = 17 - t \quad \text{lit}.$$

Remark. Because the tanks each have a capacity of 50 liters, we have the restrictions

$$0 \le V_1(t) = 18 + 2t \le 50, \qquad 0 \le V_2(t) = 17 - t \le 50.$$

These restrictions are

$$-9 \le t \le 16, \qquad -33 \le t \le 17,$$

which combine to give the restrictions

$$-9 \le t \le 16.$$

Notice that these restrictions happen when the first tank is either empty or full.

Solution (b). You are asked to give an initial-value problem that governs $S_1(t)$ and $S_2(t)$. These are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 6{\cdot}8 + \frac{S_2}{17-t}\,5 - \frac{S_1}{18+2t}\,7 - \frac{S_1}{18+2t}\,4\,, \qquad && S_1(0) = 54\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{18+2t}\,7 - \frac{S_2}{17-t}\,5 - \frac{S_2}{17-t}\,3\,, && S_2(0) = 34\,. \end{aligned}$$

You could leave the answer in the above form. It can however be simplified to

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 48 + \frac{5}{17-t}S_2 - \frac{11}{18+2t}S_1\,, \qquad && S_1(0) = 54\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{7}{18+2t}S_1 - \frac{8}{17-t}S_2\,, && S_2(0) = 34\,. \end{aligned}$$

Solution (c). You are asked to give the interval of definition for the solution of this initial-value problem. This can be done because the differential equation is *linear*. Its coefficients are undefined either at $t = -9$ or at $t = 17$ and are continuous elsewhere. Its forcing is constant, so is continuous everywhere. Therefore the natural interval of definition of this initial-value problem is $(-9, 17)$ because:

- the initial time $t = 0$ is in $(-9, 17)$;
- all the coefficients and the forcing are continuous over $(-11, 17)$;
- two coefficients are undefined at $t = -9$;
- two coefficients are undefined at $t = 17$.

However, this interval is not consistent with the restictions given earlier because the first tank overflows when $t = 16$. Therefore one acceptable answer is $(-9, 16]$.

We can also argue that the interval of definition for the solution of this initial-value problem is $[0, 16]$ because the word problem starts at $t = 0$.

(3) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t^3 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 4 + t^5 \end{pmatrix}$.

(a) [2] Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{B}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{B}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system found in part (b).

(d) [3] Compute the Green matrix associated with the system found in part (b).

Solution (a). The Wronskian is

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 1 & t^2 \\ t^3 & 4 + t^5 \end{pmatrix} = 1 \cdot (4 + t^5) - t^3 \cdot t^5 = 4 + t^5 - t^5 = 4\,.$$

Solution (b). If $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions for the system $\mathbf{x}' = \mathbf{B}(t)\mathbf{x}$ then a fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 & t^2 \\ t^3 & 4 + t^5 \end{pmatrix}\,.$$

Because any fundamental matrix is invertible and satisfies $\mathbf{\Psi}'(t) = \mathbf{B}(t)\mathbf{\Psi}(t)$, we see that

$$\begin{aligned}\mathbf{B}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 0 & 2t \\ 3t^2 & 5t^4 \end{pmatrix} \begin{pmatrix} 1 & t^2 \\ t^3 & 4+t^5 \end{pmatrix}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} 0 & 2t \\ 3t^2 & 5t^4 \end{pmatrix} \begin{pmatrix} 4+t^5 & -t^2 \\ -t^3 & 1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} -2t^4 & 2t \\ 12t^2 - 2t^7 & 2t^4 \end{pmatrix} .\end{aligned}$$

Solution (c). A general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} 1 \\ t^3 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 4+t^5 \end{pmatrix} .$$

Solution (d). By using the fundamental matrix $\mathbf{\Psi}(t)$ from part (b) we find that the Green matrix is

$$\begin{aligned}\mathbf{G}(t,s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} &= \begin{pmatrix} 1 & t^2 \\ t^3 & 4+t^5 \end{pmatrix} \begin{pmatrix} 1 & s^2 \\ s^3 & 4+s^5 \end{pmatrix}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} 1 & t^2 \\ t^3 & 4+t^5 \end{pmatrix} \begin{pmatrix} 4+s^5 & -s^2 \\ -s^3 & 1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 4+s^5 - t^2s^3 & t^2 - s^2 \\ t^3(4+s^5) - (4+t^5)s^3 & 4+t^5 - t^3s^2 \end{pmatrix} .\end{aligned}$$

Remark. Remember to check that $\mathbf{G}(s,s) = \mathbf{I}$.

(4) [8] Given that 2 is an eigenvalue of the matrix

$$\mathbf{C} = \begin{pmatrix} -4 & 0 & 3 \\ 3 & 1 & 0 \\ 2 & -2 & 4 \end{pmatrix} ,$$

do the following.

(a) [4] Find all of the eigenvectors of $\mathbf{C}$ associated with 2.

(b) [4] Find the other eigenvalues of $\mathbf{C}$. (You do not need to find more eigenvectors!)

Solution (a). The eigenvectors of $\mathbf{C}$ associated with 2 are all nonzero vectors $\mathbf{v}$ such that $\mathbf{C}\mathbf{v} = 2\mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ such that $(\mathbf{C} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$, which is

$$\begin{pmatrix} -6 & 0 & 3 \\ 3 & -1 & 0 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} -6v_1 \qquad\quad + 3v_3 &= 0 . \\ 3v_1 - \ v_2 \qquad\quad &= 0 , \\ 2v_1 - 2v_2 + 2v_3 &= 0 . \end{aligned}$$

This system may be solved either by elimination or by row reduction. By any method its general solution is found to be

$$v_1 = \alpha\,, \quad v_2 = 3\alpha\,, \quad v_3 = 2\alpha\,, \qquad \text{for any constant } \alpha\,.$$

Therefore every eigenvector of $\mathbf{C}$ associated with 2 has the form

$$\alpha \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \qquad \text{for some constant } \alpha \neq 0\,.$$

Solution (b). The eigenvalues of $\mathbf{C}$ are the zeros of its characteristic polynomial $p_{\mathbf{C}}(\zeta)$, which is defined by

$$\begin{aligned} p_{\mathbf{C}}(\zeta) = \det(\zeta\mathbf{I} - \mathbf{C}) &= \det\begin{pmatrix} \zeta+4 & 0 & -3 \\ -3 & \zeta-1 & 0 \\ -2 & 2 & \zeta-4 \end{pmatrix} \\ &= (\zeta+4)(\zeta-1)(\zeta-4) + (-3)(-3)2 - (-2)(\zeta-1)(-3) \\ &= (\zeta^2-16)(\zeta-1) + 18 - 6(\zeta-1) \\ &= \zeta^3 - \zeta^2 - 22\zeta + 40\,. \end{aligned}$$

Next, check that $p_{\mathbf{C}}(2) = 8 - 4 - 44 + 40 = 0$. (If this is not true then a mistake was made in computing $p_{\mathbf{C}}(\zeta)$.) Because we know that 2 is a zero of $p_{\mathbf{C}}(\zeta)$, the others can be found either by trying factors of 40, by factoring $p_{\mathbf{C}}(\zeta)$ by polynomial division, or by factoring a translation of $p_{\mathbf{C}}(\zeta)$. By either route the other eigenvalues of $\mathbf{C}$ are found to be -5 and 4.

Trying Factors of 40**.** The factors of 40 are ±1, ±2, ±4, ±5, ±8, ±10, ±20, and ±40. Trying these in order of increasing magnitude we get

$$\begin{aligned} p_{\mathbf{C}}(1) &= 1 - 1 - 22 + 40 = 18\,, & p_{\mathbf{C}}(-1) &= -1 - 1 + 22 + 40 = 60\,, \\ p_{\mathbf{C}}(2) &= 8 - 4 - 44 + 40 = 0\,, & p_{\mathbf{C}}(-2) &= -8 - 4 + 44 + 40 = 72\,, \\ p_{\mathbf{C}}(4) &= 64 - 16 - 88 + 40 = 0\,, & p_{\mathbf{C}}(-4) &= -64 - 16 + 88 + 40 = 48\,, \\ p_{\mathbf{C}}(5) &= 125 - 25 - 110 + 40 = 30\,, & p_{\mathbf{C}}(-5) &= -125 - 25 + 110 + 40 = 0\,. \end{aligned}$$

Because $p_{\mathbf{C}}(\zeta)$ has zeros 2, 4, and -5, the other eigenvalues of $\mathbf{C}$ are 4 and -5.

Factoring $p_{\mathbf{C}}(\zeta)$ by Polynomial Division. Because $p_{\mathbf{C}}(2) = 0$, we know that $\zeta - 2$ is a factor of $p_{\mathbf{C}}(\zeta)$. Upon dividing $p_{\mathbf{C}}(\zeta)$ by $\zeta - 2$ we see that

$$p_{\mathbf{C}}(\zeta) = (\zeta-2)\big(\zeta^2 + \zeta - 20\big) = (\zeta-2)(\zeta+5)(\zeta-4)\,.$$

Because $p_{\mathbf{C}}(\zeta)$ has zeros 2, -5, and 4, the other eigenvalues of $\mathbf{C}$ are -5 and 4. Any polynomial division algorithm can be used. For example, long division gives

$$\begin{array}{r|l} & \zeta^2 + \zeta - 20 \\ \hline \zeta - 2 & \zeta^3 - \zeta^2 - 22\zeta + 40 \\ & \underline{\zeta^3 - 2\zeta^2} \\ & \quad\ \zeta^2 - 22\zeta \\ & \quad\ \underline{\zeta^2 - 2\zeta} \\ & \qquad -20\zeta + 40\,. \end{array}$$

Synthetic division is a bit faster, if you know it.

Factoring a Translation of $p_{\mathbf{C}}(\zeta)$. Because $p_{\mathbf{C}}(2) = 0$, let $q(\delta)$ be the translation of $p_{\mathbf{C}}(\zeta)$ given by

$$\begin{aligned} q(\delta) = p_{\mathbf{C}}(2+\delta) &= (2+\delta)^3 - (2+\delta)^2 - 22(2+\delta) + 40 \\ &= \big(8 + 12\delta + 6\delta^2 + \delta^3\big) - \big(4 + 4\delta + \delta^2\big) - 22(2+\delta) + 40 \\ &= -14\delta + 5\delta^2 + \delta^3 = \delta\big(\delta^2 + 5\delta - 14\big) = \delta(\delta+7)(\delta-2)\,. \end{aligned}$$

The zeros of $q(\delta)$ are 0, -7, and 2, so the zeros of $p_{\mathbf{C}}(\zeta)$ are 2, -5, and 4. Therefore the other eigenvalues of $\mathbf{C}$ are -5 and 4.

Remark. Because $p_{\mathbf{C}}(\zeta) = q(\zeta - 2)$, by setting $\delta = \zeta - 2$ into the factorization of $q(\delta)$ found here we get the factorization of $p_{\mathbf{C}}(\zeta)$ found by polynomial division.

(5) [8] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -5 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Remark. Because this is an initial-value problem, computing the matrix exponential gives a direct route to the answer. The fastest way to compute a 2×2 matrix exponential is by using the appropriate formula.

Solution by Formula. The characteristic polynomial of $\begin{pmatrix} -2 & -5 \\ 1 & -4 \end{pmatrix}$ is

$$p(\zeta) = \zeta^2 - \mathrm{tr}(\mathbf{A})\zeta + \det(\mathbf{A}) = \zeta^2 + 6\zeta + 13 = (\zeta+3)^2 + 2^2\,.$$

This is a sum of squares with $\mu = -3$ and $\nu = 2$. Then by formula

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-3t}\left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}\,(\mathbf{A} - (-3)\mathbf{I})\right] \\ &= e^{-3t}\left[\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2}\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\right] \\ &= e^{-3t}\begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) & -\frac{5}{2}\sin(2t) \\ \frac{1}{2}\sin(2t) & \cos(2t) - \frac{1}{2}\sin(2t) \end{pmatrix}. \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A} + 3\mathbf{I}) = 0$!) Therefore the solution of the initial-value problem is

$$\begin{aligned} \mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^I &= e^{-3t}\begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) & -\frac{5}{2}\sin(2t) \\ \frac{1}{2}\sin(2t) & \cos(2t) - \frac{1}{2}\sin(2t) \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} 3\cos(2t) + \frac{3}{2}\sin(2t) \\ \frac{3}{2}\sin(2t) \end{pmatrix}. \end{aligned}$$

(6) [8] A real 3×3 matrix $\mathbf{H}$ has the eigenpairs

$$\left(0\,, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\right), \qquad \left(i6\,, \begin{pmatrix} 1 - i2 \\ 2 + i2 \\ 2 - i \end{pmatrix}\right), \qquad \left(-i6\,, \begin{pmatrix} 1 + i2 \\ 2 - i2 \\ 2 + i \end{pmatrix}\right).$$

(a) [4] Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{H} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $\mathbf{H}$!)

(b) [4] Give a real fundamental matrix for the system $\mathbf{x}' = \mathbf{H}\mathbf{x}$.

Solution (a). One choice for $\mathbf{V}$ and $\mathbf{D}$ is

$$\mathbf{V} = \begin{pmatrix} 2 & 1-i2 & 1+i2 \\ 1 & 2+i2 & 2-i2 \\ -2 & 2-i & 2+i \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i6 & 0 \\ 0 & 0 & -i6 \end{pmatrix}.$$

Remark. There are 5 other choices for $\mathbf{D}$. (Can you find them all?)

Solution (b). Use the given eigenpairs to construct the real eigensolutions

$$\mathbf{x}_1(t) = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{x}_2(t) = \mathrm{Re}\left(e^{i6t} \begin{pmatrix} 1-i2 \\ 2+i2 \\ 2-i \end{pmatrix} \right), \quad \mathbf{x}_3(t) = \mathrm{Im}\left(e^{i6t} \begin{pmatrix} 1-i2 \\ 2+i2 \\ 2-i \end{pmatrix} \right).$$

Because

$$\begin{aligned} e^{i6t} \begin{pmatrix} 1-i2 \\ 2+i2 \\ 2-i \end{pmatrix} &= \big(\cos(6t) + i\sin(6t)\big) \begin{pmatrix} 1-i2 \\ 2+i2 \\ 2-i \end{pmatrix} \\ &= \begin{pmatrix} \cos(6t) + 2\sin(6t) \\ 2\cos(6t) - 2\sin(6t) \\ 2\cos(6t) + \sin(6t) \end{pmatrix} + i \begin{pmatrix} \sin(6t) - 2\cos(6t) \\ 2\sin(6t) + 2\cos(6t) \\ 2\sin(6t) - \cos(6t) \end{pmatrix}, \end{aligned}$$

we see that a real fundamental matrix for the system is

$$\boldsymbol{\Psi}(t) = \big(\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)\big) = \begin{pmatrix} 2 & \cos(6t) + 2\sin(6t) & \sin(6t) - 2\cos(6t) \\ 1 & 2\cos(6t) - 2\sin(6t) & 2\sin(6t) + 2\cos(6t) \\ -2 & 2\cos(6t) + \sin(6t) & 2\sin(6t) - \cos(6t) \end{pmatrix}.$$

(7) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution by Eigensolutions. The characteristic polynomial of $\mathbf{B} = \begin{pmatrix} 1 & 6 \\ 4 & 3 \end{pmatrix}$ is

$$p(\zeta) = \zeta^2 - \mathrm{tr}(\mathbf{B})\zeta + \det(\mathbf{B}) = \zeta^2 - 4\zeta - 21 = (\zeta + 3)(\zeta - 7).$$

The eigenvalues of $\mathbf{B}$ are the roots of this polynomial, which are -3 and 7. Consider the matrices

$$\mathbf{B} + 3\mathbf{I} = \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix}, \qquad \mathbf{B} - 7\mathbf{I} = \begin{pmatrix} -6 & 6 \\ 4 & -4 \end{pmatrix}.$$

After checking that the determinant of each matrix is zero, we can read off that eigenpairs of $\mathbf{B}$ are

$$\left(-3, \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right), \qquad \left(7, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right),$$

whereby two eigensolutions are

$$\mathbf{x}_1(t) = e^{-3t} \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore a real general solution of the system is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{-3t}\begin{pmatrix} 3 \\ -2 \end{pmatrix} + c_2 e^{7t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution by Formula. The characteristic polynomial of $\mathbf{B} = \begin{pmatrix} 1 & 6 \\ 4 & 3 \end{pmatrix}$ is

$$p(\zeta) = \zeta^2 - \mathrm{tr}(\mathbf{B})\zeta + \det(\mathbf{B}) = \zeta^2 - 4\zeta - 21 = (\zeta-2)^2 - 4 - 21 = (\zeta-2)^2 - 5^2.$$

This is a difference of squares with $\mu = 2$ and $\nu = 5$. Then

$$\begin{aligned} e^{t\mathbf{B}} &= e^{2t}\left[\cosh(5t)\mathbf{I} + \frac{\sinh(5t)}{5}\big(\mathbf{B} - 2\mathbf{I}\big)\right] \\ &= e^{2t}\left[\cosh(5t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(5t)}{5}\begin{pmatrix} -1 & 6 \\ 4 & 1 \end{pmatrix}\right] \\ &= e^{2t}\begin{pmatrix} \cosh(5t) - \frac{1}{5}\sinh(5t) & \frac{6}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) & \cosh(5t) + \frac{1}{5}\sinh(5t) \end{pmatrix}. \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{B} - 2\mathbf{I}) = 0$!) Therefore a real general solution of the system is

$$\begin{aligned} \mathbf{x}(t) = e^{t\mathbf{B}}\mathbf{c} &= e^{2t}\begin{pmatrix} \cosh(5t) - \frac{1}{5}\sinh(5t) & \frac{6}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) & \cosh(5t) + \frac{1}{5}\sinh(5t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{2t}\begin{pmatrix} \cosh(5t) - \frac{1}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} \frac{6}{5}\sinh(5t) \\ \cosh(5t) + \frac{1}{5}\sinh(5t) \end{pmatrix}. \end{aligned}$$

(8) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -8 & -3 \\ 3 & -2 \end{pmatrix}$ is

$$p(\zeta) = \zeta^2 - \mathrm{tr}(\mathbf{A})\zeta + \det(\mathbf{A}) = \zeta^2 + 10\zeta + 25 = (\zeta+5)^2.$$

This is a pserfect square with $\mu = -5$. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-5t}\left[\mathbf{I} + t\big(\mathbf{A} - (-5)\mathbf{I}\big)\right] \\ &= e^{-5t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\right] = e^{-5t}\begin{pmatrix} 1-3t & -3t \\ 3t & 1+3t \end{pmatrix}. \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A} + 5\mathbf{I}) = 0$!) Therefore a real general solution of the system is

$$\begin{aligned} \mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} &= e^{-5t}\begin{pmatrix} 1-3t & -3t \\ 3t & 1+3t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{-5t}\begin{pmatrix} 1-3t \\ 3t \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} -3t \\ 1+3t \end{pmatrix}. \end{aligned}$$

Solution by Eigensolutions. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -8 & -3 \\ 3 & -2 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 10z + 25 = (z+5)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is the double root -5. Consider the matrix

$$\mathbf{A} + 5\mathbf{I} = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} .$$

After checking that the determinant of this matrix is zero, we can read off that an eigenpair of $\mathbf{A}$ is

$$\left(-5 \,, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) ,$$

whereby an eigensolution is

$$\mathbf{x}_1(t) = e^{-5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} .$$

A second solution can be constructed by the formula

$$\mathbf{x}_2(t) = e^{-5t} \big[\mathbf{I} + t\,(\mathbf{A} + 5\mathbf{I}) \big] \mathbf{w} \,,$$

where $\mathbf{w}$ is any nonzero vector that is not an eigenvector. If $\mathbf{w} = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}}$ then

$$\begin{aligned} \mathbf{x}_2(t) &= e^{-5t} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= e^{-5t} \begin{pmatrix} 1-3t & -3t \\ 3t & 1+3t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-5t} \begin{pmatrix} 1-3t \\ 3t \end{pmatrix} . \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A} + 5\mathbf{I}) = 0$!) Therefore a real general solution of the system is

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) = c_1 e^{-5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} 1-3t \\ 3t \end{pmatrix} .$$

(9) [10] Find the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4 + 3\mathrm{D}^2 - 4$. You may refer to the table on the last page.

Solution. The characteristic polynomial of $\mathrm{L} = \mathrm{D}^4 + 3\mathrm{D}^2 - 4$ is $p(s) = s^4 + 3s^2 - 4$. Therefore its Green function $g(t)$ is given by

$$g(t) = \mathcal{L}^{-1}\!\left[\frac{1}{p(s)} \right]\!(t) = \mathcal{L}^{-1}\!\left[\frac{1}{s^4 + 3s^2 - 4} \right]\!(t) .$$

Because $p(s)$ factors as $p(s) = (s^2 - 1)(s^2 + 4)$ we have the partial fraction identity

$$\frac{1}{s^4 + 3s^2 - 4} = \frac{1}{(s^2-1)(s^2+4)} = \frac{\frac{1}{5}}{s^2 - 1} + \frac{-\frac{1}{5}}{s^2 + 4} .$$

Because $s^2 - 1$ factors as $s^2 - 1 = (s-1)(s+1)$ we have the partial fraction identity

$$\frac{1}{s^2 - 1} = \frac{1}{(s-1)(s+1)} = \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}}{s+1} .$$

By combining the above partial fraction identities we obtain

$$\frac{1}{s^4+3s^2-4} = \tfrac{1}{10}\frac{1}{s-1} - \tfrac{1}{10}\frac{1}{s+1} - \tfrac{1}{5}\frac{1}{s^2+4}\,.$$

Referring to the table on the last page, item 1 with $a = 1$ and $n = 0$ and with $a = -1$ and $n = 0$ gives

$$\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) = e^t\,, \qquad \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = e^{-t}\,,$$

while item 3 with $a = 0$ and $b = 4$ gives

$$\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right](t) = \sin(2t)\,.$$

Therefore the Green function $g(t)$ is given by

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^4+3s^2-4}\right](t) \\ &= \tfrac{1}{10}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) - \tfrac{1}{10}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) - \tfrac{1}{10}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right](t) \\ &= \tfrac{1}{10}e^t - \tfrac{1}{10}e^{-t} - \tfrac{1}{10}\sin(2t)\,. \end{aligned}$$

Then because we see the characteristic polynomial as

$$p(s) = s^4 + 0s^3 + 3s^2 + 0s - 4\,,$$

the natrual fundamental set for $t = 0$ is found by

$$\begin{aligned} N_3(t) &= g(t) = \tfrac{1}{10}e^t - \tfrac{1}{10}e^{-t} - \tfrac{1}{10}\sin(2t)\,, \\ N_2(t) &= N_3'(t) + 0g(t) = \tfrac{1}{10}e^t + \tfrac{1}{10}e^{-t} - \tfrac{1}{5}\cos(2t)\,, \\ N_1(t) &= N_2'(t) + 3g(t) \\ &\quad = \tfrac{1}{10}e^t - \tfrac{1}{10}e^{-t} + \tfrac{2}{5}\sin(2t) + \tfrac{3}{10}e^t - \tfrac{3}{10}e^{-t} - \tfrac{3}{10}\sin(2t)\,, \\ &\quad = \tfrac{2}{5}e^t - \tfrac{2}{5}e^{-t} + \tfrac{1}{10}\sin(2t)\,, \\ N_0(t) &= N_1'(t) + 0g(t) = \tfrac{2}{5}e^t + \tfrac{2}{5}e^{-t} + \tfrac{1}{5}\cos(2t)\,. \end{aligned}$$

Remark. The calculation of the natural fundamental set is a bit simpler if the Green function is expressed in terms of hyperbolic functions. It becomes

$$\begin{aligned} N_3(t) &= g(t) = \tfrac{1}{5}\sinh(t) - \tfrac{1}{10}\sin(2t)\,, \\ N_2(t) &= N_3'(t) + 0g(t) = \tfrac{1}{5}\cosh(t) - \tfrac{1}{5}\cos(2t)\,, \\ N_1(t) &= N_2'(t) + 3g(t) \\ &\quad = \tfrac{1}{5}\sinh(t) + \tfrac{2}{5}\sin(2t) + \tfrac{3}{5}\sinh(t) - \tfrac{3}{10}\sin(2t)\,, \\ &\quad = \tfrac{4}{5}\sinh(t) + \tfrac{1}{10}\sin(2t)\,, \\ N_0(t) &= N_1'(t) + 0g(t) = \tfrac{4}{5}\cosh(t) + \tfrac{1}{5}\cos(2t)\,. \end{aligned}$$

Solution from General Initial-Value Problem. For the operator $\mathrm{D}^4 + 3\mathrm{D}^2 - 4$ the general initial-value problem for initial-time 0 is

$$y'''' + 3y'' - 4y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \quad y''(0) = y_2\,, \quad y'''(0) = y_3\,.$$

Its characteristic polynomial is

$$p(z) = z^4 + 3z^2 - 4 = (z^2 - 1)(z^2 + 4) = (z - 1)(z + 1)(z^2 + 2^2)\,,$$

which has roots 1, -1, $i2$ and $-i2$. Therefore a real general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos(2t) + c_4 \sin(2t)\,.$$

Because

$$\begin{aligned} y'(t) &= c_1 e^t - c_2 e^{-t} - 2c_3 \sin(2t) + 2c_4 \cos(2t)\,, \\ y''(t) &= c_1 e^t + c_2 e^{-t} - 4c_3 \cos(2t) - 4c_4 \sin(2t)\,, \\ y'''(t) &= c_1 e^t - c_2 e^{-t} + 8c_3 \sin(2t) - 8c_4 \cos(2t)\,, \end{aligned}$$

the general initial conditions yield the linear algebraic system

$$\begin{aligned} y_0 &= y(0) = c_1 e^0 + c_2 e^0 + c_3 \cos(0) + c_4 \sin(0) = c_1 + c_2 + c_3\,. \\ y_1 &= y'(0) = c_1 e^0 - c_2 e^0 - 2c_3 \sin(0) + 2c_4 \cos(0) = c_1 - c_2 + 2c_4\,, \\ y_2 &= y''(0) = c_1 e^0 + c_2 e^0 - 4c_3 \cos(0) - 4c_4 \sin(0) = c_1 + c_2 - 4c_3\,, \\ y_3 &= y'''(t) = c_1 e^0 - c_2 e^0 + 8c_3 \sin(0) - 8c_4 \cos(0) = c_1 - c_2 - 8c_4\,. \end{aligned}$$

This can be viewed as decoupling into the two systems

$$\begin{aligned} y_0 &= c_1 + c_2 + c_3\,, \qquad & y_1 &= c_1 - c_2 + 2c_4\,, \\ y_2 &= c_1 + c_2 - 4c_3\,, \qquad & y_3 &= c_1 - c_2 - 8c_4\,, \end{aligned}$$

where the system on the left is for $c_1 + c_2$ and c_3 while the system on the right is for $c_1 - c_2$ and c_4. The solutions of these systems are

$$\begin{aligned} c_1 + c_2 &= \frac{4y_0 + y_2}{5}\,, \qquad & c_1 - c_2 &= \frac{4y_1 + y_3}{5}\,, \\ c_3 &= \frac{y_0 - y_2}{5}\,, \qquad & c_4 &= \frac{y_1 - y_3}{10}\,. \end{aligned}$$

The top two equations then yield

$$c_1 = \frac{4y_0 + 4y_1 + y_2 + y_3}{10}\,, \qquad c_2 = \frac{4y_0 - 4y_1 + y_2 - y_3}{10}\,.$$

Therefore the solution of the general initial-value problem is

$$\begin{aligned} y &= \frac{4y_0 + 4y_1 + y_2 + y_3}{10}\, e^t + \frac{4y_0 - 4y_1 + y_2 - y_3}{10}\, e^{-t} \\ &\quad + \frac{y_0 - y_2}{5}\, \cos(2t) + \frac{y_1 - y_3}{10}\, \sin(2t) \\ &= y_0\big(\tfrac{2}{5}e^t + \tfrac{2}{5}e^{-t} + \tfrac{1}{5}\cos(2t)\big) + y_1\big(\tfrac{2}{5}e^t - \tfrac{2}{5}e^{-t} + \tfrac{1}{10}\sin(2t)\big) \\ &\quad + y_2\big(\tfrac{1}{10}e^t + \tfrac{1}{10}e^{-t} - \tfrac{1}{5}\cos(2t)\big) + y_3\big(\tfrac{1}{10}e^t - \tfrac{1}{10}e^{-t} - \tfrac{1}{10}\sin(2t)\big)\,. \end{aligned}$$

We can read off from this that the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4 + 3\mathrm{D}^2 - 4$ is

$$\begin{aligned} N_0(t) &= \tfrac{2}{5}e^t + \tfrac{2}{5}e^{-t} + \tfrac{1}{5}\cos(2t)\,, \qquad & N_1(t) &= \tfrac{2}{5}e^t - \tfrac{2}{5}e^{-t} + \tfrac{1}{10}\sin(2t)\,, \\ N_2(t) &= \tfrac{1}{10}e^t + \tfrac{1}{10}e^{-t} - \tfrac{1}{5}\cos(2t)\,, \qquad & N_3(t) &= \tfrac{1}{10}e^t - \tfrac{1}{10}e^{-t} - \tfrac{1}{10}\sin(2t)\,. \end{aligned}$$

(10) [8] Compute the Laplace transform of $f(t) = u(t-3)\, e^{-i2t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} u(t-3)\, e^{-i2t}\, \mathrm{d}t = \lim_{T\to\infty} \int_3^T e^{-(s+i2)t}\, \mathrm{d}t\,.$$

For every $T > 3$ we have

$$\int_3^T e^{-(s+i2)t}\, \mathrm{d}t = -\frac{e^{-(s+i2)t}}{s+i2}\bigg|_3^T = -\frac{e^{-(s+i2)T}}{s+i2} + \frac{e^{-(s+i2)3}}{s+i2}\,,$$

whereby the definition of the Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \left[-\frac{e^{-(s+i2)T}}{s+i2} + \frac{e^{-(s+i2)3}}{s+i2} \right] = \begin{cases} \dfrac{e^{-(s+i2)3}}{s+i2} & \text{for } s > 0\,, \\ \text{undefined} & \text{for } s \le 0\,. \end{cases}$$

(11) [10] Consider the following MATLAB commands.

```
>> syms t x(t) s X
>> f = t^2 + heaviside(t − 2)*(6 − t − t^2) + heaviside(t − 6)*(t − 6);
>> diffeqn = diff(x, 2) + 4*diff(x, 1) + 29*x(t) == f;
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, ...
                  [laplace(x(t), t, s), x(0), subs(diff(x(t), t), t, 0)], [X, 3, −4]);
>> xtrans = simplify(solve(algeqn, X));
>> x = ilaplace(xtrans, s, t)
```

(a) [2] Give the initial-value problem for $x(t)$ that is being solved.

(b) [8] Find the Laplace transform $X(s)$ of the solution $x(t)$. (Just solve for $X(s)$! DO NOT take the inverse Laplace transform of $X(s)$ to solve for $x(t)$!) You may refer to the table below.

Solution (a). The initial-value problem for $x(t)$ that is being solved is

$$x'' + 4x' + 29x = f(t)\,, \qquad x(0) = 3\,, \quad x'(0) = -4\,,$$

where the forcing $f(t)$ can be expressed either as the piecewise-defined function

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 2\,, \\ 6 - t & \text{for } 2 \le t < 6\,, \\ 0 & \text{for } 6 \le t\,, \end{cases}$$

or in terms of the unit step function as

$$f(t) = t^2 + u(t-2)(6 - t - t^2) + u(t-6)(t-6)\,.$$

Solution (b). The Laplace transform of the differential equation is

$$\mathcal{L}[x''](s) + 4\mathcal{L}[x'](s) + 29\mathcal{L}[x](s) = \mathcal{L}[f](s)\,,$$

where the initial conditions give

$$\begin{aligned}\mathcal{L}[x](s) &= X(s)\,,\\ \mathcal{L}[x'](s) &= s\,\mathcal{L}[x](s) - x(0) = s\,X(s) - 3\,,\\ \mathcal{L}[x''](s) &= s\,\mathcal{L}[x'](s) - x'(0) = s^2X(s) - 3s + 4\,.\end{aligned}$$

Therefore the Laplace transform of the initial-value problem is

$$\big(s^2X(s) - 3s + 4\big) + 4\big(s\,X(s) - 3\big) + 29X(s) = F(s)\,,$$

where $F(s) = \mathcal{L}[f](s)$. This simplifies to

$$(s^2 + 4s + 29)X(s) - 3s - 8 = F(s)\,,$$

whereby

$$X(s) = \frac{1}{s^2 + 4s + 29}\big(3s + 8 + F(s)\big)\,.$$

To compute $F(s)$, we write $f(t)$ as

$$\begin{aligned}f(t) &= t^2 + u(t-2)(6 - t - t^2) + u(t-6)(t-6)\\ &= t^2 + u(t-2)j_1(t-2) + u(t-6)j_2(t-6)\,,\end{aligned}$$

where upon by setting $j_1(t-2) = 6 - t - t^2$ and $j_2(t-6) = t - 6$, the shifty step method gives

$$\begin{aligned}&j_1(t) = 6 - (t+2) - (t+2)^2 = 6 - t - 2 - t^2 - 4t - 4 = -t^2 - 5t\,,\\ &j_2(t) = (t+6) - 6 = t\,.\end{aligned}$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 1$, and with $a = 0$ and $n = 2$ shows that

$$\mathcal{L}[t](s) = \frac{1}{s^2}\,, \qquad \mathcal{L}[t^2](s) = \frac{2}{s^3}\,,$$

whereby item 7 with $c = 2$ and $j(t) = j_1(t) = -t^2 - 5t$ and with $c = 6$ and $j(t) = j_2(t) = t$ shows that

$$\begin{aligned}&\mathcal{L}\big[u(t-2)j_1(t-2)\big](s) = e^{-2s}\mathcal{L}[j_1](s) = -e^{-2s}\mathcal{L}[t^2 + 5t](s) = -e^{-2s}\left(\frac{2}{s^3} + \frac{5}{s^2}\right)\,,\\ &\mathcal{L}\big[u(t-6)j_2(t-6)\big](s) = e^{-6s}\mathcal{L}[j_2](s) = e^{-6s}\mathcal{L}[t](s) = e^{-6s}\frac{1}{s^2}\,.\end{aligned}$$

Therefore

$$\begin{aligned}F(s) &= \mathcal{L}\big[t^2 + u(t-2)j_1(t-2) + u(t-6)j_2(t-6)\big](s)\\ &= \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{5}{s^2}\right) + e^{-6s}\frac{1}{s^2}\,.\end{aligned}$$

Upon placing this result into the expression for $X(s)$ found earlier, we obtain

$$X(s) = \frac{1}{s^2 + 4s + 29}\left(3s + 8 + \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{5}{s^2}\right) + e^{-6s}\frac{1}{s^2}\right)\,.$$

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[Y(s)](t)$ of the function

$$Y(s) = e^{-4s}\,\frac{2s+9}{s^2-6s+34}\,.$$

You may refer to the table below.

Solution. Referring to the table on the last page, item 7 with $c = 4$ shows that

$$\mathcal{L}^{-1}\big[e^{-4s}\,J(s)\big] = u(t-4)j(t-4)\,, \qquad \text{where} \qquad j(t) = \mathcal{L}^{-1}[J(s)](t)\,.$$

We apply this formula to

$$J(s) = \frac{2s+9}{s^2-6s+34}\,.$$

Complete the square in the denominator to get $(s-3)^2+5^2$. We have the partial fraction identity

$$J(s) = \frac{2s+9}{s^2-6s+34} = \frac{2(s-3)+15}{(s-3)^2+5^2} = \frac{2(s-3)}{(s-3)^2+5^2} + \frac{15}{(s-3)^2+5^2}\,.$$

Referring to the table on the last page, items 2 and 3 with $a = 3$ and $b = 5$ give

$$\mathcal{L}^{-1}\left[\frac{s-3}{(s-3)^2+5^2}\right](t) = e^{3t}\cos(5t)\,, \qquad \mathcal{L}^{-1}\left[\frac{5}{(s-3)^2+5^2}\right](t) = e^{3t}\sin(5t)\,.$$

The above formulas and the linearity of the inverse Laplace transform yield

$$\begin{aligned} j(t) = \mathcal{L}^{-1}[J(s)](t) &= \mathcal{L}^{-1}\left[\frac{2s+9}{s^2-6s+34}\right](t) \\ &= \mathcal{L}^{-1}\left[\frac{2(s-3)}{(s-3)^2+5^2} + \frac{15}{(s-3)^2+5^2}\right](t) \\ &= 2\mathcal{L}^{-1}\left[\frac{s-3}{(s-3)^2+5^2}\right](t) + 3\mathcal{L}^{-1}\left[\frac{5}{(s-3)^2+5^2}\right](t) \\ &= 2e^{3t}\cos(5t) + 3e^{3t}\sin(5t)\,. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}^{-1}\big[Y(s)\big](t) &= \mathcal{L}^{-1}[e^{-4s}J(s)](t) \\ &= u(t-4)j(t-4) \\ &= u(t-4)\Big(2e^{3(t-4)}\cos\big(5(t-4)\big) + 3e^{3(t-4)}\sin\big(5(t-4)\big)\Big)\,. \end{aligned}$$

Table of Laplace Transforms

$\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$	for $s > a$.
$\mathcal{L}[e^{at}\cos(bt)](s) = \dfrac{s-a}{(s-a)^2+b^2}$	for $s > a$.
$\mathcal{L}[e^{at}\sin(bt)](s) = \dfrac{b}{(s-a)^2+b^2}$	for $s > a$.
$\mathcal{L}[j'(t)](s) = sJ(s) - j(0)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[e^{at} j(t)](s) = J(s-a)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s)$	where $J(s) = \mathcal{L}[j(t)](s)$, $c \geq 0$, and u is the unit step function.
$\mathcal{L}[\delta(t-c)j(t)](s) = e^{-cs}j(c)$	where $c \geq 0$ and δ is the unit impulse.