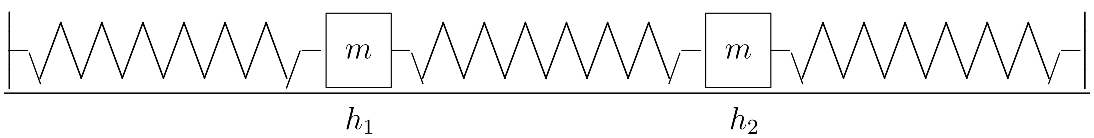
**Math 246 Exam 3**
**Professor David Levermore**
**Thursday, 19 November 2020**
**due by 4:00pm Friday, 20 November**

- This is an "open-book exam". You may refer to the on-line text or the solutions to any of our past quizzes or group work.
- **Work on your own!** Do not consult with anyone or with any interactive resource.
- **On the top of the first page of your answers please hand copy and sign the University Honor Pledge.**

  **University Honor Pledge:** *I pledge on my honor that I have not given or received any unauthorized assistance on this examination.*
- **Answers should be handwritten.** Indicate where the answer to each part of each problem is located. Cross out work that you do not want considered.
- Use the **Table of Laplace Transforms** found at the end of the exam.
- **Your reasoning must be given for full credit!**
- Upload your answers to ELMS as a pdf file by **4:00pm on Friday, November 20.**

(1) [6] Two masses are connected by springs and slide along a frictionless horizontal track as illustrated by the following schematic diagram.




Their motion is governed by the second-order system

$$\ddot{h}_1 = -4h_1 - 2(h_1 - h_2)\,, \qquad \ddot{h}_2 = -2(h_2 - h_1) - 3h_2\,,$$

where $h_1$ and $h_2$ are the horizontal displacements of the masses from their respective equilibrium positions. Recast this system as a first-order system in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

(2) [8] Two connected tanks, each with a capacity of 50 liters, contain brine (salt water). Initially the first tank contains 18 liters of brine with a salt concentration of 3 grams per liter and the second contains 17 liters of brine with a salt concentration of 2 grams per liter. At $t = 0$ brine with a salt concentration of 6 grams per liter flows into the first tank at 8 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 4 liter per hour, and from the second into a drain at 3 liters per hour.

(a) [2] Determine the volume (liters) of brine in each tank as a function of time.
(b) [4] Give an initial-value problem that governs the amount (grams) of salt in each tank as a function of time.
(c) [2] Give the interval of definition for the solution of this initial-value problem.

**More problems are on the next page.**

(3) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t^3 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 4 + t^5 \end{pmatrix}$.

(a) [2] Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{B}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{B}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system found in part (b).

(d) [3] Compute the Green matrix associated with the system found in part (b).

(4) [8] Given that 2 is an eigenvalue of the matrix

$$\mathbf{C} = \begin{pmatrix} -4 & 0 & 3 \\ 3 & 1 & 0 \\ 2 & -2 & 4 \end{pmatrix},$$

do the following.

(a) [4] Find all of the eigenvectors of $\mathbf{C}$ associated with 2.

(b) [4] Find the other eigenvalues of $\mathbf{C}$. (You do not need to find more eigenvectors!)

(5) [8] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -5 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

(6) [8] A real $3 \times 3$ matrix $\mathbf{H}$ has the eigenpairs

$$\left(0, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\right), \qquad \left(i6, \begin{pmatrix} 1 - i2 \\ 2 + i2 \\ 2 - i \end{pmatrix}\right), \qquad \left(-i6, \begin{pmatrix} 1 + i2 \\ 2 - i2 \\ 2 + i \end{pmatrix}\right).$$

(a) [4] Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{H} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $\mathbf{H}$!)

(b) [4] Give a real fundamental matrix for the system $\mathbf{x}' = \mathbf{H}\mathbf{x}$.

(7) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(8) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(9) [10] Find the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4 + 3\mathrm{D}^2 - 4$. You may refer to the table on the last page.

**More problems are on the next page.**

(10) [8] Compute the Laplace transform of $f(t) = u(t-3)\, e^{-i2t}$ from its definition. (Here $u$ is the unit step function.)

(11) [10] Consider the following MATLAB commands.

```
>> syms t x(t) s X
>> f = t^2 + heaviside(t - 2)*(6 - t - t^2) + heaviside(t - 6)*(t - 6);
>> diffeqn = diff(x, 2) + 4*diff(x, 1) + 29*x(t) == f;
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, ...
                    [laplace(x(t), t, s), x(0), subs(diff(x(t), t), t, 0)], [X, 3, -4]);
>> xtrans = simplify(solve(algeqn, X));
>> x = ilaplace(xtrans, s, t)
```

(a) [2] Give the initial-value problem for $x(t)$ that is being solved.

(b) [8] Find the Laplace transform $X(s)$ of the solution $x(t)$. (Just solve for $X(s)$! DO NOT take the inverse Laplace transform of $X(s)$ to solve for $x(t)$!) You may refer to the table below.

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[Y(s)](t)$ of the function

$$Y(s) = e^{-4s}\,\frac{2s+9}{s^2-6s+34}\,.$$

You may refer to the table below.

## Table of Laplace Transforms

| | |
|---|---|
| $\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$ | for $s > a$. |
| $\mathcal{L}[e^{at}\cos(bt)](s) = \dfrac{s-a}{(s-a)^2+b^2}$ | for $s > a$. |
| $\mathcal{L}[e^{at}\sin(bt)](s) = \dfrac{b}{(s-a)^2+b^2}$ | for $s > a$. |
| $\mathcal{L}[j'(t)](s) = sJ(s) - j(0)$ | where $J(s) = \mathcal{L}[j(t)](s)$. |
| $\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s)$ | where $J(s) = \mathcal{L}[j(t)](s)$. |
| $\mathcal{L}[e^{at} j(t)](s) = J(s-a)$ | where $J(s) = \mathcal{L}[j(t)](s)$. |
| $\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s)$ | where $J(s) = \mathcal{L}[j(t)](s)$, $c \geq 0$, and $u$ is the unit step function. |
| $\mathcal{L}[\delta(t-c)j(t)](s) = e^{-cs}j(c)$ | where $c \geq 0$ and $\delta$ is the unit impulse. |