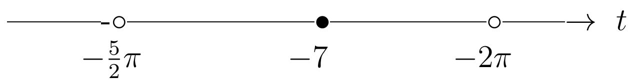
Math 246 Exam 2 Solutions
Professor David Levermore
Thursday, 22 October 2020
due by 4:00pm Friday, 23 October

(1) [4] Give the interval of definition for the solution of the initial-value problem

$$y''' + \frac{e^{3t}}{\sin(2t)}\, y'' + \frac{2+t}{8-t}\, y = \frac{\cos(4t)}{9-t^2}\,, \qquad y(-7) = y'(-7) = y''(-7) = 5\,.$$

Solution. The equation is linear and is already in normal form. Notice the following.
- ⋄ The coefficient of y'' is undefined at $t = n\frac{\pi}{2}$ for every integer n and is continuous elsewhere.
- ⋄ The coefficient of y is undefined at $t = 8$ and is continuous elsewhere.
- ⋄ The forcing is undefined at $t = \pm 3$ and is continuous elsewhere.
- ⋄ The initial time is $t = -7$.

Plotting these points on a time-line near the initial time $t = -7$ gives



Therefore the interval of definition is $(-\frac{5}{2}\pi, -2\pi)$ because:
- the initial time $t = -7$ is in $(-\frac{5}{2}\pi, -2\pi)$;
- all the coefficients and the forcing are continuous over $(-\frac{5}{2}\pi, -2\pi)$;
- the coefficient of y'' is undefined at $t = -\frac{5}{2}\pi$;
- the coefficient of y'' is undefined at $t = -2\pi$.

Remark. All four reasons must be given for full credit.
- ∘ The first two are why a (unique) solution exists over the interval $(-\frac{5}{2}\pi, -2\pi)$.
- ∘ The last two are why this solution does not exist over a larger interval.

(2) [12] The functions e^{7t} and e^{-7t} are a fundamental set of solutions to $v'' - 49v = 0$.

(a) [8] Solve the general initial-value problem

$$v'' - 49v = 0\,, \qquad v(0) = v_0\,, \quad v'(0) = v_1\,.$$

(b) [4] Find the associated natural fundamental set of solutions to $v'' - 49v = 0$.

Solution (a). Because we are given that e^{7t} and e^{-7t} are a fundamental set of solutions to $v'' - 49v = 0$, a general solution is

$$v = c_1 e^{7t} + c_2 e^{-7t}\,.$$

Because $v' = 7c_1 e^{7t} - 7c_2 e^{-7t}$, the initial conditions imply

$$v_0 = v(0) = c_1 + c_2\,, \qquad v_1 = v'(0) = 7c_1 - 7c_2\,.$$

We solve these equations to obtain

$$c_1 = \tfrac{1}{2}v_0 + \tfrac{1}{14}v_1\,, \qquad c_2 = \tfrac{1}{2}v_0 - \tfrac{1}{14}v_1\,.$$

Therefore the solution to the general initial-value problem is

$$v(t) = \left(\tfrac{1}{2}v_0 + \tfrac{1}{14}v_1\right)e^{7t} - \left(\tfrac{1}{2}v_0 - \tfrac{1}{14}v_1\right)e^{-7t}\,.$$

Solution (b). The solution found in part (a) can be written as

$$v(t) = v_0 \frac{e^{7t} + e^{-7t}}{2} + v_1 \frac{e^{7t} - e^{-7t}}{14} .$$

We can read off from this that the associated natural fundamental set of solutions is

$$N_0(t) = \frac{e^{7t} + e^{-7t}}{2} , \qquad N_1(t) = \frac{e^{7t} - e^{-7t}}{14} .$$

Remark. These may be expressed in terms of hyperbolic functions as

$$N_0(t) = \cosh(7t) , \qquad N_1(t) = \tfrac{1}{7} \sinh(7t) .$$

(3) [4] Suppose that $Z_1(t)$, $Z_2(t)$, $Z_3(t)$, and $Z_4(t)$ solve the differential equation

$$z'''' + 5z''' + e^{3t} z'' + \sin(5t) z' + t^2 z = 0 ,$$

Suppose we know that $\mathrm{Wr}[Z_1, Z_2, Z_3, Z_4](3) = 4$. Find $\mathrm{Wr}[Z_1, Z_2, Z_3, Z_4](t)$.

Solution. The Abel Theorem says that $w(t) = \mathrm{Wr}[Z_1, Z_2, Z_3, Z_4](t)$ satisfies

$$w' + 5w = 0 .$$

We see that $w(t) = ce^{-5t}$ for some c. Because $w(t)$ satisfies the initial condition

$$w(3) = \mathrm{Wr}[Z_1, Z_2, Z_3, Z_4](3) = 4 ,$$

we have $w(0) = ce^{-5 \cdot 3} = 4$, whereby $c = 4e^{5 \cdot 3}$. Therefore $w(t) = 4e^{-5(t-3)}$, which shows that

$$\mathrm{Wr}[Z_1, Z_2, Z_3, Z_4](t) = 4e^{-5(t-3)} .$$

(4) [12] Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2 + i4$, $-2 + i4$, $-2 + i4$, $-2 - i4$, $-2 - i4$, $-2 - i4$, -3, -3, 0, 0, 0.

(a) [1] Give the order of L. (Give your reasoning!)
(b) [6] Give a real general solution of the homogeneous equation $\mathrm{L}u = 0$.
(c) [5] Write down the form for the particular solution needed to start the Undetermined Coefficients method for the equation $\mathrm{L}v = t^2 e^{-2t} \cos(4t)$.

Solution (a). Because 11 roots are listed, the degree of the characteristic polynomial must be 11, whereby the order of L is 11.

Solution (b). A fundamental set of eleven real-valued solutions is built as follows.

- ⋄ The conjugate pair of triple roots $-2 \pm i4$ contributes

$$e^{-2t} \cos(4t) , \qquad e^{-2t} \sin(4t) , \qquad t\, e^{-2t} \cos(4t) , \qquad t\, e^{-2t} \sin(4t) ,$$
$$t^2 e^{-2t} \cos(4t) , \qquad \text{and} \qquad t^2 e^{-2t} \sin(4t) .$$

- ⋄ The double real root -3 contributes

$$e^{-3t} \qquad \text{and} \qquad t\, e^{-3t} .$$

- ⋄ The triple real root 0 contributes

$$1 , \qquad t , \qquad \text{and} \qquad t^2 .$$

Therefore a real general solution of the homogeneous equation $\mathrm{L}u = 0$ is

$$\begin{aligned} u = {} & c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t) + c_3 t\, e^{-2t}\cos(4t) + c_4 t\, e^{-2t}\sin(4t) \\ & + c_5 t^2 e^{-2t}\cos(4t) + c_6 t^2 e^{-2t}\sin(4t) \\ & + c_7 e^{-3t} + c_8 t\, e^{-3t} + c_9 + c_{10} t + c_{11} t^2 \,. \end{aligned}$$

Solution (c). The forcing of the nonhomogeneous linear equation $\mathrm{L}v = t^2 e^{-2t}\cos(4t)$ has degree $d = 2$ and characteristic $\mu + i\nu = -2 + i4$. Because the characteristic $\mu + i\nu = -2 + i4$ is listed as a triple root of the characteristic polynomial, it has multiplicity $m = 3$. Therefore, we have

$$d = 2\,, \qquad \mu + i\nu = -2 + i4\,, \qquad m = 3\,.$$

Because $m + d = 5$, $m = 3$, and $\mu + i\nu = -2 + i4$, the form for the particular solution needed to start the Undetermined Coefficients method is

$$\begin{aligned} v_p = {} & \left(A_0 t^5 + A_1 t^4 + A_2 t^3\right) e^{-2t}\cos(4t) \\ & + \left(B_0 t^5 + B_1 t^4 + B_2 t^3\right) e^{-2t}\sin(4t)\,. \end{aligned}$$

(5) [8] Find a real general solution of the equation $y'''' + 12y'' + 36y = 36\cos(3t)$.

Solution. This is a *nonhomogeneous* linear equation with *constant coefficients*. Its linear differential operator is $\mathrm{L} = \mathrm{D}^4 + 12\mathrm{D}^2 + 36$. Its characteristic polynomial is

$$p(z) = z^4 + 12z^2 + 36 = (z^2 + 6)^2\,,$$

which has the conjugate pair of double roots $\pm i\sqrt{6}$. The forcing $36\cos(3t)$ has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = i3$, which has multiplicity $m = 0$. Therefore we can use either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients to find a particular solution. Each of these methods gives the real particular solution

$$y_P(t) = 4\cos(3t)\,.$$

Therefore a real general solution is

$$y(t) = c_1\cos\!\left(\sqrt{6}\,t\right) + c_2\sin\!\left(\sqrt{6}\,t\right) + c_3 t\,\cos\!\left(\sqrt{6}\,t\right) + c_4 t\,\sin\!\left(\sqrt{6}\,t\right) + 4\cos(3t)\,.$$

Key Identity Evaluations. Because $m = m + d = 0$ and $\mu + i\nu = i3$, we only need to evaluate the Key Identity at $z = i3$. The Key Identity is

$$\mathrm{L}(e^{zt}) = (z^4 + 12z^2 + 36)\cdot e^{zt}\,.$$

When this is evaluated at $z = i3$ we find that

$$\mathrm{L}(e^{i3t}) = \left((i3)^4 + 12\cdot(i3)^2 + 36\right)\cdot e^{i3t} = (81 - 12\cdot 9 + 36)e^{i3t} = 9e^{i3t}\,.$$

Because the forcing $36\cos(3t)$ has the phasor form $\mathrm{Re}(36e^{i3t})$, we multiply the above by 4 to obtain

$$\mathrm{L}(e^{i3t}) = 36e^{i3t}\,.$$

The real part of this equation shows that a particular solution is

$$y_P(t) = \mathrm{Re}(4e^{i3t}) = 4\cos(3t)\,.$$

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the phasor form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big)\,,$$

this formula gives the particular solution

$$y_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)}\,e^{i\nu t}\right)\,.$$

For this problem the forcing has the phasor form

$$f(t) = 36\cos(3t) = \mathrm{Re}\big(36e^{i3t}\big)\,,$$

with characteristic $\mu + i\nu = i3$ and phasor $\alpha - i\beta = 36$. Because the characteristic polynomial is $p(z) = z^4 + 12z^2 + 36$ and $m = 0$, we have

$$p^{(m)}(\mu + i\nu) = p(i3) = (i3)^4 + 12\cdot(i3)^2 + 36 = 81 - 12\cdot 9 + 36 = 9\,.$$

Therefore the particular solution becomes

$$y_P(t) = \mathrm{Re}\left(\frac{36}{9}\,e^{i3t}\right) = 4\cos(3t)\,.$$

Undetermined Coefficients. Because $m + d = m = 0$ and $\mu + i\nu = i3$, there is a particular solution in the form

$$y_P(t) = A\cos(3t) + B\sin(3t)\,.$$

Because

$$\begin{aligned} y_P'(t) &= -3A\sin(3t) + 3B\cos(3t)\,,\\ y_P''(t) &= -9A\cos(3t) - 9B\sin(3t)\,,\\ y_P'''(t) &= 27A\sin(3t) - 27B\cos(3t)\,,\\ y_P''''(t) &= 81A\cos(3t) + 81B\sin(3t)\,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}y_P(t) &= y_P''(t) + 12y_P'(t) + 36y_P(t)\\ &= \big[81A\cos(3t) + 81B\sin(3t)\big] + 12\big[-9A\cos(3t) - 9B\sin(3t)\big]\\ &\qquad\qquad + 36\big[A\cos(3t) + B\sin(3t)\big]\\ &= 9A\cos(3t) + 9B\sin(3t)\,. \end{aligned}$$

By setting $\mathrm{L}y_P(t) = 36\cos(3t)$, the linear independence of $\cos(3t)$ and $\sin(3t)$ implies that $9A = 36$ and $9B = 0$, whereby the particular solution becomes

$$y_P(t) = 4\cos(3t)\,.$$

(6) [8] What answer will be produced by the following Matlab commands?

```
>> syms x(t)
>> ode = diff(x,t,2) + 3*diff(x,t) − 10*x == 28*t*exp(2*t);
>> xSol(t) = dsolve(ode)
```

Solution. The commands ask MATLAB for a real general solution of the equation

$$\mathrm{D}^2x + 3\mathrm{D}x - 10x = 28t\,e^{2t}\,,\qquad \text{where}\quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

While your answer did not have to be given in MATLAB format, MATLAB will produce something equivalent to

2*t^2*exp(2*t) − (4/7)*t*exp(2*t) + C1*exp(−5*t) + C2*exp(2*t)

This can be seen as follows. This is a *nonhomogeneous* linear equation for $x(t)$ with *constant coefficients*. Its linear differential operator is $\mathrm{L} = \mathrm{D}^2 + 3\mathrm{D} - 10$. Its characteristic polynomial is

$$p(z) = z^2 + 3z - 10 = (z+5)(z-2)\,,$$

which has the two real roots -5 and 2. Therefore a real general solution of the associated homogeneous problem is

$$x_H(t) = c_1 e^{-5t} + c_2 e^{2t}\,.$$

The forcing $28t\,e^{2t}$ has degree $d = 1$, characteristic $\mu + i\nu = 2$, and multiplicity $m = 1$. A particular solution $x_P(t)$ can be found by using either Key Identity Evaluations or Undetermined Coefficients. Below we show that each of these methods yields the particular solution

$$x_P(t) = 2t^2 e^{2t} - \tfrac{4}{7} t\, e^{2t}\,.$$

Therefore a real general solution is

$$x = c_1 e^{-5t} + c_2 e^{2t} + 2t^2 e^{2t} - \tfrac{4}{7} t\, e^{2t}\,.$$

Up to notational differences, this is the answer that MATLAB produces.

Key Identity Evaluations. Because $m = 1$, $m + d = 2$, and $\mu + i\nu = 2$, we need to evaluate the first and second derivative of the Key Identity with respect to z at $z = 2$. The Key Identity and its first two derivatives with respect to z are

$$\begin{aligned}
\mathrm{L}\big(e^{zt}\big) &= (z^2 + 3z - 10) \cdot e^{zt}\,,\\
\mathrm{L}\big(t\, e^{zt}\big) &= (z^2 + 3z - 10) \cdot t\, e^{zt} + (2z + 3) e^{zt}\,,\\
\mathrm{L}\big(t^2 e^{zt}\big) &= (z^2 + 3z - 10) \cdot t^2 e^{zt} + 2(2z + 3) t\, e^{zt} + 2e^{zt}\,.
\end{aligned}$$

When the first and second derivatives are evaluated at $z = 2$ we find

$$\begin{aligned}
\mathrm{L}\big(t\, e^{2t}\big) &= (2 \cdot 2 + 3) e^{2t} = 7e^{2t}\,,\\
\mathrm{L}\big(t^2 e^{2t}\big) &= 2(2 \cdot 2 + 3) t\, e^{2t} + 2e^{2t} = 14t\, e^{2t} + 2e^{2t}\,.
\end{aligned}$$

Because the forcing is $28t\,e^{2t}$, we multiply the first equation by $\frac{2}{7}$ and subtract it from the second to obtain

$$\mathrm{L}\big(t^2 e^{2t} - \tfrac{2}{7} t\, e^{2t}\big) = 14t\, e^{2t}\,.$$

We then multiply this equation by 2 to get

$$\mathrm{L}\big(2t^2 e^{2t} - \tfrac{4}{7} t\, e^{2t}\big) = 28t\, e^{2t}\,.$$

We can read off from this that a particular solution is

$$x_P(t) = 2t^2 e^{2t} - \tfrac{4}{7} t\, e^{2t}\,.$$

Undetermined Coefficients. Because $m + d = 2$, $m = 1$, and $\mu + i\nu = 2$, there is a particular solution in the form

$$x_P(t) = \big(A_0 t^2 + A_1 t\big)\, e^{2t}\,.$$

Because

$$\begin{aligned} x_P'(t) &= 2\left(A_0t^2 + A_1t\right)e^{2t} + (2A_0t + A_1)\,e^{2t} \\ &= \left(2A_0t^2 + (2A_0 + 2A_1)t + A_1\right)e^{2t}\,, \\ x_P''(t) &= 2\left(2A_0t^2 + (2A_0 + 2A_1)t + A_1\right)e^{2t} + (4A_0t + 2A_0 + 2A_1)\,e^{2t} \\ &= \left(4A_0t^2 + (8A_0 + 4A_1)t + 2A_0 + 4A_1\right)e^{2t}\,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}x_P(t) &= x_P''(t) + 3x_P'(t) - 10x_P(t) \\ &= \left(4A_0t^2 + (8A_0 + 4A_1)t + 2A_0 + 4A_1\right)e^{2t} \\ &\quad + 3\left(2A_0t^2 + (2A_0 + 2A_1)t + A_1\right)e^{2t} - 10\left(A_0t^2 + A_1t\right)e^{2t} \\ &= 0A_0t^2e^{2t} + (14A_0 + 0A_1)t\,e^{2t} + (2A_0 + 7A_1)e^{2t}\,. \end{aligned}$$

Setting $\mathrm{L}x_P(t) = 28t\,e^{2t}$, the linear independence of $t\,e^{2t}$ and e^{2t} implies that

$$14A_0 = 28\,, \qquad 2A_0 + 7A_1 = 0\,.$$

This system has solution $A_0 = 2$, $A_1 = -\frac{7}{4}$, whereby the particular solution is

$$x_P(t) = 2t^2e^{2t} - \tfrac{4}{7}t\,e^{2t}\,.$$

(7) [8] Compute the Green function $g(t)$ associated with the differential operator

$$\mathrm{D}^2 + 6\mathrm{D} + 45\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. Because the linear differential operator has constant coefficients, its Green function $g(t)$ satisfies

$$\mathrm{D}^2g + 6\mathrm{D}g + 45g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The characteristic polynomial is

$$p(z) = z^2 + 6z + 45 = (z+3)^2 + 6^2\,,$$

which has the conjugate pair of simple roots $-3 + i6$. Hence, a general solution of the equation is

$$g(t) = c_1e^{-3t}\cos(6t) + c_2e^{-3t}\sin(6t)\,.$$

The first initial condition implies $0 = g(0) = c_1$, whereby

$$g(t) = c_2e^{-3t}\sin(6t)\,.$$

Because

$$g'(t) = 6c_2e^{-3t}\cos(6t) - 3c_2e^{-3t}\sin(6t)\,.$$

the second initial condition implies $1 = g'(0) = 6c_2$, whereby $c_2 = \frac{1}{6}$. Therefore the Green function associated with the differential operator is

$$g(t) = \tfrac{1}{6}e^{-3t}\sin(6t)\,.$$

(8) [8] Solve the initial-value problem

$$h'' + 6h' + 45h = \frac{72e^{-3t}}{\sin(6t)}\,, \qquad h(\tfrac{\pi}{4}) = h'(\tfrac{\pi}{4}) = 0\,.$$

Solution. This is a *nonhomogeneous* linear equation with *constant coefficients*. Because its forcing does *not have characteristic form*, we cannot use either Key Identity Evaluations or Undetermined Coefficients. Because this is an initial-value problem with *homogeneous initial conditions*, we will use the Green function method, which leads directly to the answer.

By the previous problem the Green function for this problem is $g(t) = \frac{1}{6}e^{-3t}\sin(6t)$. Because the equation is in normal form, the initial time is $\frac{\pi}{4}$, and both of the initial values are 0, the solution to this inital-value problem is given by the Green formula

$$\begin{aligned} h(t) = \int_{\frac{\pi}{4}}^{t} g(t-s)f(s)\,\mathrm{d}s &= \int_{\frac{\pi}{4}}^{t} \tfrac{1}{6}e^{-3(t-s)}\sin\big(6(t-s)\big)\,\frac{72e^{-3s}}{\sin(6s)}\,\mathrm{d}s \\ &= 12e^{-3t}\int_{\frac{\pi}{4}}^{t} \frac{\sin(6t-6s)}{\sin(6s)}\,\mathrm{d}s \\ &= 12e^{-3t}\int_{\frac{\pi}{4}}^{t} \frac{\sin(6t)\cos(6s)-\cos(6t)\sin(6s)}{\sin(6s)}\,\mathrm{d}s \\ &= 12e^{-3t}\sin(6t)\int_{\frac{\pi}{4}}^{t}\frac{\cos(6s)}{\sin(6s)}\,\mathrm{d}s - 12e^{-3t}\cos(6t)\int_{\frac{\pi}{4}}^{t}\mathrm{d}s\,. \end{aligned}$$

Because

$$\int_{\frac{\pi}{4}}^{t}\frac{\cos(6s)}{\sin(6s)}\,\mathrm{d}s = \log(|\sin(6s)|)\Big|_{\frac{\pi}{4}}^{t} = \log(|\sin(6t)|) - \log(|\sin(\tfrac{3\pi}{2})|) = \log(|\sin(6t)|)\,,$$

$$\int_{\frac{\pi}{4}}^{t}\mathrm{d}s = s\Big|_{\frac{\pi}{4}}^{t} = t - \tfrac{\pi}{4}\,,$$

we obtain

$$h(t) = 2e^{-3t}\sin(6t)\log(|\sin(6t)|) - 12e^{-3t}\cos(6t)\big(t-\tfrac{\pi}{4}\big)\,.$$

Remark. The fact that the interval of definition for this solution is $(\frac{\pi}{6},\frac{\pi}{3})$ can be read off directly from the initial-value problem beforehand. Because $\sin(6t) < 0$ over this interval, the absolute value inside the log is needed.

Remark. This problem can also be solved by the general Green function method, but that approach is less efficient because it does not use the Green function $g(t)$ that was the solution of the previous problem. The integrals end up being the same. The formula for the general Green function gives

$$G(t,s) = \frac{e^{-3t}\sin(6t)e^{-3s}\cos(6s) - e^{-3t}\cos(6t)e^{-3s}\sin(6s)}{6e^{-6s}}\,.$$

Remark. This problem can also be solved by variation of parameters, but that approach is less efficient because it does not directly solve the initial-value problem. Rather, it yields a general solution after which the parameters c_1 and c_2 in it must be determined to satisfy the initial conditions. The integrals end up being the same.

(9) [10] Consider the nonhomogeneous initial-value problem

$$t\,p'' - (1+2t)p' + 2p = \frac{24t^2}{1+2t}\,, \qquad p(2) = p'(2) = 0\,.$$

(a) [3] Show that $1+2t$ and e^{2t} are a fundamental set of solutions for the associated homogeneous equation.

(b) [7] Solve the nonhomogeneous initial-value problem.

Solution (a). The Wronskian of $1+2t$ and e^{2t} is

$$\begin{aligned}\mathrm{Wr}[1+2t, e^{2t}](t) = \det\begin{pmatrix} 1+2t & e^{2t} \\ 2 & 2e^{2t}\end{pmatrix} &= (1+2t)2e^{2t} - 2e^{2t} \\ &= 2e^{2t} + 4te^{2t} - 2e^{2t} = 4te^{2t}\,.\end{aligned}$$

Because $\mathrm{Wr}[1+2t, e^{2t}](t) \neq 0$ for $t > 0$, the functions $1+2t$ and e^{2t} are linearly independent.

Solution (b). The *nonhomogeneous* equation for p has *variable coefficients*, so we must use either the variation of parameters method or the general Green function method to solve it. Because this is an initial-value problem, the general Green function method should be favored. To apply either method we must first bring the initial-value problem into its normal form,

$$p'' - \frac{1+2t}{t}\,p' + \frac{2}{t}\,p = \frac{24t}{1+2t}\,, \qquad p(2) = p'(2) = 0\,.$$

We see from this that the interval of definition for its solution is $(0,\infty)$. Because $1+2t$ and e^{2t} are linearly independent, they constitute a fundamental set of solutions to the associated homogeneous equation.

General Green Function. The Green function $G(t,s)$ is given by

$$G(t,s) = \frac{1}{\mathrm{Wr}[1+2s, e^{2s}](s)}\det\begin{pmatrix} 1+2s & e^{2s} \\ 1+2t & e^{2t}\end{pmatrix} = \frac{e^{2t}(1+2s) - (1+2t)e^{2s}}{4se^{2s}}\,.$$

Because this initial-value problem has *homogeneous initial conditions*, its solution is given by the Green Formula with initial time 2. Specifically, the Green Formula gives

$$\begin{aligned}p(t) = \int_2^t G(t,s)\,f(s)\,\mathrm{d}s &= \int_2^t \frac{e^{2t}(1+2s) - (1+2t)e^{2s}}{4se^{2s}}\,\frac{24s}{1+2s}\,\mathrm{d}s \\ &= e^{2t}\int_2^t 6e^{-2s}\,\mathrm{d}s - (1+2t)\int_2^t \frac{6}{1+2s}\,\mathrm{d}s\,.\end{aligned}$$

Because

$$\int_2^t 6e^{-2s}\,\mathrm{d}s = -3e^{-2s}\Big|_2^t = 3e^{-4} - 3e^{-2t}\,,$$

$$\int_2^t \frac{6}{1+2s}\,\mathrm{d}s = 3\log(1+2s)\Big|_2^t = 3\log(1+2t) - 3\log(5) = 3\log\left(\frac{1+2t}{5}\right)\,,$$

the solution of the initial-value problem is

$$p = e^{2t}\big(3e^{-4} - 3e^{-2t}\big) - 3(1+2t)\log\left(\frac{1+2t}{5}\right)\,.$$

Variation of Parameters. Because $1+2t$ and e^{2t} constitute a fundamental set of solutions to the associated homogeneous equation, we seek a general solution of the nonhomogeneous equation in the form

$$y(t) = (1+2t)u_1(t) + e^{2t}u_2(t)\,,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$\begin{aligned}(1+2t)u_1'(t) + e^{2t}u_2'(t) &= 0\,,\\ 2u_1'(t) + 2e^{2t}u_2'(t) &= \frac{24t}{1+2t}\,.\end{aligned}$$

The solution of this system is

$$u_1'(t) = -\frac{6}{1+2t}\,, \qquad u_2'(t) = 6e^{-2t}\,.$$

Integrate these equations over $t > 0$ to obtain

$$\begin{aligned}u_1(t) &= \int \frac{-6}{1+2t}\mathrm{d}t = c_1 - 3\log(1+2t)\,,\\ u_2(t) &= \int 6e^{-2t}\,\mathrm{d}t = c_2 - 3e^{-2t}\,.\end{aligned}$$

Therefore a general solution of the nonhomogeneous equation over $t > 0$ is

$$\begin{aligned}p(t) &= (1+2t)u_1(t) + e^{2t}u_2(t)\\ &= (1+2t)\big(c_1 - 3\log(1+2t)\big) + e^{2t}\big(c_2 - 3e^{-2t}\big)\\ &= (1+2t)c_1 + e^{2t}c_2 - 3\big((1+2t)\log(1+2t) + 1\big)\,.\end{aligned}$$

Next, we determine c_1 and c_2 from the initial conditions. Because

$$p'(t) = 2c_i + 2e^{2t}c_2 - 3\big(2\log(1+2t) + 2\big)\,,$$

the initial conditions imply

$$\begin{aligned}0 = p(2) &= 5c_1 + e^4c_2 - 3\big(5\log(5) + 1\big)\,,\\ 0 = p'(2) &= 2c_i + 2e^4c_2 - 3\big(2\log(5) + 2\big)\,.\end{aligned}$$

Therefore c_1 and c_2 satisfy the linear algebraic system

$$5c_1 + e^4c_2 = 3\big(5\log(5) + 1\big)\,, \qquad 2c_i + 2e^4c_2 = 3\big(2\log(5) + 2\big)\,.$$

The solution of this system is

$$c_1 = 3\log(5)\,, \qquad c_2 = 3e^{-4}\,.$$

Therefore the solution of the inital-value problem is

$$p = 3e^{-4}e^{2t} - 3 - 3(1+2t)\log\left(\frac{1+2t}{5}\right)\,.$$

(10) [8] Give a real general solution of the equation

$$D^2v - 5Dv - 36v = 12\cos(2t) - 5\sin(2t)\,, \qquad \text{where} \quad D = \frac{d}{dt}\,.$$

Solution. This is a *nonhomogeneous* linear equation with *constant coefficients*. Its linear differential operator is $L = D^2 - 5D - 36$. Its characteristic polynomial is

$$p(z) = z^2 - 5z - 36 = (z+4)(z-9)\,,$$

which has the simple real roots -4 and 9. The forcing $12\cos(2t) - 5\sin(2t)$ has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = i2$, which has multiplicity $m = 0$. Therefore we can use either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients to find a particular solution. Each of these methods gives the real particular solution

$$v_P(t) = -\tfrac{53}{170}\cos(2t) + \tfrac{8}{170}\sin(2t)\,.$$

Therefore a real general solution is

$$v(t) = c_1e^{-4t} + c_2e^{9t} - \tfrac{53}{170}\cos(2t) + \tfrac{8}{170}\sin(2t)\,.$$

Key Identity Evaluations. Because $m = m + d = 0$ and $\mu + i\nu = i2$, we only need to evaluate the Key Identity at $z = i2$. The Key Identity is

$$L(e^{zt}) = (z^2 - 5z - 36)\cdot e^{zt}\,.$$

When this is evaluated at $z = i2$ we find that

$$L(e^{i2t}) = \big((i2)^2 - 5\cdot(i2) - 36\big)\cdot e^{i2t} = (-4 - i10 - 36)e^{i2t} = (-40 - i10)e^{i2t}\,.$$

Because the forcing has the phasor form

$$12\cos(2t) - 5\sin(2t) = \operatorname{Re}\big((12 + i5)e^{i2t}\big)\,,$$

we multiply the previous equation by $12 + i5$ and divide by $-40 - i10$ to obtain

$$L\left(\frac{12 + i5}{-40 - i10}\,e^{i2t}\right) = (12 + i5)e^{i2t}\,.$$

The real part of this equation gives the particular solution

$$\begin{aligned}
v_P(t) &= \operatorname{Re}\left(\frac{12 + i5}{-40 - i10}\,e^{i2t}\right)\\
&= -\tfrac{1}{10}\operatorname{Re}\left(\frac{12 + i5}{4 + i}\,e^{i2t}\right) = -\tfrac{1}{10}\operatorname{Re}\left(\frac{12 + i5}{4 + i}\,\frac{4 - i}{4 - i}\,e^{i2t}\right)\\
&= -\tfrac{1}{170}\operatorname{Re}\big((12 + i5)\,(4 - i)\,e^{i2t}\big) = -\tfrac{1}{170}\operatorname{Re}\big((53 + i8)\,e^{i2t}\big)\\
&= -\tfrac{1}{170}\operatorname{Re}\big((53 + i8)\,\big(\cos(2t) + i\sin(2t)\big)\big) = -\tfrac{53}{170}\cos(2t) + \tfrac{8}{170}\sin(2t)\,.
\end{aligned}$$

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the phasor form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\Big((\alpha - i\beta)e^{i\nu t}\Big)\,,$$

this formula gives the particular solution

$$v_P(t) = t^m e^{\mu t}\,\mathrm{Re}\bigg(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)}\,e^{i\nu t}\bigg)\,.$$

For this problem the forcing has the phasor form

$$f(t) = 12\cos(2t) - 5\sin(2t) = \mathrm{Re}\Big((12 + i5)e^{i2t}\Big)\,,$$

with characteristic $\mu + i\nu = i2$ and phasor $\alpha - i\beta = 12 + i5$. Because the characteristic polynomial is $p(z) = z^2 - 5z - 36$ and $m = 0$, we have

$$p^{(m)}(\mu + i\nu) = p(i2) = (i2)^2 - 5\cdot(i2) - 36 = -4 - i10 - 36 = -40 - i10\,.$$

Therefore the particular solution becomes

$$\begin{aligned}
v_P(t) &= \mathrm{Re}\bigg(\frac{12 + i5}{-40 - i10}\,e^{i2t}\bigg)\\
&= -\tfrac{1}{10}\,\mathrm{Re}\bigg(\frac{12 + i5}{4 + i}\,e^{i2t}\bigg) = -\tfrac{1}{10}\,\mathrm{Re}\bigg(\frac{12 + i5}{4 + i}\,\frac{4 - i}{4 - i}\,e^{i2t}\bigg)\\
&= -\tfrac{1}{170}\,\mathrm{Re}\Big((12 + i5)\,(4 - i)\,e^{i2t}\Big) = -\tfrac{1}{170}\,\mathrm{Re}\Big((53 + i8)\,e^{i2t}\Big)\\
&= -\tfrac{1}{170}\,\mathrm{Re}\Big((53 + i8)\,\big(\cos(2t) + i\sin(2t)\big)\Big) = -\tfrac{53}{170}\cos(2t) + \tfrac{8}{170}\sin(2t)\,.
\end{aligned}$$

Undetermined Coefficients. Because $m + d = m = 0$ and $\mu + i\nu = i2$, there is a particular solution in the form

$$v_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$\begin{aligned}
v_P'(t) &= -2A\sin(2t) + 2B\cos(2t)\,,\\
v_P''(t) &= -4A\cos(2t) - 4B\sin(2t)\,,
\end{aligned}$$

we see that

$$\begin{aligned}
\mathrm{L}v_P(t) &= v_P''(t) - 5v_P'(t) - 36v_P(t)\\
&= \big[-4A\cos(2t) - 4B\sin(2t)\big] - 5\big[-2A\sin(2t) + 2B\cos(2t)\big]\\
&\qquad\qquad\qquad\qquad - 36\big[A\cos(2t) + B\sin(2t)\big]\\
&= (-40A - 10B)\cos(2t) + (10A - 40B)\sin(2t)\,.
\end{aligned}$$

By setting $\mathrm{L}v_P(t) = 12\cos(2t) - 5\sin(2t)$, the linear independence of $\cos(2t)$ and $\sin(2t)$ implies that A and B solve the linear algebric system

$$-40A - 10B = 12\,, \qquad 10A - 50B = -5\,.$$

This implies that $A = -\frac{53}{170}$ and $B = \frac{8}{170}$, whereby the particular solution becomes

$$v_P(t) = -\tfrac{53}{170}\cos(2t) + \tfrac{8}{170}\sin(2t)\,.$$

(11) [8] The vertical displacement of a spring-mass system is governed by the equation

$$\ddot{h} + 40\dot{h} + 481h = a\cos(\omega t - \phi)\,,$$

where $a > 0$, $\omega > 0$, and $0 \leq \phi < 2\pi$. Assume CGS units.

(a) [2] Give the natural frequency and period of the system.

(b) [2] Show the system is under damped and give its damping rate.

(c) [4] Give the steady state solution in its phasor form $\mathrm{Re}(\Gamma\, e^{i\omega t})$.

Solution (a). The natural frequency is

$$\omega_o = \sqrt{481} \quad 1/\mathrm{sec}\,.$$

The natural period is then

$$T_o = \frac{2\pi}{\omega_o} = \frac{2\pi}{\sqrt{481}} \quad \mathrm{sec}\,.$$

Solution (b). The characteristic polynomial of the equation is

$$\begin{aligned} p(z) = z^2 + 40z + 481 &= (z+20)^2 + 481 - 400 \\ &= (z+20)^2 + 81 = (z+20)^2 + 9^2\,. \end{aligned}$$

This has the conjugate pair of roots $-20 \pm i9$. Therefore the system is *under damped*. The damping rate is $\eta = 20 \quad 1/\mathrm{sec}$, which is minus the real part of these roots.

Alternative Solution (b). The system is *under damped* because the damping rate $\eta = 40/2 = 20 \quad 1/\mathrm{sec}$ is less than the natural frequency $\omega_o = \sqrt{481} \quad 1/\mathrm{sec}$.

Remark. The damped frequency is $\omega_\eta = 9 \quad 1/\mathrm{sec}$, which is the imaginary part of the roots of the characteristic polynomial. Alternatively, it is

$$\omega_\eta = \sqrt{\omega_o^2 - \eta^2} = \sqrt{481 - 400} = \sqrt{81} = 9 \quad 1/\mathrm{sec}\,.$$

The damped period T_η is then

$$T_\eta = \frac{2\pi}{\omega_\eta} = \frac{2\pi}{\sqrt{81}} = \frac{2\pi}{9} \quad \mathrm{sec}\,.$$

Solution (c). The forcing $f(t) = a\cos(\omega t - \phi)$ has the phasor form

$$f(t) = \mathrm{Re}\big(\gamma\, e^{i\omega t}\big)\,, \qquad \text{where the phasor is} \quad \gamma = ae^{-i\phi}\,.$$

Therefore the steady state solution has the phasor form

$$h_P(t) = \mathrm{Re}\big(\Gamma\, e^{i\omega t}\big)\,, \qquad \text{where the phasor is} \quad \Gamma = \frac{\gamma}{p(i\omega)}\,.$$

Because $\gamma = ae^{-i\phi}$ and $p(z) = z^2 + 40z + 481$, the phasor Γ is

$$\Gamma = \frac{ae^{-i\phi}}{481 - \omega^2 + i40\omega}\,.$$

We are not asked to give the solution in either its Cartesian or its polar phasor form, so we can stop here.

(12) [10] When a 10 gram mass is hung vertically from a spring, at rest it stretches the spring 20 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) A damper imparts a damping force of 560 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. Assume that the spring force is proportional to displacement, that the damping force is proportional to velocity, and that there are no other forces. At $t = 0$ the mass is displaced 3 cm below its rest position and is released with an upward velocity of 2 cm/sec.

(a) [6] Give an initial-value problem that governs the displacement $h(t)$ for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

(b) [2] Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

(c) [2] Find the damped frequency and damped period of the system. (Give your reasoning!)

Solution (a). Let $h(t)$ be the displacement in centimeters at time t in seconds of the mass from its rest position, with upward displacements being positive. Because there is no external forcing, the governing initial-value problem has the form

$$m\ddot{h} + c\dot{h} + kh = 0\,, \qquad h(0) = -3\,, \quad \dot{h}(0) = 2\,,$$

where m is the mass, c is the damping coefficient, and k is the spring constant. The problem says that $m = 10$ grams. The damping coefficient c is found by equating the damping force imparted by the damper when the speed of the mass is 4 cm/sec, which is $c\,4$ dynes, with the force of 560 dynes. This gives $c\,4 = 560$, or

$$c = \frac{560}{4} = 140 \quad \text{dynes sec/cm}\,.$$

The spring constant k is found by equating the force of the spring when it is stetched 20 cm, which is $k\,20$ dynes, with the weight of the mass, which is $mg = 10 \cdot 980$ dynes. This gives $k\,20 = 10 \cdot 980$, or

$$k = \frac{10 \cdot 980}{20} = 490 \quad \text{dynes/cm}\,.$$

Therefore the governing initial-value problem is

$$10\ddot{h} + 140\,\dot{h} + 490h = 0\,, \qquad h(0) = -3\,, \quad \dot{h}(0) = 2\,.$$

Remark. With the equation in normal form the answer is

$$\ddot{h} + 14\,\dot{h} + 49h = 0\,, \qquad h(0) = -3\,, \quad \dot{h}(0) = 2\,.$$

Remark. If we had chosen downward displacements to be positive then the governing initial-value problem would be the same except for the initial conditions, which would be $h(0) = 3$ and $\dot{h}(0) = -2$.

Solution (b). The damping rate is $\eta = 14/2 = 7$. Because $\eta^2 = 49 = \omega_o^2$, the system is *critically damped*.

Alternative Solution (b). The characteristic polynomial is

$$p(z) = z^2 + 14z + 49 = (z+7)^2\,.$$

This polynomial has the double negative root -7, so the system is *critically damped*.

Solution (c). A critically damped system has no damped frequency or period.