MATH 416 Homework 12
Due Friday, 15 May 2020

Haar wavelets were the only example given in class of a multiresolution analysis for which the mother scaling and wavelet functions were expressed by simple formulas. Here we develop another such example, the Shannon wavelets, which have the scaling function $\phi(t) = \operatorname{sinc}(t)$. These wavelets have no practical application as far as I know because they decay very slowly. However, they do illustrate multiresolution analysis.

1. Show that $\phi(t) = \operatorname{sinc}(t)$ is a scaling function. This means:
   a. show that $\phi \in L^2(\mathbb{R})$;
   b. show that $\{\phi_k : k \in \mathbb{Z}\}$ is an orthonormal set, where $\phi_k \in L^2(\mathbb{R})$ is defiined by
   $$\phi_k(t) = \phi(t-k) \quad \text{for every } t \in \mathbb{R}\,.$$

2. For every $j, k \in \mathbb{Z}$ define $\phi_{jk} \in L^2(\mathbb{R})$ by
   $$\phi_{jk}(t) = 2^{\frac{j}{2}}\phi(2^j t - k) \quad \text{for every } t \in \mathbb{R}\,.$$
   For every $j \in \mathbb{Z}$ define the scaling subspace V_j by $V_j = \overline{\operatorname{span}}\{\phi_{jk} : k \in \mathbb{Z}\}$.
   a. Show for every $j \in \mathbb{Z}$ that $\{\phi_{jk} : k \in \mathbb{Z}\}$ is an orthonormal set.
   b. Show for every $j \in \mathbb{Z}$ that
   $$u \in V_j \quad \Longleftrightarrow \quad \operatorname{supp}(\hat{u}) \subset [-2^{j-1}, 2^{j-1}]\,,$$
   where $\hat{u} = \mathcal{F}u$ is the Fourier transform of u.

   Remark. Here $\operatorname{supp}(\hat{u})$ denotes the support of $\hat{u}$, which is $\{\xi \in \mathbb{R} : \hat{u}(\xi) \neq 0\}$.

3. Find $\{h_\ell\}_{\ell\in\mathbb{Z}}$ such that ϕ satisfies a two-scale formula in the form
   $$\phi(t) = \sum_{\ell\in\mathbb{Z}} h_\ell \sqrt{2}\phi(2t-\ell)\,.$$
   Hint: Use the Shannon Sampling Theorem. The series will not be finite.

4. The associated wavelet function is given by
   $$\psi(t) = \sum_{\ell\in\mathbb{Z}} g_\ell \sqrt{2}\phi(2t-\ell)\,,$$
   where $g_\ell = (-1)^\ell h_{1-\ell}$ for every $\ell \in \mathbb{Z}$. Find $\psi(t)$.

5. For every $j, k \in \mathbb{Z}$ define $\psi_{jk} \in L^2(\mathbb{R})$ by
   $$\psi_{jk}(t) = 2^{\frac{j}{2}}\psi(2^j t - k) \quad \text{for every } t \in \mathbb{R}\,.$$
   For every $j \in \mathbb{Z}$ define the wavelet subspace W_j by $W_j = \overline{\operatorname{span}}\{\psi_{jk} : k \in \mathbb{Z}\}$.
   a. Show for every $j \in \mathbb{Z}$ that $\{\psi_{jk} : k \in \mathbb{Z}\}$ is an orthonormal set.
   b. For every $j \in \mathbb{Z}$ find the closed $S_j \subset \mathbb{R}$ such that
   $$v \in W_j \quad \Longleftrightarrow \quad \operatorname{supp}(\hat{v}) \subset S_j\,,$$
   where $\hat{v} = \mathcal{F}v$ is the Fourier transform of v.