MATH 416 Homework 11
Due Friday, 15 May 2020

Let $\mathcal{F} : L^2 \to L^2(\mathbb{R})$ denote the Fourier transform given by

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}} e^{-i2\pi\xi t} u(t)\,\mathrm{d}t \qquad \text{for every } u \in L^2(\mathbb{R})\,.$$

If $u \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ then $\mathcal{F}u \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$.

1. Let $u \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\mathcal{F}u$ is nonzero.
   a. Suppose there exists $\beta \in \mathbb{R}$ such that
   $$\mathcal{F}u(\xi) = e^{-i2\alpha} e^{-i4\pi\beta\xi} \mathcal{F}u(-\xi) \qquad \text{for every } \xi \in \mathbb{R}\,.$$
   Show that either $e^{-i2\alpha} = 1$ and $u(\beta + t) = u(\beta - t)$, or $e^{-i2\alpha} = -1$ and $u(\beta + t) = -u(\beta - t)$. (This says that u must have either even or odd symmetry about β.)
   b. Conversely, suppose that $u(\beta + t) = \pm u(\beta - t)$ for some $\beta \in \mathbb{R}$. Show that
   $$\mathcal{F}u(\xi) = \pm e^{-i4\pi\beta\xi} \mathcal{F}u(-\xi) \qquad \text{for every } \xi \in \mathbb{R}\,.$$

2. Let $u \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\mathcal{F}u$ is nonzero.
   a. Suppose there exists $\alpha, \beta \in \mathbb{R}$ such that
   $$\mathcal{F}u(\xi) = e^{-i2\alpha} e^{-i4\pi\beta\xi} \overline{\mathcal{F}u(-\xi)} \qquad \text{for every } \xi \in \mathbb{R}\,.$$
   Show that $\beta = 0$ and $e^{i\alpha}u$ is real-valued.
   b. Conversely, suppose that $e^{i\alpha}u$ is real-valued for some $\alpha \in \mathbb{R}$. Show that
   $$\mathcal{F}u(\xi) = e^{-i2\alpha} \overline{\mathcal{F}u(-\xi)} \qquad \text{for every } \xi \in \mathbb{R}\,.$$

3. Let $\hat{\phi}$ be given by
   $$\hat{\phi}(\xi) = \begin{cases} e^{-\left(\log(\xi)\right)^2} & \text{for } \xi > 0\,, \\ 0 & \text{for } \xi \leq 0\,. \end{cases}$$
   a. Graph $\hat{\phi}(\xi)$ over $[0, 20]$.
   b. What does $\phi(t) = \mathcal{F}^{-1}\hat{\phi}(t)$ look like?

4. The Haar scaling function $\phi : \mathbb{R} \to \mathbb{R}$ is
   $$\phi(t) = \begin{cases} 1 & \text{for } t \in [0, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$
   (a) Show that $\{\phi_k : k \in \mathbb{Z}\}$ is an orthonormal set in $L^2(\mathbb{R})$, where $\phi_k(t) = \phi(t - k)$.
   (b) Show that ϕ satisfies the two-scale relation
   $$\phi(t) = \phi(2t) + \phi(2t - 1)\,.$$
   (c) For every $j, k \in \mathbb{Z}$ let $\phi_{jk}(t) = 2^{\frac{j}{2}} \phi(2^j t - k)$ for every $t \in \mathbb{R}$. For every $j \in \mathbb{Z}$ define
   $$V_j = \overline{\operatorname{span}}\{\phi_{jk} : k \in \mathbb{Z}\}\,.$$
   Show for every $j \in \mathbb{Z}$ that $V_{j-1} \subset V_j$.

5. The Haar wavelet function $\psi : \mathbb{R} \to \mathbb{R}$ is

$$\psi(t) = \begin{cases} 1 & \text{for } t \in [0, \frac{1}{2})\,, \\ -1 & \text{for } t \in [\frac{1}{2}, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

(a) Show that $\{\psi_k \,:\, k \in \mathbb{Z}\}$ is an orthonormal set in $L^2(\mathbb{R})$, where $\psi_k(t) = \psi(t-k)$.

(b) Show that ψ is given by

$$\psi(t) = \phi(2t) - \phi(2t-1)\,.$$

(c) Define the subspaces V_0 and W_0 by

$$V_0 = \overline{\text{span}}\{\phi_k \,:\, k \in \mathbb{Z}\}\,, \qquad W_0 = \overline{\text{span}}\{\psi_k \,:\, k \in \mathbb{Z}\}\,.$$

Show that the subspaces V_0 and W_0 are orthogonal.

(d) Let V_1 be as in the previous problem. Show that

$$V_1 = V_0 + W_0 = \{v + w \,:\, v \in V_0\,,\, w \in W_0\}\,.$$