MATH 416 Homework 10
Due Wednesday, 13 May 2020

The Haar wavelet function $w : \mathbb{R} \to \mathbb{R}$ is

$$w(t) = \begin{cases} 1 & \text{for } t \in (0, \frac{1}{2})\,, \\ -1 & \text{for } t \in (\frac{1}{2}, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

It has primitive $W : \mathbb{R} \to \mathbb{R}$ given by

$$W(t) = \begin{cases} \frac{1}{2} - |t - \frac{1}{2}| & \text{for } t \in (0, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For each $j, k \in \mathbb{Z}$ define $w_{jk} : \mathbb{R} \to \mathbb{R}$ by

$$w_{jk}(t) = 2^{\frac{j}{2}} w(2^j t - k)\,.$$

1. Consider the set

$$S = \left\{ w_{jk} \,:\, j \in \{0, 1, \cdots\}\,,\, k \in \{0, 1, \cdots, 2^j - 1\} \right\}.$$

   (a) Show that S is an orthonormal set in $L^2([0, 1])$.
   (b) Show that every constant function is orthogonal to $\operatorname{span}\{S\}$.

2. Let $b \in (0, 1)$ and set $u(t) = \chi_{[0,b)}(t)$ where

$$\chi_{[0,b)}(t) = \begin{cases} 1 & \text{if } t \in [0, b)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

   Show that

$$u(t) = b + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j - 1} \langle w_{jk}\,,\, u \rangle w_{jk}(t)\,.$$

3. Let $\mathbb{A}$ be the parametric representation of the affine group. Then $\mathbb{A} = \mathbb{R}_+ \times \mathbb{R}$ equipped with the binary operation $\circ$ defined by

$$(a, b) \circ (c, d) = (ac, ad + b) \quad \text{for every } (a, b), (c, d) \in \mathbb{A}\,.$$

   For every $(a, b) \in \mathbb{A}$ define $\sigma_{(a,b)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\sigma_{(a,b)} u(t) = \frac{1}{\sqrt{a}}\, u\!\left(\frac{t - b}{a} \right) \quad \text{for every } u \in L^2(\mathbb{R})\,.$$

   (a) Show that $\sigma_{(a,b)} \sigma_{(c,d)} = \sigma_{(a,b)\circ(c,d)}$ for every $(a, b), (c, d) \in \mathbb{A}$.
   (b) Compute $\sigma^*_{(a,b)}$ for every $(a, b) \in \mathbb{A}$.
   (c) Show that $\sigma_{(a,b)}$ is unitary for every $(a, b) \in \mathbb{A}$.

4. Let $w : \mathbb{R} \to \mathbb{R}$ be the Haar wavelet function. For every $(a, b) \in \mathbb{A}$ define
$$w_{ab}(t) = \sigma_{(a,b)} w(t)\,.$$
The associated continuous wavelet transform $\mathcal{W} : L^2(\mathbb{R}) \to L^2(\mathbb{A})$ is defined by
$$\mathcal{W}u(a, b) = \langle w_{ab}\,,\, u\rangle \quad \text{for every } u \in L^2(\mathbb{R})\,.$$
Compute $\mathcal{W}u$ for $u(t) = 1/(1 + t^2)$.

5. Let $w : \mathbb{R} \to \mathbb{R}$ be the Haar wavelet function. Let $\mathcal{F} : L^2 \to L^2(\mathbb{R})$ denote the Fourier transform given by
$$\mathcal{F}u(\xi) = \int_{\mathbb{R}} e^{-i2\pi\xi t} u(t)\,\mathrm{d}t \quad \text{for every } u \in L^2(\mathbb{R})\,.$$
   (a) Compute $\mathcal{F}w(\xi)$.
   (b) Verify that
$$\int_0^\infty \frac{|\mathcal{F}w(\xi)|^2}{\xi}\,\mathrm{d}\xi < \infty\,.$$