MATH 416 Homework 9
Due Wednesday, 13 May 2020

The following problems consider the "hat" and "sinc" functions defined over $\mathbb{R}$ by

$$\mathrm{hat}(t) = \begin{cases} 1 - |t| & \text{for } t \in (-1,1)\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad \mathrm{sinc}(t) = \begin{cases} \dfrac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0\,, \\ 1 & \text{if } t = 0\,. \end{cases}$$

Each of these functions satisfies the *interpolation condition*,

$$\phi(0) = 1\,, \qquad \phi(k) = 0 \quad \text{for every } k \in \mathbb{Z} - \{0\}\,.$$

For any $\phi : \mathbb{R} \to \mathbb{R}$ that satisfies this condition define $\phi_k : \mathbb{R} \to \mathbb{R}$ by $\phi_k(t) = \phi(t-k)$. For every real sequence $\{c_k\}_{k\in\mathbb{Z}}$ over $\mathbb{Z}$ and every $m, n \in \mathbb{Z}$ with $m \leq n$ define $u_{mn} : \mathbb{R} \to \mathbb{R}$ by

$$u_{mn}(t) = \sum_{k=m}^{n} c_k\, \phi_k(t)\,. \tag{1}$$

We say that $\{u_{mn} : m, n \in \mathbb{Z}\,,\, m \leq n\}$ has the *Cauchy property* with respect to a norm $\|\cdot\|$ if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that every $m, n \in \mathbb{Z}$ with $m \leq n$ we have

$$n \leq -N_\epsilon \quad \text{or} \quad N_\epsilon \leq m \qquad \Longrightarrow \qquad \|u_{mn}\| < \epsilon\,.$$

1. Let $\phi(t) = \mathrm{hat}(t)$. Let $\{c_k\}_{k\in\mathbb{Z}}$ be a real sequence over $\mathbb{Z}$. Define u_{mn} by (1). Show that $\{u_{mn}\}$ has the Cauchy property with respect to the $L^\infty(\mathbb{R})$ norm if and only if
$$\lim_{k\to-\infty} |c_k| = 0 \quad \text{and} \quad \lim_{k\to+\infty} |c_k| = 0\,.$$

2. Let $\phi(t) = \mathrm{hat}(t)$. Let $\{c_k\}_{k\in\mathbb{Z}}$ be a real sequence over $\mathbb{Z}$. Define u_{mn} by (1). Show that $\{u_{mn}\}$ has the Cauchy property with respect to the $L^2(\mathbb{R})$ norm if and only if
$$\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty\,.$$

3. Let $\phi(t) = \mathrm{hat}(t)$. Let $\{c_k\}_{k\in\mathbb{Z}}$ be a real sequence over $\mathbb{Z}$. Define u_{mn} by (1). Show that $\{u_{mn}\}$ has the Cauchy property with respect to the $L^1(\mathbb{R})$ norm if and only if
$$\sum_{k\in\mathbb{Z}} |c_k| < \infty\,.$$

4. Let $\phi(t) = \mathrm{sinc}(t)$. Let $\{c_k\}_{k\in\mathbb{Z}}$ be a real sequence over $\mathbb{Z}$. Define u_{mn} by (1). Show that $\{u_{mn}\}$ has the Cauchy property with respect to the $L^2(\mathbb{R})$ norm if and only if
$$\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty\,.$$

5. Let $\hat{u}(\xi) = \mathrm{hat}(\xi)$.
   a. Compute $u(t) = \mathcal{F}^{-1}\hat{u}(t)$.
   b. Find the sequence $\{c_k\}_{k\in\mathbb{Z}}$ such that
$$u(t) = \sum_{k\in\mathbb{Z}} c_k\, \mathrm{sinc}(2t - k)\,.$$