MATH 416 Homework 7
Due Wednesday, 1 April 2020

Let $N \in \mathbb{Z}_+$. For each $n \in \mathbb{Z}_+$ define the windowing function $w_n : \mathbb{Z} \to [0, \infty)$ by

$$w_n(j) = \begin{cases} \dfrac{1 + p_n\big(\cos(\frac{\pi}{N}\,j)\big)}{2} & \text{for } j \in [-N, N]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $p_n(z)$ is the unique odd polynomial determined by

$$p_n'(z) = c_n(1 - z^2)^{n-1}\,, \qquad p(0) = 0\,, \qquad p(1) = 1\,.$$

These $p_n(z)$ are the same ones we used in the continuous case.

1. Give explicit expressions for $w_1(j)$, $w_2(j)$, $w_3(j)$, and $w_4(j)$. Show that each of these four functions satifies the replication condition

$$\sum_{k \in \mathbb{Z}} w_n(j + kN) = 1\,.$$

Given any windowing function $w : \mathbb{Z} \to [0, \infty)$ that satisfies the replication condition

$$\sum_{k \in \mathbb{Z}} w(j + kN) = 1\,,$$

the N-periodization of the localization wf of any $f : \mathbb{Z} \to \mathbb{C}$ is given by

$$f_N(j) = \sum_{k \in \mathbb{Z}} w(j + kN)\, f(j + kN)\,.$$

In class we showed that the k^{th} Fourier coefficient of this periodization is

$$\hat{f}_N(k) = \frac{1}{N} \sum_{j \in \mathbb{Z}_N} \overline{e_k(j)}\, f_N(j) = \frac{1}{N} \sum_{j=-N}^{N} \overline{e_k(j)}\, w(j)\, f(j)\,,$$

where $e_k(j) = \omega_N^{kj}$ for every $k \in \mathbb{Z}_N$.

2. Compute $\hat{f}_N(k)$ for every $k \in \mathbb{Z}_N$ when $f(j) = \exp(i\omega j)$ for some $\omega \in \mathbb{R}$ and $w(j) = w_1(j)$. Use the Euler identity $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and the formula for the sum of a finite geometric series.

3. Compute $\hat{f}_N(k)$ for every $k \in \mathbb{Z}_N$ when $f(t) = \exp(i\omega j)$ for some $\omega \in \mathbb{R}$ and $w(j) = w_2(j)$. The trig identity $(\cos(\theta))^3 = \frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos(\theta)$ can be helpful.

Remark. If $\omega = m\frac{2\pi}{N}$ for some $m \in \mathbb{Z}$ then $f(j) = \exp(i\omega j)$ is N-periodic and, because $w(j)$ satisfies the replication condition, we have $f_N(j) = f(j) = e_m(j)$. In that case the orthonomality of the Fourier basis $\{e_k(j)\}$ implies that $\hat{f}_N(k) = \delta_{km}$. Therefore the interesting case is when the frequency ω is not a multiple of $\frac{2\pi}{N}$. In that case the periodization of $f(j) = \exp(i\omega j)$ will be distributed amongst the Fourier basis. This is the phenomenon of *aliasing*. We will investigate it in the next two problems.

4. Let $\hat{f}_N(k)$ for every $k \in \mathbb{Z}_N$ when $f(j) = \exp(i\omega j)$ for some $\omega \in \mathbb{R}$ and $w(j) = w_1(j)$. Set $N = 100$ and plot $\hat{f}_N(k)$ versus k for $\omega = \frac{\pi}{100}, \frac{\pi}{200}, \frac{\pi}{400}$, and $\frac{\pi}{800}$. What do you see?

5. Let $\hat{f}_N(k)$ for every $k \in \mathbb{Z}_N$ when $f(j) = \exp(i\omega j)$ for some $\omega \in \mathbb{R}$ and $w(j) = w_2(j)$. Set $N = 100$ and plot $\hat{f}_N(k)$ versus k for $\omega = \frac{\pi}{100}, \frac{\pi}{200}, \frac{\pi}{400}$, and $\frac{\pi}{800}$. What do you see? How does this compare with your results for the previous problem?