MATH 416 Homework 6
Due Wednesday, 1 April 2020

For every $N \in \mathbb{Z}_+$ with $N \geq 2$ let $\mathbb{Z}_N$ denote the integers mod N and let $\omega_N = \exp(i2\pi/N)$. The discrete Fourier basis for $\mathbb{C}^N$ is then $\{e_k\}_{k\in\mathbb{Z}_N}$ where for every $k \in \mathbb{Z}_N$ we define e_k by

$$e_k(j) = \omega_N^{kj} \qquad \text{for every } j \in \mathbb{Z}_N \,,$$

so that we can view each e_k as the column vector

$$e_k = \begin{pmatrix} 1 \\ \omega_N^k \\ \omega_N^{2k} \\ \vdots \\ \omega_N^{(N-1)k} \end{pmatrix} .$$

The associated Fourier matrix F_N is then

$$\begin{aligned} F_N &= \begin{pmatrix} e_0 & e_1 & e_2 \cdots & e_{N-1} \end{pmatrix} = \begin{pmatrix} \omega_N^{kj} \end{pmatrix}_{k,j\in\mathbb{Z}_N} \\ &= \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2N-2} \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2N-2} & \cdots & \omega_N^{(N-1)^2} \end{pmatrix} . \end{aligned}$$

1. Find ω_N and write out F_N for $N = 4$, $N = 6$, and $N = 8$.

2. Let $L^2(\mathbb{Z}_N)$ denote $\mathbb{C}^N$ equipped with the inner product

$$\langle u \,,\, v\rangle = \frac{1}{N} \sum_{j\in\mathbb{Z}_N} \overline{u(j)}v(j) \,.$$

   Show that $\{e_k\}_{k\in\mathbb{Z}_N}$ is an orthonormal basis of $L^2(\mathbb{Z}_N)$.

3. Show that every $v \in L^2(\mathbb{Z}_N)$ can be expressed as

$$v = \sum_{k\in\mathbb{Z}_N} \hat{v}(k)\, e_k \,, \quad \text{where} \quad \hat{v}(k) = \langle e_k \,,\, v\rangle \,.$$

4. Compute $\det(F_N)$ for $N = 4$, $N = 6$, and $N = 8$. (Hint: Vandermonde)

5. Complete the calulation that we started in lecture by using the Fast Fourier Transform (FFT) algorithm to calculate $\hat{v}(k)$ for $N = 8$. Recall that this calculation started as

$$\begin{pmatrix} v(0)\\ v(1)\\ v(2)\\ v(3)\\ v(4)\\ v(5)\\ v(6)\\ v(7) \end{pmatrix} \mapsto \begin{matrix} \begin{pmatrix} v(0)\\ v(2)\\ v(4)\\ v(6) \end{pmatrix} \\ \\ \begin{pmatrix} v(1)\\ v(3)\\ v(5)\\ v(7) \end{pmatrix} \end{matrix} \mapsto \begin{matrix} \begin{pmatrix} v(0)\\ v(4) \end{pmatrix} \\ \begin{pmatrix} v(2)\\ v(6) \end{pmatrix} \\ \begin{pmatrix} v(1)\\ v(5) \end{pmatrix} \\ \begin{pmatrix} v(3)\\ v(7) \end{pmatrix} \end{matrix} \mapsto \begin{matrix} \begin{pmatrix} v(0)+v(4)\\ v(0)-v(4) \end{pmatrix} \\ \begin{pmatrix} v(2)+v(6)\\ v(2)-v(6) \end{pmatrix} \\ \begin{pmatrix} v(1)+v(5)\\ v(1)-v(5) \end{pmatrix} \\ \begin{pmatrix} v(3)+v(7)\\ v(3)-v(7) \end{pmatrix} \end{matrix} \mapsto \begin{matrix} \begin{pmatrix} \big(v(0)+v(4)\big)+\big(v(2)+v(6)\big)\\ \big(v(0)-v(4)\big)-i\big(v(2)-v(6)\big)\\ \big(v(0)+v(4)\big)-\big(v(2)+v(6)\big)\\ \big(v(0)-v(4)\big)+i\big(v(2)-v(6)\big) \end{pmatrix} \\ \\ \begin{pmatrix} \big(v(1)+v(5)\big)+\big(v(3)+v(7)\big)\\ \big(v(1)-v(5)\big)-i\big(v(3)-v(7)\big)\\ \big(v(1)+v(5)\big)-\big(v(3)+v(7)\big)\\ \big(v(1)-v(5)\big)+i\big(v(3)-v(7)\big) \end{pmatrix} \end{matrix}.$$

You may express the answer in terms of ω_8.