MATH 416 Homework 5
Due Wednesday, 4 March 2020

Let $T > 0$. For each $n \in \mathbb{Z}_+$ define the windowing function $w_n : \mathbb{R} \to [0, \infty)$ by

$$w_n(t) = \begin{cases} \dfrac{1 + p_n\big(\cos(\frac{\pi}{T}\, t)\big)}{2} & \text{for } t \in [-T, T]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $p_n(z)$ is the unique odd polynomial determined by

$$p_n'(z) = c_n(1 - z^2)^{n-1}\,, \qquad p(0) = 0\,, \qquad p(1) = 1\,.$$

1. The polynomials $p_1(z)$, $p_2(z)$, $p_3(z)$, and $p_4(z)$ were given in class. Find $p_5(z)$ and $p_6(z)$.

2. Give explicit expressions for $w_1(t)$, $w_2(t)$, $w_3(t)$, and $w_4(t)$. Show that each of these four functions satifies the replication condition

$$\sum_{k \in \mathbb{Z}} w_n(t + kT) = 1\,.$$

   Remark. $w_1(t)$ is the centered, wide Hanning function.

3. Set $T = \pi$ and plot on a single graph $w_n(t)$ versus t over $[-2\pi, 2\pi]$ for $n = 1, 2, 3, 4$. Show that $w_4(t)$ satisfies

$$\begin{aligned} w_4^{(k)}(\pm\pi) &= 0 \qquad \text{for every } k \in \{0, 1, 2, 3, 4, 5, 6, 7\}\,, \\ w_4^{(k)}(0) &= 0 \qquad \text{for every } k \in \{1, 2, 3, 4, 5, 6, 7\}\,. \end{aligned}$$

Given any windowing function $w : \mathbb{R} \to [0, \infty)$ that satisfies the replication condition

$$\sum_{k \in \mathbb{Z}} w(t + kT) = 1\,,$$

the T-periodization of the localization wf of any $f : \mathbb{R} \to \mathbb{C}$ is given by

$$f_T(t) = \sum_{k \in \mathbb{Z}} w(t + kT)\, f(t + kT)\,.$$

In class we showed that the k^{th} Fourier coefficient of this periodization is

$$c_k = \frac{1}{T} \int_0^T \overline{e_k(t)}\, f_T(t)\, \mathrm{d}t = \frac{1}{T} \int_{-T}^{T} \overline{e_k(t)}\, w(t)\, f(t)\, \mathrm{d}t\,,$$

where $e_k(t) = \exp(ik\frac{2\pi}{T}t)$ for every $k \in \mathbb{Z}$.

4. Compute c_k for every $k \in \mathbb{Z}$ when $f(t) = \exp(i\omega t)$ for some $\omega \in \mathbb{R}$ and $w(t) = w_1(t)$. Integration by parts can be avoided by using the Euler identity $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

5. Compute c_k for every $k \in \mathbb{Z}$ when $f(t) = \exp(i\omega t)$ for some $\omega \in \mathbb{R}$ and $w(t) = w_2(t)$. The trig identity $(\cos(\theta))^3 = \frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos(\theta)$ can be helpful.

Remark. If $\omega = m\frac{2\pi}{T}$ for some $m \in \mathbb{Z}$ then $f(t) = \exp(i\omega t)$ is T-periodic and, because $w(t)$ satisfies the replication condition, we have $f_T(t) = f(t) = e_m(t)$. In that case the orthonomality of the Fourier basis $\{e_k(t)\}$ implies that $c_k = \delta_{km}$. Therefore the interesting case is when the frequency ω is not a multiple of $\frac{2\pi}{T}$. In that case the periodization of $f(t) = \exp(i\omega t)$ will be distributed amongst the Fourier basis. This phenomenon is called *aliasing*. We will see it again.