MATH 416 Homework 4
Due Wednesday, 26 February 2020

1. Let $T > 0$ and define $e_k(t) = \exp(\mathrm{i}\frac{2\pi}{T}t)$ for every $k \in \mathbb{Z}$ and $t \in \mathbb{R}$. Let $C(\mathbb{T}_T)$ denote the set of all continuous, complex-valued, T-periodic functions over $\mathbb{R}$. Equip $C(\mathbb{T}_T)$ with the inner product
$$\langle g\,,\,f\rangle = \frac{1}{T}\int_0^T \overline{g(t)}\,f(t)\,\mathrm{d}t\,.$$
In class we showed that $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal set with respect to this inner product. For every $n \in \mathbb{Z}_+$ define the linear mapping $\mathcal{P}_n : C(\mathbb{T}_T) \to C(\mathbb{T}_T)$ by
$$\mathcal{P}_n f(t) = \sum_{k=-n}^{n} e_k(t)\langle e_k\,,\,f\rangle\,, \qquad \text{for every } f \in C(\mathbb{T}_T)\,.$$
Show that $\mathcal{P}_n$ is an orthogonal projection with respect to this inner product.

2. Let $L > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic such that
$$f(x) = \frac{|x|}{L} - 1 \qquad \text{for every } x \in [-L, L]\,.$$
Compute the coefficients $\{a_k\}$ in its cosine expansion
$$f(x) = \sum_{k=0}^{\infty} a_k \cos\!\Big(k\frac{\pi}{L}x\Big)\,.$$
(Set $T = 2L$ in some of the formulas developed in class and evaluate the definite integrals that arise. Use the fact that the intgrands are even.) Determime whether or not
$$\sum_{k=0}^{n} |a_k| \qquad \text{converges}\,.$$
(If this series converges then the Weierstrass M-Test implies that the cosine expansion converges uniformly.)

3. Let $f : \mathbb{R} \to \mathbb{R}$ be as in the previous problem. For every $n \in \mathbb{Z}_+$ define
$$f_n(x) = \sum_{k=0}^{n} a_k \cos\!\Big(k\frac{\pi}{L}x\Big)\,,$$
where $\{a_k\}$ are the coefficients computed in the previous problem. Set $L = \pi$. Use Matlab to plot on a single graph
   - $f(x)$ versus x over $x \in [-2\pi, 2\pi]$,
   - $f_2(x)$ versus x over $x \in [-2\pi, 2\pi]$,
   - $f_4(x)$ versus x over $x \in [-2\pi, 2\pi]$,
   - $f_8(x)$ versus x over $x \in [-2\pi, 2\pi]$,
   - $f_{16}(x)$ versus x over $x \in [-2\pi, 2\pi]$.

   Based upon these plots, does it look like f_n converges to f uniformly? If not then does it look like f_n converges to f pointwise?

4. Let $L > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic such that

$$f(x) = \begin{cases} 1 & \text{for } x \in (0, L)\,, \\ -1 & \text{for } x \in (-L, 0)\,, \\ 0 & \text{for } x = 0 \text{ or } x = \pm L\,. \end{cases}$$

Compute the coefficients $\{b_k\}$ in its sine expansion

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\Big(k\frac{\pi}{L}x\Big)\,.$$

(Set $T = 2L$ in some of the formulas developed in class and evaluate the definite integrals that arise. Use the fact that the integrands are even.) Determime whether or not

$$\sum_{k=1}^{n} |b_k| \qquad \text{converges}\,.$$

(If this series converges then the Weierstrass M-Test implies that the sine expansion converges uniformly.)

5. Let $f : \mathbb{R} \to \mathbb{R}$ be as in the previous problem. For every $n \in \mathbb{Z}_+$ define

$$f_n(x) = \sum_{k=1}^{n} b_k \sin\Big(k\frac{\pi}{L}x\Big)\,,$$

where $\{b_k\}$ are the coefficients computed in the previous problem. Set $L = \pi$. Use Matlab to plot on a single graph

- $f(x)$ versus x over $x \in [-2\pi, 2\pi]$,
- $f_2(x)$ versus x over $x \in [-2\pi, 2\pi]$,
- $f_4(x)$ versus x over $x \in [-2\pi, 2\pi]$,
- $f_8(x)$ versus x over $x \in [-2\pi, 2\pi]$,
- $f_{16}(x)$ versus x over $x \in [-2\pi, 2\pi]$.

Based upon these plots, does it look like f_n converges to f uniformly? If not then does it look like f_n converges to f pointwise?