MATH 416 Homework 1
Due Wednesday, 1 April 2020

1. Let $\mathbf{X}$ be an inner product space over a field $\mathbb{F}$ with inner product $\langle \cdot\,, \cdot \rangle$. Its associated norm is given by
$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}\,, \mathbf{x} \rangle}\,.$$
Show that this norm satisfies the parallelagram identity — namely, show that for every $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ we have
$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} = \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2\,.$$

2. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denote the usual Euclidean basis given by
$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\,, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\,.$$
Let $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be given by
$$\mathbf{b}_1 = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}\,, \qquad \mathbf{b}_2 = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}\,, \qquad \mathbf{b}_3 = \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}\,.$$
   a. Let $\mathbb{E}^3$ denote $\mathbb{R}^3$ equipped with the usual Euclidean inner product, $\langle \mathbf{x}\,, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{y}$. Show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthogonal basis for $\mathbb{E}^3$.
   b. Express each member of the Euclidean basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as a linear combination of $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

3. Let $\mathbf{X}$ be an inner product space over a field $\mathbb{F}$ with inner product $\langle \cdot\,, \cdot \rangle$. Let $\{\mathbf{v}_j\}_{j=1}^N$ be an orthonormal set in $\mathbf{X}$. For every $\mathbf{x} \in \mathbf{X}$ define $P : \mathbf{X} \to \mathbf{X}$ by
$$P(\mathbf{x}) = \sum_{j=1}^{N} \mathbf{v}_j \langle \mathbf{v}_j\,, \mathbf{x} \rangle\,.$$
Show that P is an orthogonal projection onto $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N\}$.

4. Let $\mathbf{X}$ be an inner product space over a field $\mathbb{F}$ with inner product $\langle \cdot\,, \cdot \rangle$. Let $\{\mathbf{v}_j\}_{j=1}^N$ be an orthonormal set in $\mathbf{X}$. Show that for every $\mathbf{x} \in \mathbf{X}$ we have
$$\sum_{j=1}^{N} |\langle \mathbf{v}_j\,, \mathbf{x} \rangle|^2 \leq \|\mathbf{x}\|^2\,.$$

5. Let $C([-1,1];\mathbb{R})$ denote the linear space of real-valued functions that are continuous over the interval $[-1,1]$. Equip $C([-1,1];\mathbb{R})$ with the inner product
$$\langle g\,, f \rangle = \frac{1}{2}\int_{-1}^{1} g(x)\, f(x)\, x^4 \mathrm{d}x\,.$$
Use the Gram-Schmidt algorithm to generate an orthonomal basis for the subspace $\operatorname{Span}\{1, x, x^2, x^3, x^4\}$.