MATH 416 Take-Home Exam 2 Solutions
Due 11:59pm Thursday, 21 May 2020

Be sure to show all work and to make your reasoning clear.

1. [20] Consider the function $f(x) = 1/\sqrt{1+x^2}$ over $\mathbb{R}$.

   a. [8] Compute the polynomial of degree at most 7 that interpolates the values $f(x)$ at the uniform nodes $\{-7, -5, -3, -1, 1, 3, 5, 7\}$.

   b. [8] Compute the polynomial of degree at most 7 that interpolates the values $f(x)$ at the Chebyshev nodes $\{8r : T_8(r) = 0\}$.

   c. [4] Plot $f(x)$ and these two interpolants over the interval $[-8, 8]$. Which interpolant gives a better approximation to $f(x)$ over $[-8, 8]$? Why?

Remark. In parts (a) and (b) the nodes are symmetric about the origin (if x_k is a node then so is $-x_k$), while the function $f(x) = 1/\sqrt{1+x^2}$ has even symmetry over $\mathbb{R}$ ($f(-x) = f(x)\ \forall x \in \mathbb{R}$). These symmetries imply that the interpolating polynomials will each be even. This observation greatly simplifies any approach to these problems. They each have eight nodes, which we can denote as

$$-x_4 < -x_3 < -x_2 < -x_1 < 0 < x_1 < x_2 < x_3 < x_4 \,.$$

Because every even polynomial of degree at most 7 must have degree at most 6, each interpolating polynomial $p(x)$ must have the form

$$p(x) = a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6 \,,$$

where the polynomial $q(y)$ given by

$$q(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 \,,$$

interpolates the values of $g(y) = 1/\sqrt{1+y}$ at the nodes $\{y_k = x_k^2 : k = 1, 2, 3, 4\}$. If we use the Vandermonde approach then $\{a_0, a_1, a_2, a_3\}$ solve the linear algebraic system

$$\begin{pmatrix} 1 & x_1^2 & x_1^4 & x_1^6 \\ 1 & x_1^2 & x_1^4 & x_1^6 \\ 1 & x_1^2 & x_1^4 & x_1^6 \\ 1 & x_1^2 & x_1^4 & x_1^6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{pmatrix} .$$

If we use the Lagrange approach then we can directly write down

$$\begin{aligned} p(x) = {} & \frac{(x_2^2 - x^2)(x_3^2 - x^2)(x_4^2 - x^2)}{(x_2^2 - x_1^2)(x_3^2 - x_1^2)(x_4^2 - x_1^2)}\, f(x_1) + \frac{(x^2 - x_1^2)(x_3^2 - x^2)(x_4^2 - x^2)}{(x_2^2 - x_1^2)(x_3^2 - x_2^2)(x_4^2 - x_2^2)}\, f(x_2) \\ & + \frac{(x^2 - x_1^2)(x^2 - x_2^2)(x_4^2 - x^2)}{(x_3^2 - x_1^2)(x_3^2 - x_2^2)(x_4^2 - x_3^2)}\, f(x_3) + \frac{(x^2 - x_1^2)(x^2 - x_2^2)(x^2 - x_3^2)}{(x_4^2 - x_1^2)(x_4^2 - x_2^2)(x_4^2 - x_3^2)}\, f(x_4) \,. \end{aligned}$$

We will use the Lagrange approach below.

Solution (a). For the uniform nodes we have $x_1 = 1$, $x_2 = 3$, $x_3 = 5$, and $x_4 = 7$, so

$$\begin{aligned} p_{\mathrm{a}}(x) = {} & \frac{(9 - x^2)(25 - x^2)(49 - x^2)}{8 \cdot 24 \cdot 48} \frac{1}{\sqrt{2}} + \frac{(x^2 - 1)(25 - x^2)(49 - x^2)}{8 \cdot 16 \cdot 40} \frac{1}{\sqrt{10}} \\ & + \frac{(x^2 - 1)(x^2 - 9)(49 - x^2)}{24 \cdot 16 \cdot 24} \frac{1}{\sqrt{26}} + \frac{(x^2 - 1)(x^2 - 9)(x^2 - 25)}{48 \cdot 40 \cdot 24} \frac{1}{\sqrt{50}} \,. \end{aligned}$$

$\square$

Solution (b). The Chebyshev nodes are $\{8r : T_8(r) = 0\}$. Because

$$\cos(8\theta) = T_8\big(\cos(\theta)\big)\,,$$

we see that $T_8(r) = 0$ for some $r \in [-1, 1]$ if and only if $r = \cos(\theta)$ and $8\theta = k\pi - \frac{\pi}{2}$ for some $k \in \mathbb{Z}$. All the roots in $(0, 1)$ are given by $\theta = \frac{(2k-1)\pi}{16}$ for $k = 1, 2, 3,$ and 4. Therefore the positive Chebyshev nodes are

$$x_1 = 8\cos(\tfrac{7\pi}{16})\,, \qquad x_2 = 8\cos(\tfrac{5\pi}{16})\,, \qquad x_3 = 8\cos(\tfrac{3\pi}{16})\,, \qquad x_4 = 8\cos(\tfrac{\pi}{16})\,.$$

Placing these nodes into the Lagrange formula for $p(x)$ given above yields $p_{\mathrm{b}}(x)$. $\square$

Remark. While not required here, these nodes can be expressed as algebric numbers. Starting with $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, two applications of the cosine half-angle identity yield

$$\cos(\tfrac{\pi}{8}) = \sqrt{\frac{1 + \cos(\frac{\pi}{4})}{2}} = \frac{\sqrt{2 + \sqrt{2}}}{2}\,,$$

$$\cos(\tfrac{\pi}{16}) = \sqrt{\frac{1 + \cos(\frac{\pi}{8})}{2}} = \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2}\,.$$

Starting with $\cos(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$, two applications of the cosine half-angle identity yield

$$\cos(\tfrac{3\pi}{8}) = \sqrt{\frac{1 + \cos(\frac{3\pi}{4})}{2}} = \frac{\sqrt{2 - \sqrt{2}}}{2}\,,$$

$$\cos(\tfrac{3\pi}{16}) = \sqrt{\frac{1 + \cos(\frac{3\pi}{8})}{2}} = \frac{\sqrt{2 + \sqrt{2 - \sqrt{2}}}}{2}\,.$$

Finally, we have

$$\cos(\tfrac{5\pi}{16}) = \sin(\tfrac{3\pi}{16}) = \sqrt{1 - \big(\cos(\tfrac{3\pi}{16})\big)^2} = \frac{\sqrt{2 - \sqrt{2 - \sqrt{2}}}}{2}\,,$$

$$\cos(\tfrac{7\pi}{16}) = \sin(\tfrac{\pi}{16}) = \sqrt{1 - \big(\cos(\tfrac{\pi}{16})\big)^2} = \frac{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{2}\,.$$

Therefore the positive Chebychev nodes are

$$x_1 = 4\sqrt{2 - \sqrt{2 + \sqrt{2}}}\,, \qquad x_2 = 4\sqrt{2 - \sqrt{2 - \sqrt{2}}}\,,$$

$$x_3 = 4\sqrt{2 + \sqrt{2 - \sqrt{2}}}\,, \qquad x_4 = 4\sqrt{2 + \sqrt{2 + \sqrt{2}}}\,.$$

Solution (c). You are asked to plot $f(t)$, $p_{\mathrm{a}}(t)$ and $p_{\mathrm{b}}(t)$ versus t over $[-8, 8]$. You should see that due to the coarseness of these approximations, neither is great. However, $p_{\mathrm{b}}(t)$ maintains positivity, almost recovers the correct monotonicity, and is significantly quantitatively better over the outer region $4 \le |t| \le 8$. On the other hand, $p_{\mathrm{a}}(t)$ is quantitatively better only over the inner region $|t| \le 1$. On balance, $p_{\mathrm{b}}(t)$ is better. $\square$

2. [20] Let $\phi(t) = \mathrm{hat}(t)$, where the "hat" function is defined over $\mathbb{R}$ by

$$\mathrm{hat}(t) = \begin{cases} 1 - |t| & \text{for } t \in (-1, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This function satisfies the *interpolation condition*,

$$\phi(0) = 1\,, \qquad \phi(k) = 0 \quad \text{for every } k \in \mathbb{Z} - \{0\}\,.$$

Define $\phi_k : \mathbb{R} \to \mathbb{R}$ by $\phi_k(t) = \phi(t - k)$. Let $\{c_k\}_{k \in \mathbb{Z}}$ be any real sequence over $\mathbb{Z}$. For every $m, n \in \mathbb{Z}$ with $m \leq n$ define $u_{mn} : \mathbb{R} \to \mathbb{R}$ by

$$u_{mn}(t) = \sum_{k=m}^{n} c_k \, \phi_k(t)\,.$$

The set $\{u_{mn} \,:\, m, n \in \mathbb{Z}\,,\, m \leq n\}$ has the *Cauchy property* with respect to a norm $\|\cdot\|$ if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for every $m, n \in \mathbb{Z}$ with $m \leq n$ we have

$$n \leq -N_\epsilon \quad \text{or} \quad N_\epsilon \leq m \qquad \Longrightarrow \qquad \|u_{mn}\| < \epsilon\,.$$

Consider the $L^4(\mathbb{R})$ norm defined by

$$\|v\|_{L^4(\mathbb{R})} = \left(\int_{\mathbb{R}} |v(t)|^4 \, \mathrm{d}t \right)^{\frac{1}{4}}\,.$$

a. [4] Evaluate

$$\|u_{mn}\|_{L^4(\mathbb{R})}^4 = \int_{\mathbb{R}} |u_{mn}(t)|^4 \, dt\,.$$

b. [8] Show that

$$\tfrac{1}{5} \sum_{k=m}^{n} |c_k|^4 \leq \|u_{mn}\|_{L^4(\mathbb{R})}^4 \leq \sum_{k=m}^{n} |c_k|^4\,.$$

c. [8] Prove that $\{u_{mn} \,:\, m, n \in \mathbb{Z}\,,\, m \leq n\}$ has the Cauchy property with respect to the $L^4(\mathbb{R})$ norm if and only if

$$\sum_{k \in \mathbb{Z}} |c_k|^4 < \infty\,.$$

Solution (a). Because each u_{mn} is the continuous piecewise linear interpolant that satisfies $u_{mn}(k) = c_k$ for every $k \in \mathbb{Z}$ with $k \in [m, n]$ and $u_{mn}(k) = 0$ for every $k \in \mathbb{Z}$ with $k \notin [m, n]$, we see that for every $k \in \mathbb{Z}$ and $t \in [k, k+1]$ we have

$$u_{mn}(t) = \begin{cases} (t - m + 1)c_m & \text{if } [k, k+1] = [m-1, m]\,, \\ (k + 1 - t)c_k + (t - k)c_{k+1} & \text{if } [k, k+1] \subset [m, n]\,, \\ (n + 1 - t)c_n & \text{if } [k, k+1] = [n, n+1]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Because the binomial expansion yields

$$\begin{aligned}\big((k+1-t)c_k + (t-k)c_{k+1}\big)^4 &= (k+1-t)^4 c_k^4 + 4(k+1-t)^3(t-k)c_k^3 c_{k+1} \\ &\quad + 6(k+1-t)^2(t-k)^2 c_k^2 c_{k+1}^2 \\ &\quad + 4(k+1-t)(t-k)^3 c_k c_{k+1}^3 + (t-k)^4 c_{k+1}^4\,,\end{aligned}$$

while elementary integrations yield

$$\int_k^{k+1} (k+1-t)^4\,\mathrm{d}t = \int_k^{k+1} (t-k)^4\,\mathrm{d}t = \frac{1}{5}\,, \qquad \int_k^{k+1} (k+1-t)^2(t-k)^2\,\mathrm{d}t = \frac{1}{30}\,,$$
$$\int_k^{k+1} (k+1-t)^3(t-k)\,\mathrm{d}t = \int_k^{k+1} (k+1-t)(t-k)^3\,\mathrm{d}t = \frac{1}{20}\,,$$

a direct calculation shows that

$$\int_k^{k+1} |u_{mn}(t)|^4\,dt = \begin{cases} \dfrac{c_m^4}{5} & \text{if } [k,k+1] = [m-1,m]\,, \\ \dfrac{c_k^4 + c_k^3 c_{k+1} + c_k^2 c_{k+1}^2 + c_k c_{k+1}^3 + c_{k+1}^5}{5} & \text{if } [k,k+1] \subset [m,n]\,, \\ \dfrac{c_n^4}{5} & \text{if } [k,k+1] = [n,n+1]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Therefore

$$\begin{aligned}\|u_{mn}\|_{L^4(\mathbb{R})}^4 &= \int_{\mathbb{R}} |u_{mn}(t)|^4\,dt = \sum_{k\in\mathbb{Z}} \int_k^{k+1} |u_{mn}(t)|^4\,dt \\ &= \frac{c_m^4}{5} + \sum_{k=m}^{n-1} \frac{c_k^4 + c_k^3 c_{k+1} + c_k^2 c_{k+1}^2 + c_k c_{k+1}^3 + c_{k+1}^4}{5} + \frac{c_n^4}{5}\,.\end{aligned}$$

This completes part a. ☐

Solution (b). By using the facts that $c_k c_{k+1} \le \frac{1}{2}(c_k^2 + c_{k+1}^2)$ and $c_k^2 c_{k+1}^2 \le \frac{1}{2}(c_k^4 + c_{k+1}^4)$, the quantity inside the sum can be bounded above by

$$\frac{c_k^4 + c_k^3 c_{k+1} + c_k^2 c_{k+1}^2 + c_k c_{k+1}^3 + c_{k+1}^4}{5} \le \frac{3c_k^4 + 4c_k^2 c_{k+1}^2 + 3c_{k+1}^4}{10} \le \frac{c_k^4 + c_{k+1}^4}{2}\,.$$

We thereby obtain the upper bound

$$\|u_{mn}\|_{L^4(\mathbb{R})}^4 \le \frac{c_m^4}{5} + \sum_{k=m}^{n-1} \frac{c_k^4 + c_{k+1}^4}{2} + \frac{c_n^4}{5} \le \sum_{k=m}^{n} c_k^4\,.$$

Similarly, By using the fact that $c_k c_{k+1} \ge -\frac{1}{2}(c_k^2 + c_{k+1}^2)$ on its second and fourth terms, the quantity inside the sum can be bounded below by

$$\frac{c_k^4 + c_k^3 c_{k+1} + c_k^2 c_{k+1}^2 + c_k c_{k+1}^3 + c_{k+1}^4}{5} \ge \frac{c_k^4 + c_{k+1}^4}{10}\,.$$

We thereby obtain the lower bound

$$\|u_{mn}\|_{L^4(\mathbb{R})}^4 \ge \frac{c_m^4}{5} + \sum_{k=m}^{n-1} \frac{c_k^4 + c_{k+1}^4}{10} + \frac{c_n^4}{5} \ge \frac{1}{5} \sum_{k=m}^{n} c_k^4\,.$$

This lower bound combined with the upper bound yields the assertion of Part b. ☐

Remark. The upper bound could also be derived using the inequalities

$$c_k^3 c_{k+1} \le \tfrac{3}{4} c_k^4 + \tfrac{1}{4} c_{k+1}^4\,, \qquad c_k^2 c_{k+1}^2 \le \tfrac{1}{2} c_k^4 + \tfrac{1}{2} c_{k+1}^4\,, \qquad c_k c_{k+1}^3 \le \tfrac{1}{4} c_k^4 + \tfrac{3}{4} c_{k+1}^4\,.$$

These follow from the Young inequality that for every $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ says

$$|xy| \le \tfrac{1}{p}|x|^p + \tfrac{1}{q}|y|^q \quad \text{for every } x, y \in \mathbb{R}\,.$$

For example, the first inequality follows by taking $x = c_k^3$, $y = c_{k+1}$, $p = \frac{4}{3}$, and $q = 4$.

Solution (c). First suppose that $\{c_k : k \in \mathbb{Z}\}$ satisfies the sum condition

$$\sum_{k \in \mathbb{Z}} c_k^4 < \infty\,.$$

We want to show that $\{u_{mn} : m, n \in \mathbb{Z}\,,\ m \le n\}$ has the Cauchy property with respect to the $L^4(\mathbb{R})$ norm.

Let $\epsilon > 0$. The fact that $\{c_k : k \in \mathbb{Z}\}$ satisfies the sum condition implies that there exists $N_\epsilon \in \mathbb{N}$ such that for every $m, n \in \mathbb{Z}$ with $m \le n$ we have

$$n \le -N_\epsilon \quad \text{or} \quad N_\epsilon \le m \qquad \Longrightarrow \qquad \sum_{k=m}^{n} c_k^4 < \epsilon^4\,.$$

But then our upper bound implies that

$$n \le -N_\epsilon \quad \text{or} \quad N_\epsilon \le m \qquad \Longrightarrow \qquad \|u_{mn}\|_{L^4(\mathbb{R})}^4 < \epsilon^4\,.$$

Hence, $\{u_{mn} : m, n \in \mathbb{Z}\,,\ m \le n\}$ has the Cauchy property with respect to the $L^4(\mathbb{R})$ norm.

Now suppose that $\{u_{mn} : m, n \in \mathbb{Z}\,,\ m \le n\}$ has the Cauchy property with respect to the $L^4(\mathbb{R})$ norm. We want to show that $\{c_k : k \in \mathbb{Z}\}$ satisfies the sum condition

$$\sum_{k \in \mathbb{Z}} c_k^4 < \infty\,.$$

We do this by showing that for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for every $m, n \in \mathbb{Z}$ with $m \le n$ we have

$$n \le -N_\epsilon \quad \text{or} \quad N_\epsilon \le m \qquad \Longrightarrow \qquad \sum_{k=m}^{n} c_k^4 < \epsilon\,.$$

Let $\epsilon > 0$. The fact that $\{u_{mn} : m, n \in \mathbb{Z}\,,\ m \le n\}$ has the Cauchy property with respect to the $L^4(\mathbb{R})$ norm implies that there exists $N_\epsilon \in \mathbb{N}$ such that for every $m, n \in \mathbb{Z}$ with $m \le n$ we have

$$n \le -N_\epsilon \quad \text{or} \quad N_\epsilon \le m \qquad \Longrightarrow \qquad \|u_{mn}\|_{L^4(\mathbb{R})}^4 < \tfrac{1}{5}\epsilon\,.$$

But then our lower bound implies that

$$n \le -N_\epsilon \quad \text{or} \quad N_\epsilon \le m \qquad \Longrightarrow \qquad \sum_{k=m}^{n} c_k^4 < \epsilon\,.$$

But this implies that the sum condition is satisfied. This completes Part c. □

3. [20] The Haar wavelet function $\psi : \mathbb{R} \to \mathbb{R}$ is

$$\psi(t) = \begin{cases} 1 & \text{for } t \in [0, \frac{1}{2})\,, \\ -1 & \text{for } t \in [\frac{1}{2}, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

It has a primitive $\Psi : \mathbb{R} \to \mathbb{R}$ given by

$$\Psi(t) = \begin{cases} \min\{t, 1-t\} & \text{for } t \in (0,1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For each $j, k \in \mathbb{Z}$ define $\psi_{jk} : \mathbb{R} \to \mathbb{R}$ and $\Psi_{jk} : \mathbb{R} \to \mathbb{R}$ by

$$\psi_{jk}(t) = 2^{\frac{j}{2}} \psi(2^j t - k)\,, \qquad \Psi_{jk}(t) = 2^{-\frac{j}{2}} \Psi(2^j t - k)\,.$$

Let $S \subset L^2([0,1])$ be given by

$$S = \big\{ \psi_{jk} \;:\; j \in \{0, 1, \cdots\}\,,\, k \in \{0, 1, \cdots, 2^j - 1\} \big\}\,.$$

Problem 1 of Homework 10 showed that S is an orthonormal set in $L^2([0,1])$ that is orthogonal to every constant function. For every $J \in \mathbb{Z}_+$ let $\mathcal{P}_J : L^2([0,1]) \to L^2([0,1])$ be the orthogonal projection given by

$$\mathcal{P}_J u(t) = \langle 1\,,\, u \rangle + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle \psi_{jk}\,,\, u \rangle\, \psi_{jk}(t) \qquad \text{for every } u \in L^2([0,1])\,.$$

Let $b \in (0,1)$ and set $v(t) = \chi_{[0,b)}(t)$ where

$$\chi_{[0,b)}(t) = \begin{cases} 1 & \text{if } t \in [0,b)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

a. [8] Show for every $J \in \mathbb{Z}_+$ that

$$\mathcal{P}_J v(t) = b + \sum_{j=0}^{J-1} \min\big\{ 2^j b - \lfloor 2^j b \rfloor\,,\, \lceil 2^j b \rceil - 2^j b \big\}\, \psi\big(2^j t - \lfloor 2^j b \rfloor\big)\,,$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the "floor" and "ceiling" functions, which are defined for every $x \in \mathbb{R}$ by

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} \;:\; k \le x\}\,, \qquad \lceil x \rceil = \min\{k \in \mathbb{Z} \;:\; x \le k\}\,.$$

b. [8] Use induction on J to prove for every $J \in \mathbb{Z}_+$ that

$$\mathcal{P}_J v(t) = \begin{cases} 1 & \text{for } t \in [0, \underline{b}_J)\,, \\ 2^J b - \lfloor 2^J b \rfloor & \text{for } t \in [\underline{b}_J, \overline{b}_J)\,, \\ 0 & \text{for } t \in [\overline{b}_J, 1)\,, \end{cases}$$

where $\underline{b}_J = \lfloor 2^J b \rfloor / 2^J$ and $\overline{b}_J = \lceil 2^J b \rceil / 2^J$.

c. [4] Show for every $J \in \mathbb{Z}_+$ that

$$\|\mathcal{P}_J v - v\|^2_{L^2([0,1])} = \frac{\big(\lceil 2^J b \rceil - 2^J b\big)\big(2^J b - \lfloor 2^J b \rfloor\big)}{2^J} \le \frac{1}{2^{J+2}}\,.$$

Solution (a). We need to compute the coefficients in the projection $\mathcal{P}_J v$ of the function $v(t) = \chi_{[0,b)}(t)$ for every $J \in \mathbb{Z}_+$. Therefore we must compute

$$\langle 1\,,\, v\rangle \qquad \text{and} \qquad \langle \psi_{jk}\,,\, v\rangle \quad \text{for every } j \in \mathbb{N} \text{ and } k \in \{0, 1, \cdots, 2^{j-1}\}\,.$$

The easiest step is

$$\langle 1\,,\, v\rangle = \int_0^1 \chi_{[0,b)}(t)\,\mathrm{d}t = \int_0^b \mathrm{d}t = b\,.$$

Next, let $j \in \mathbb{N}$ and $k \in \{0, 1, \cdots, 2^{j-1}\}$. Because $\Psi_{jk}(t)$ is the primitive of $\psi_{jk}(t)$ that satisfies $\Psi_{jk}(0) = 0$, we have

$$\langle \psi_{jk}\,,\, v\rangle = \int_0^1 \psi_{jk}(t)\,\chi_{[0,b)}(t)\,\mathrm{d}t = \int_0^b \psi_{jk}(t)\,\mathrm{d}t = \Psi_{jk}(b)\,.$$

Because $\Psi_{jk}(t) = 2^{-\frac{j}{2}}\Psi(2^j t - k)$, the given formula for $\Psi(t)$ yields

$$\begin{aligned}\langle \psi_{jk}\,,\, v\rangle &= 2^{-\frac{j}{2}}\Psi(2^j b - k)\\ &= \begin{cases} 2^{-\frac{j}{2}}\min\{2^j b - k\,,\, k + 1 - 2^j b\} & \text{if } 2^j b - k \in (0,1)\,,\\ 0 & \text{otherwise}\,.\end{cases}\end{aligned}$$

But $2^j b - k \in (0,1)$ holds if and only if $k = \lfloor 2^j b\rfloor$. If $k = \lfloor 2^j b\rfloor$ and $2^j b > \lfloor 2^j b\rfloor$ then $\lceil 2^j b\rceil = k + 1$ and we see that

$$\langle \psi_{jk}\,,\, v\rangle = \begin{cases} 2^{-\frac{j}{2}}\min\big\{2^j b - \lfloor 2^j b\rfloor\,,\, \lceil 2^j b\rceil - 2^j b\big\} & \text{if } k = \lfloor 2^j b\rfloor\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

If $k = \lfloor 2^j b\rfloor$ and $2^j b = \lfloor 2^j b\rfloor$ then this formula still holds even though $\lceil 2^j b\rceil = k$ because its right-hand side vanishes.

Finally, we place the above results into the definition of the orthogonal projection $\mathcal{P}_J$ and use the fact that $\psi_{jk}(t) = 2^{\frac{j}{2}}\psi(2^j t - k)$ to obtain

$$\begin{aligned}\mathcal{P}_J v(t) &= \langle 1\,,\, v\rangle + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\langle \psi_{jk}\,,\, v\rangle\,\psi_{jk}(t)\\ &= b + \sum_{j=0}^{J-1}\min\big\{2^j b - \lfloor 2^j b\rfloor\,,\, \lceil 2^j b\rceil - 2^j b\big\}\,2^{-\frac{j}{2}}\psi_{j\lfloor 2^j b\rfloor}(t)\\ &= b + \sum_{j=0}^{J-1}\min\big\{2^j b - \lfloor 2^j b\rfloor\,,\, \lceil 2^j b\rceil - 2^j b\big\}\,\psi\big(2^j t - \lfloor 2^j b\rfloor\big)\,.\end{aligned}$$

This completes Part a. □

Solution (b). For every $J \in \mathbb{Z}_+$ let $\mathcal{P}_J v \in L^2([0,1])$ be as in the assertion in Part a and let $v_J \in L^2([0,1])$ be defined by

$$v_J(t) = \begin{cases} 1 & \text{if } t \in [0, \underline{b}_J)\,,\\ 2^J b - \lfloor 2^J b\rfloor & \text{if } t \in [\underline{b}_J, \overline{b}_J)\,,\\ 0 & \text{if } t \in [\overline{b}_J, 1)\,.\end{cases}$$

We want to show for every $J \in \mathbb{Z}_+$ that $\mathcal{P}_J v(t) = v_J(t)$ for every $t \in [0,1)$.

We begin the induction at $J = 1$. For every $t \in [0, 1)$ we have

$$\mathcal{P}_{1} v(t) = b + \min\{b\,,\,1 - b\}\,\psi(t) = \begin{cases} 1 & \text{if } b \in [\frac{1}{2}, 1) \text{ and } t \in [0, \frac{1}{2})\,, \\ 2b - 1 & \text{if } b \in [\frac{1}{2}, 1) \text{ and } t \in [\frac{1}{2}, 1)\,, \\ 2b & \text{if } b \in [0, \frac{1}{2}) \text{ and } t \in [0, \frac{1}{2})\,, \\ 0 & \text{if } b \in [0, \frac{1}{2}) \text{ and } t \in [\frac{1}{2}, 1)\,. \end{cases}$$

Next, notice that

$$\underline{b}_1 = \frac{\lfloor 2b \rfloor}{2} = \begin{cases} 0 & \text{if } b \in [0, \frac{1}{2})\,, \\ \frac{1}{2} & \text{if } b \in [\frac{1}{2}, 1)\,, \end{cases} \qquad \overline{b}_1 = \frac{\lceil 2b \rceil}{2} = \begin{cases} \frac{1}{2} & \text{if } b \in (0, \frac{1}{2}]\,, \\ 1 & \text{if } b \in (\frac{1}{2}, 1]\,, \end{cases}$$

whereby for every $t \in [0, 1)$ we have

$$\mathcal{P}_{1} v(t) = \begin{cases} 1 & \text{if } t \in [0, \underline{b}_1)\,, \\ 2b - \lfloor 2b \rfloor & \text{if } t \in [\underline{b}_1, \overline{b}_1)\,, \\ 0 & \text{if } t \in [\overline{b}_1, 1)\,, \end{cases}$$

which shows that $\mathcal{P}_{1} v(t) = v_{1}(t)$ for every $t \in [0, 1)$.

Now suppose for some $J > 1$ we know that $\mathcal{P}_{J-1} v(t) = v_{J-1}(t)$ for every $t \in [0, 1)$. Observe that $b \in [\underline{b}_{J-1}, \overline{b}_{J-1}]$ and that $\psi\big(2^{J-1}t - \lfloor 2^{J-1}b \rfloor\big) = 0$ outside $[\underline{b}_{J-1}, \overline{b}_{J-1})$. Let $b_{J-1} = \frac{1}{2}(\underline{b}_{J-1} + \overline{b}_{J-1})$. Then for every $t \in [\underline{b}_{J-1}, \overline{b}_{J-1})$ we have

$$\begin{aligned} \mathcal{P}_{J} v(t) &= \mathcal{P}_{J-1} v(t) + \min\big\{2^{J-1}b - \lfloor 2^{J-1}b \rfloor\,,\, \lceil 2^{J-1}b \rceil - 2^{J-1}b\big\}\,\psi\big(2^{J-1}t - \lfloor 2^{J-1}b \rfloor\big) \\ &= v_{J-1}(t) + \begin{cases} \lceil 2^{J-1}b \rceil - 2^{J-1}b & \text{if } b \in [b_{J-1}, \overline{b}_{J-1}) \text{ and } t \in [\underline{b}_{J-1}, b_{J-1})\,, \\ 2^{J-1}b - \lceil 2^{J-1}b \rceil & \text{if } b \in [b_{J-1}, \overline{b}_{J-1}) \text{ and } t \in [b_{J-1}, \overline{b}_{J-1})\,, \\ 2^{J-1}b - \lfloor 2^{J-1}b \rfloor & \text{if } b \in [\underline{b}_{J-1}, b_{J-1}) \text{ and } t \in [\underline{b}_{J-1}, b_{J-1})\,, \\ \lfloor 2^{J-1}b \rfloor - 2^{J-1}b & \text{if } b \in [\underline{b}_{J-1}, b_{J-1}) \text{ and } t \in [b_{J-1}, \overline{b}_{J-1})\,, \end{cases} \\ &= \begin{cases} 1 & \text{if } b \in [b_{J-1}, \overline{b}_{J-1}) \text{ and } t \in [\underline{b}_{J-1}, b_{J-1})\,, \\ 2^J b - 2^J b_{J-1} & \text{if } b \in [b_{J-1}, \overline{b}_{J-1}) \text{ and } t \in [b_{J-1}, \overline{b}_{J-1})\,, \\ 2^J b - 2^J \underline{b}_{J-1} & \text{if } b \in [\underline{b}_{J-1}, b_{J-1}) \text{ and } t \in [\underline{b}_{J-1}, b_{J-1})\,, \\ 0 & \text{if } b \in [\underline{b}_{J-1}, b_{J-1}) \text{ and } t \in [b_{J-1}, \overline{b}_{J-1})\,. \end{cases} \end{aligned}$$

Next, notice that

$$\underline{b}_J = \frac{\lfloor 2^J b \rfloor}{2^J} = \begin{cases} \underline{b}_{J-1} & \text{if } b \in [\underline{b}_{J-1}, b_{J-1})\,, \\ b_{J-1} & \text{if } b \in [b_{J-1}, \overline{b}_{J-1})\,, \end{cases}$$

$$\overline{b}_J = \frac{\lceil 2^J b \rceil}{2^J} = \begin{cases} b_{J-1} & \text{if } b \in (\underline{b}_{J-1}, b_{J-1}]\,, \\ \overline{b}_{J-1} & \text{if } b \in (b_{J-1}, \overline{b}_{J-1}]\,. \end{cases}$$

Therefore for every $t \in [0, 1)$ we have

$$\mathcal{P}_{J} v(t) = \begin{cases} 1 & \text{if } t \in [0, \underline{b}_J)\,, \\ 2^J b - \lfloor 2^J b \rfloor & \text{if } t \in [\underline{b}_J, \overline{b}_J)\,, \\ 0 & \text{if } t \in [\overline{b}_J, 1)\,, \end{cases}$$

whereby $\mathcal{P}_{J} v(t) = v_{J}(t)$ for every $t \in [0, 1)$. The induction proof is thereby complete. $\square$

Remark. This result shows that if $2^J b = \lfloor 2^J b \rfloor$ for some $J \in \mathbb{Z}_+$ that then $\mathcal{P}_J v = v$.

Solution (c). First, treat the case when $\lfloor 2^J b \rfloor < \lceil 2^J b \rceil$. Because $b \in [\underline{b}_J, \bar{b}_J]$, we have

$$\begin{aligned}
\|\mathcal{P}_J v - v\|^2_{L^2([0,1])} &= \int_{\underline{b}_J}^{\bar{b}_J} \big(\mathcal{P}_J v - v\big)^2 \,\mathrm{d}t = \int_{\underline{b}_J}^{b} \big(1 - \mathcal{P}_J v\big)^2 \,\mathrm{d}t + \int_{b}^{\bar{b}_J} \big(\mathcal{P}_J v\big)^2 \,\mathrm{d}t \\
&= \int_{\underline{b}_J}^{b} \big(\lceil 2^J b \rceil - 2^J b\big)^2 \,\mathrm{d}t + \int_{b}^{\bar{b}_J} \big(2^J b - \lfloor 2^J b \rfloor\big)^2 \,\mathrm{d}t \\
&= \big(\lceil 2^J b \rceil - 2^J b\big)^2 \big(b - \underline{b}_J\big) + \big(2^J b - \lfloor 2^J b \rfloor\big)^2 \big(\bar{b}_J - b\big) \\
&= \frac{\big(\lceil 2^J b \rceil - 2^J b\big)^2 \big(2^J b - \lfloor 2^J b \rfloor\big)}{2^J} + \frac{\big(2^J b - \lfloor 2^J b \rfloor\big)^2 \big(\lceil 2^J b \rceil - 2^J b\big)}{2^J} \\
&= \frac{\big(\lceil 2^J b \rceil - 2^J b\big)\big(2^J b - \lfloor 2^J b \rfloor\big)}{2^J} \,.
\end{aligned}$$

Next, this equality still holds when $\lfloor 2^J b \rfloor = \lceil 2^J b \rceil$ because then $\lfloor 2^J b \rfloor = 2^J b = \lceil 2^J b \rceil$, whereby both of its sides vanish. Therefore the equality in the assertion of Part c holds. The inequality in the assertion of Part c holds because $(1-x)x \leq \frac{1}{4}$ for every $x \in \mathbb{R}$. $\square$

Remark. This result shows that

$$\lim_{J \to \infty} \|\mathcal{P}_J v - v\|_{L^2([0,1])} = 0 \,,$$

which says that for every $b \in [0,1]$ we have

$$\chi_{[0,b)} \in \overline{\mathrm{span}}\big\{1 \,,\, \psi_{jk} \,:\, j \in \{0,1,\cdots\} \,,\, k \in \{0,1,\cdots 2^j - 1\}\big\} \,.$$

For every $[a,b) \subset [0,1]$ we have $\chi_{[a,b)} = \chi_{[0,b)} - \chi_{[0,a)}$, whereby

$$\chi_{[a,b)} \in \overline{\mathrm{span}}\big\{1 \,,\, \psi_{jk} \,:\, j \in \{0,1,\cdots\} \,,\, k \in \{0,1,\cdots 2^j - 1\}\big\} \,.$$

We thereby have the inclusions

$$\begin{aligned}
\overline{\mathrm{span}}\big\{\chi_{[a,b)} \,:\, [a,b) \subset [0,1]\big\} &\subset \overline{\mathrm{span}}\big\{1 \,,\, \psi_{jk} \,:\, j \in \{0,1,\cdots\} \,,\, k \in \{0,1,\cdots 2^j - 1\}\big\} \\
&\subset L^2([0,1]) \,.
\end{aligned}$$

This lays the groundwork for a proof that

$$L^2([0,1]) = \overline{\mathrm{span}}\big\{1 \,,\, \psi_{jk} \,:\, j \in \{0,1,\cdots\} \,,\, k \in \{0,1,\cdots 2^j - 1\}\big\} \,,$$

which implies that the orthonormal set $\big\{1 \,,\, \psi_{jk} \,:\, j \in \{0,1,\cdots\} \,,\, k \in \{0,1,\cdots 2^j - 1\}\big\}$ is a basis for $L^2([0,1])$. This orthonormal set is the so-called Haar basis for $L^2([0,1])$. The step needed to complete this proof is to show that

$$L^2([0,1]) \subset \overline{\mathrm{span}}\big\{\chi_{[a,b)} \,:\, [a,b) \subset [0,1]\big\} \,.$$

This step requires knowledge about definite integrals. A partial step in that direction is to show that for every function u that is Riemann integrable over $[0,1]$ we have

$$u \in \overline{\mathrm{span}}\big\{\chi_{[a,b)} \,:\, [a,b) \subset [0,1]\big\} \,.$$

This can be proved without knowledge of the Lebesgue integral. The conclusion uses the fact that $L^2([0,1])$ can be identified with the completion of the Riemann integrable functions with respect to the $L^2([0,1])$ norm, which is a fact about the Lebesgue integral.

4. [20] The Haar scaling function $\phi : \mathbb{R} \to \mathbb{R}$ and wavelet function $\psi : \mathbb{R} \to \mathbb{R}$ are

$$\phi(t) = \begin{cases} 1 & \text{for } t \in [0, 1)\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad \psi(t) = \begin{cases} 1 & \text{for } t \in [0, \frac{1}{2})\,, \\ -1 & \text{for } t \in [\frac{1}{2}, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

They satisfy the two-scale relations

$$\phi(t) = \phi(2t) + \phi(2t-1)\,, \qquad \psi(t) = \phi(2t) - \phi(2t-1)\,.$$

For every $j, k \in \mathbb{Z}$ define $\phi_{jk} : \mathbb{R} \to \mathbb{R}$ and $\psi_{jk} : \mathbb{R} \to \mathbb{R}$ by

$$\phi_{jk}(t) = 2^{\frac{j}{2}}\phi(2^j t - k)\,, \qquad \psi_{jk}(t) = 2^{\frac{j}{2}}\psi(2^j t - k)\,.$$

For every $j \in \mathbb{Z}$ define the subspaces V_j and W_j by

$$V_j = \overline{\text{span}}\{\phi_{jk} \,:\, k \in \mathbb{Z}\}\,, \qquad W_j = \overline{\text{span}}\{\psi_{jk} \,:\, k \in \mathbb{Z}\}\,.$$

a. [8] Show for every $j \in \mathbb{Z}$ that V_j and W_j are orthogonal subspaces.

b. [8] Show for every $j \in \mathbb{Z}$ that

$$V_{j+1} = V_j + W_j = \big\{v + w \,:\, v \in V_j\,,\, w \in W_j\big\}\,.$$

c. [4] Show for every $j \in \mathbb{Z}_+$ that

$$V_{j+1} = V_0 + W_0 + \cdots + W_j\,.$$

Solution (a). Let $j \in \mathbb{Z}$. For every k_1, $k_2 \in \mathbb{Z}$ we have $\phi_{jk_1}(t) = 2^{\frac{j}{2}}\phi(2^j t - k_1)$ and $\psi_{jk_1}(t) = 2^{\frac{j}{2}}\psi(2^j t - k_2)$, whereby

$$\begin{aligned} \langle \phi_{jk_1}\,,\, \psi_{jk_2} \rangle &= \int_{\mathbb{R}} \phi_{jk_1}(t)\, \psi_{jk_1}(t)\, \mathrm{d}t = 2^j \int_{\mathbb{R}} \phi(2^j t - k_1)\, \psi(2^j t - k_2)\, \mathrm{d}t \\ &= \int_R \phi(t)\, \psi(t + k_1 - k_2)\, \mathrm{d}t = \int_0^1 \psi(t + k_1 - k_2)\, \mathrm{d}t = 0\,. \end{aligned}$$

So each member of the basis for V_j is orthogonal to every member of the basis for W_j. Therefore V_j and W_j are orthogonal subspaces. □

Solution (b). Let $j \in \mathbb{Z}$. The two-scale relation for ϕ implies that for every $k \in \mathbb{Z}$ we have

$$2^{\frac{j}{2}}\phi(2^j t - k) = 2^{\frac{j}{2}}\phi(2^{j+1}t - 2k) + 2^{\frac{j}{2}}\phi(2^{j+1}t - 2k - 1)\,.$$

Because $\phi_{jk}(t) = 2^{\frac{j}{2}}\phi(2^j t - k)$, this is equivalent to

$$\phi_{jk}(t) = \tfrac{1}{\sqrt{2}}\phi_{(j+1)(2k)}(t) + \tfrac{1}{\sqrt{2}}\phi_{(j+1)(2k+1)}(t)\,.$$

So each member of the basis for V_j is in V_{j+1}. Therefore $V_j \subset V_{j+1}$.

Similarly, The two-scale relation for ψ implies that for every $k \in \mathbb{Z}$ we have

$$2^{\frac{j}{2}}\psi(2^j t - k) = 2^{\frac{j}{2}}\phi(2^{j+1}t - 2k) - 2^{\frac{j}{2}}\phi(2^{j+1}t - 2k - 1)\,.$$

Because $\psi_{jk}(t) = 2^{\frac{j}{2}}\psi(2^j t - k)$, this is equivalent to

$$\psi_{jk}(t) = \tfrac{1}{\sqrt{2}}\,\phi_{(j+1)(2k)}(t) - \tfrac{1}{\sqrt{2}}\,\phi_{(j+1)(2k+1)}(t)\,.$$

So each member of the basis for W_j is in V_{j+1}. Therefore $W_j \subset V_{j+1}$.

Because $V_j \subset V_{j+1}$, $W_j \subset V_{j+1}$, and V_{j+1} is a linear subspace, we conclude that $V_j + W_j \subset V_{j+1}$. What remains to be shown is that $V_{j+1} \subset V_j + W_j$.

The two-scale relations imply that

$$2\phi(2t) = \phi(t) + \psi(t)\,, \qquad 2\phi(2t-1) = \phi(t) - \psi(t)\,.$$

It follows that for every $k \in \mathbb{Z}$ we have

$$\begin{aligned} 2^{\frac{j+2}{2}}\phi(2^{j+1}t - 2k) &= 2^j\phi(2^jt - k) + 2^j\psi(2^jt - k)\,,\\ 2^{\frac{j+2}{2}}\phi(2^{j+1}t - 2k - 1) &= 2^j\phi(2^jt - k) + 2^j\psi(2^jt - k)\,, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \phi_{(j+1)(2k)}(t) &= \tfrac{1}{\sqrt{2}}\,\phi_{jk}(t) + \tfrac{1}{\sqrt{2}}\,\psi_{jk}(t)\,,\\ \phi_{(j+1)(2k+1)}(t) &= \tfrac{1}{\sqrt{2}}\,\phi_{jk}(t) - \tfrac{1}{\sqrt{2}}\,\psi_{jk}(t)\,. \end{aligned}$$

So each member of the basis for V_{j+1} is in $V_j + W_j$. Therefore $V_{j+1} \subset V_j + W_j$. When this is combined with our earlier inclusion result, we conclude that $V_{j+1} = V_j + W_j$. □

Solution (c). We proceed by induction on j. By Part b we know that

$$V_1 = V_0 + W_0\,, \quad \text{and} \quad V_2 = V_1 + W_1\,.$$

Therefore

$$V_2 = V_1 + W_1 = (V_0 + W_0) + W_1 = V_0 + W_0 + W_1\,.$$

Therefore we have established the result for $j = 1$.

Now suppose that the result holds for j for some $j \geq 1$. This means that

$$V_{j+1} = V_0 + W_0 + \cdots + W_j\,.$$

But then by Part b we see that

$$\begin{aligned} V_{j+2} = V_{j+1} + W_{j+1} &= \big(V_0 + W_0 + \cdots + W_j\big) + W_{j+1}\\ &= V_0 + W_0 + \cdots + W_j + W_{j+1}\,. \end{aligned}$$

Therefore we have established the result for $j+1$. Therefore the result holds for every $j \in \mathbb{Z}_+$ by induction. □

Remark. The orthonormal set $\{\psi_{jk} \,:\, j,k \in \mathbb{Z}\}$ is the Haar basis for $L^2(\mathbb{R})$. The proof that it is a basis for $L^2(\mathbb{R})$ is similar to the proof of the Haar basis for $L^2([0,1])$ that was presented in the solution to Problem 3. The result of Part d can then be strengthened. We have

$$V_j = \bigoplus_{j'<j} W_{j'} \quad \text{for every } j \in \mathbb{Z}\,, \qquad \text{and} \qquad L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j\,.$$

Remark. Haar did his work over 100 years ago. The label "Haar basis" was established long before wavelet theory was developed and the term "wavelet" was introduced, but from a modern perspective the label "Haar wavelet basis" is also suitible.

5. [20] Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform given by

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}} e^{-i2\pi\xi t} u(t)\,\mathrm{d}t \qquad \text{for every } u \in L^2(\mathbb{R})\,.$$

Let $\psi : \mathbb{R} \to \mathbb{R}$ be the function given by

$$\psi(t) = 2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t)\,.$$

For every $j, k \in \mathbb{Z}$ define $\psi_{jk} : \mathbb{R} \to \mathbb{R}$ by $\psi_{jk}(t) = 2^{\frac{j}{2}}\psi(2^j t - k)$.

a. [4] Compute $\mathcal{F}\psi(\xi)$. You can use the fact that

$$\mathcal{F}\mathrm{sinc}(\xi) = \begin{cases} 1 & \text{for } |\xi| < \frac{1}{2}\,, \\ \frac{1}{2} & \text{for } |\xi| = \frac{1}{2}\,, \\ 0 & \text{for } |\xi| > \frac{1}{2}\,. \end{cases}$$

b. [4] Compute

$$\int_0^\infty \frac{|\mathcal{F}\psi(\xi)|^2}{\xi}\,\mathrm{d}\xi\,.$$

c. [4] For every $j, k \in \mathbb{Z}$ compute $\mathcal{F}\psi_{jk}(\xi)$.

d. [8] Show that $\{\psi_{jk} : j, k \in \mathbb{Z}\}$ is an orthonormal set in $L^2(\mathbb{R})$.

Solution (a). By linearity of the Fourier transform we have

$$\mathcal{F}\psi(\xi) = 2\mathcal{F}[\mathrm{sinc}(2t)](\xi) - \mathcal{F}[\mathrm{sinc}(t)](\xi)\,.$$

From the definition of $\mathcal{F}$ and the given fact we see that

$$\begin{aligned} 2\mathcal{F}[\mathrm{sinc}(2t)](\xi) &= 2\int_{\mathbb{R}} e^{-i2\pi\xi t}\mathrm{sinc}(2t)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} e^{-i\pi\xi t}\mathrm{sinc}(t)\,\mathrm{d}t = \mathcal{F}\mathrm{sinc}(\tfrac{\xi}{2}) = \begin{cases} 1 & \text{if } |\xi| < 1\,, \\ \frac{1}{2} & \text{if } |\xi| = 1\,, \\ 0 & \text{if } |\xi| > 1\,. \end{cases} \end{aligned}$$

Therefore

$$\mathcal{F}\psi(\xi) = \mathcal{F}\mathrm{sinc}(\tfrac{\xi}{2}) - \mathcal{F}\mathrm{sinc}(\xi) = \begin{cases} 1 & \text{if } \frac{1}{2} < |\xi| < 1\,, \\ \frac{1}{2} & \text{if } |\xi| = \frac{1}{2} \text{ or } |\xi| = 1\,, \\ 0 & \text{if } |\xi| < \frac{1}{2} \text{ or } |\xi| > 1\,. \end{cases}$$

□

Solution (b). We see from Part a that

$$\int_0^\infty \frac{|\mathcal{F}\psi(\xi)|^2}{\xi}\,\mathrm{d}\xi = \int_{\frac{1}{2}}^1 \frac{1}{\xi}\,\mathrm{d}\xi = \log(\xi)\Big|_{\frac{1}{2}}^1 = -\log(\tfrac{1}{2}) = \log(2)\,.$$

□

Solution (c). Because $\psi_{jk}(t) = 2^{\frac{j}{2}}\psi(2^j t - k)$, we have

$$\begin{aligned}
\mathcal{F}\psi_{jk}(\xi) &= \int_{\mathbb{R}} e^{-i2\pi\xi t}\psi_{jk}(t)\,\mathrm{d}t = 2^{\frac{j}{2}}\int_{\mathbb{R}} e^{-i2\pi\xi t}\psi(2^j t - k)\,\mathrm{d}t \\
&= 2^{-\frac{j}{2}}\int_{\mathbb{R}} e^{-i2\pi\xi\frac{t+k}{2^j}}\psi(t)\,\mathrm{d}t = 2^{-\frac{j}{2}}e^{-i2\pi k\frac{\xi}{2^j}}\mathcal{F}\psi(\tfrac{\xi}{2^j}) \\
&= \begin{cases} 2^{-\frac{j}{2}}e^{-i2\pi k\frac{\xi}{2^j}} & \text{if } 2^{j-1} < |\xi| < 2^j\,, \\ 2^{-\frac{j+2}{2}}e^{-i2\pi k\frac{\xi}{2^j}} & \text{if } |\xi| = 2^{j-1} \text{ or } |\xi| = 2^j\,, \\ 0 & \text{if } |\xi| < 2^{j-1} \text{ or } |\xi| > 2^j\,. \end{cases}
\end{aligned}$$

□

Solution (d). The Plancherel Theorem says

$$\langle \psi_{j_1k_1}\,,\,\psi_{j_2k_2}\rangle = \langle \mathcal{F}\psi_{j_1k_1}\,,\,\mathcal{F}\psi_{j_2k_2}\rangle = \int_{\mathbb{R}} \overline{\mathcal{F}\psi_{j_1k_1}(\xi)}\,\mathcal{F}\psi_{j_2k_2}(\xi)\,\mathrm{d}\xi\,.$$

We see from our solution to Part c that

$$\begin{aligned}
&\mathcal{F}\psi_{j_1k_1}(\xi) \quad \text{is supported on } 2^{j_1-1} \le |\xi| \le 2^{j_1}\,, \\
&\mathcal{F}\psi_{j_2k_2}(\xi) \quad \text{is supported on } 2^{j_2-1} \le |\xi| \le 2^{j_2}\,.
\end{aligned}$$

Thus, if $j_1 \neq j_2$ then the above integrand vanishes at all but at most two points, whereby

$$\langle \mathcal{F}\psi_{j_1k_1}\,,\,\mathcal{F}\psi_{j_2k_2}\rangle = 0\,.$$

If $j_1 = j_2 = j$ then our solution to Part c, a change of variable, the odd symmetry of sine, the even symmetry of cosine, and an elementary integration combine to show that

$$\begin{aligned}
\langle \mathcal{F}\psi_{jk_1}\,,\,\mathcal{F}\psi_{jk_2}\rangle &= 2^{-j}\int_{2^{j-1}<|\xi|<2^j} e^{-i2\pi(k_2-k_1)\frac{\xi}{2^j}}\,\mathrm{d}\xi \\
&= \int_{\frac{1}{2}<|\xi|<1} e^{-i2\pi(k_2-k_1)\xi}\,\mathrm{d}\xi = \int_{\frac{1}{2}<|\xi|<1} \cos\bigl(2\pi(k_2-k_1)\xi\bigr)\,\mathrm{d}\xi \\
&= 2\int_{\frac{1}{2}}^{1} \cos\bigl(2\pi(k_2-k_1)\xi\bigr)\,\mathrm{d}\xi = \delta_{k_1k_2}\,.
\end{aligned}$$

Putting everything together we have

$$\langle \psi_{j_1k_1}\,,\,\psi_{j_2k_2}\rangle = \langle \mathcal{F}\psi_{j_1k_1}\,,\,\mathcal{F}\psi_{j_2k_2}\rangle = \delta_{j_1j_2}\,\delta_{k_1k_2}\,.$$

Therefore $\{\psi_{jk} \,:\, j,k \in \mathbb{Z}\}$ is an orthonormal set in $L^2(\mathbb{R})$. □

Remark. The orthonormal set $\{\psi_{jk} \,:\, j,k \in \mathbb{Z}\}$ is the Shannon wavelet basis. To prove that this set is a basis we would need to show for any $u \in L^2(\mathbb{R})$ that

$$\langle \psi_{jk}\,,\,u\rangle = 0 \quad \text{for every } j,k \in \mathbb{Z} \qquad \Longrightarrow \qquad u = 0\,.$$

This can be done using the Plancherel Theorem and the result of Part c. Try it!